



# Lattice fractional calculus



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## ABSTRACT

Integration and differentiation of non-integer orders for  $N$ -dimensional physical lattices with long-range particle interactions are suggested. The proposed lattice fractional derivatives and integrals are represented by kernels of lattice long-range interactions, such that their Fourier series transformations have a power-law form with respect to components of wave vector. Continuous limits for these lattice fractional derivatives and integrals give the continuum derivatives and integrals of non-integer orders with respect to coordinates. Lattice analogs of fractional differential equations that include suggested lattice differential and integral operators can serve as an important element of microscopic approach to non-local continuum models in mechanics and physics.

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## 1. Introduction

The main approaches to describe nonlocal properties of media and materials are a macroscopic approach based on the continuum mechanics [1–5], and a microscopic approach based on the lattice mechanics [6–9]. Continuum mechanics can be considered as a continuous limit of lattice dynamics, where the sizes of continuum elements are much larger than the distances between lattice particles.

Theory of derivatives and integrals of non-integer orders [10–19] has a long history and it goes back to the famous scientist such as Leibniz, Riemann, Liouville, Letnikov, Weyl, Riesz and other. Fractional calculus and fractional differential equations have a wide application in different areas of physics [20–31]. Fractional integro-differential equations are very important to describe processes in nonlocal continua and media. Fractional integrals and differential operators with respect to coordinates allow us to describe continuously distributed system with power-law type of nonlocality. Therefore fractional calculus serve as a powerful tool in physics and mechanics of nonlocal continua. As it was shown in [41,42,25], the fractional differential equations for nonlocal continua can be directly connected to models of lattice with long-range interactions of power-law type. Interconnection between the equations for lattice with long-range interactions and the fractional differential equations for continuum is proved by special transform operator that includes a continuous limit, and the Fourier series and integral transformations [41–44]. In [55–59] this approach has been applied to lattice models of fractional nonlocal continua in one-dimensional case only. In this paper we propose a lattice fractional calculus that allows us to extend these lattice models to  $N$ -dimensional case.

Dynamics of physical lattices and discretely distributed systems with long-range interactions has been the subject of investigations in different areas of science. Effect of synchronization for nonlinear systems with long-range interactions is described in [32]. Non-equilibrium phase transitions for systems with long-range interactions are considered in [33]. Stationary states for fractional systems with long-range interactions are discussed in [46,34,35]. The evolution of soliton-like

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and breather-like structures in one-dimensional lattice of coupled oscillators with the long-range power are considered in [36]. Kinks in the Frenkel–Kontorova model with long-range particle interactions is studied in [37]. In statistical mechanics and nonlinear dynamics, solvable models with long-range interactions are described in detail in the reviews [38–40]. Different discrete systems and lattice with long-range interactions and its continuous limits are considered in [23,25]. It is important that lattice models with long-range interactions of power-law type can lead to fractional nonlocal continuum models in the continuous limit [41,42,25]. Nonlocal continuum mechanics can be considered as a continuous limit of mechanics of lattice with long-range interactions, when the sizes of continuum element are much larger than the distances between particles of lattice.

It should be note that a calculus of operators of integer orders for physical lattice models has been considered in the papers [48–50]. This lattice calculus of integer order is defined on a general triangulating graph by using discrete field quantities and differential operators roughly analogous to differential forms and exterior differential calculus. A scheme to derive lattice differential operators of integer orders from the discrete velocities and associated Maxwell–Boltzmann distributions that are used in lattice hydrodynamics has been suggested in the articles [51,52]. In this paper to formulate a lattice fractional calculus, we use other approach that is based on models of physical lattices with long-range inter-particle interactions and its continuum limit that are suggested in [41,42,25] (see also [43–47,55–59]).

In this paper, we propose lattice analogs of differentiation and integration of non-integer orders based on  $N$ -dimensional generalization of the lattice approach suggested in [41,42,25]. A general form of lattice fractional derivatives and integrals that gives continuum derivatives and integrals of non-integer orders in continuous limit is suggested. These continuum fractional operators of differentiations and integrations can be considered as fractional derivatives and integrals of the Riesz type with respect to coordinates.

## 2. Lattice fractional differential operators

### 2.1. Lattice fractional partial derivatives

Let us consider an unbounded physical lattice characterized by  $N$  non-coplanar vectors  $\mathbf{a}_i, i = 1, \dots, N$ , that are the shortest vectors by which a lattice can be displaced and be brought back into itself. For simplification, we assume that  $\mathbf{a}_i, i = 1, \dots, N$ , are mutually perpendicular primitive lattice vectors. We choose directions of the axes of the Cartesian coordinate system coincides with the vector  $\mathbf{a}_i$ . Then  $\mathbf{a}_i = a_i \mathbf{e}_i$ , where  $a_i = |\mathbf{a}_i|$  and  $\mathbf{e}_i, i = 1, \dots, N$ , is the basis of the Cartesian coordinate system for  $\mathbb{R}^N$ . This simplification means that the lattice is a primitive  $N$ -dimensional orthorhombic Bravais lattice. The position vector of an arbitrary lattice site is written  $\mathbf{r}(\mathbf{n}) = \sum_{i=1}^N n_i \mathbf{a}_i$ , where  $n_i$  are integer. In a lattice the sites are numbered by  $\mathbf{n}$ , so that the vector  $\mathbf{n} = (n_1, \dots, n_N)$  can be considered as a number vector of the corresponding lattice particle. We assume that the equilibrium positions of particles coincide with the lattice sites  $\mathbf{r}(\mathbf{n})$ . Coordinates  $\mathbf{r}(\mathbf{n})$  of lattice sites differs from the coordinates of the corresponding particles, when particles are displaced relative to their equilibrium positions. To define the coordinates of a particle, we define displacement of  $\mathbf{n}$ -particle from its equilibrium position by the scalar field  $u(\mathbf{n})$ , or the vector field  $\mathbf{u}(\mathbf{n}) = \sum_{i=1}^N u_i(\mathbf{n}) \mathbf{e}_i$ , where the vectors  $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$  form the basis of the Cartesian coordinate system. The functions  $u_i(\mathbf{n}) = u_i(n_1, \dots, n_N)$  are components of the displacement vector for lattice particle that is defined by  $\mathbf{n} = (n_1, \dots, n_N)$ . In many cases, we can assume that  $u(\mathbf{n})$  belongs to the Hilbert space  $l_2$  of square-summable sequences to apply the Fourier transformations. For simplification, we will consider differential and integral operators for the lattice functions  $u = u(\mathbf{n}) = u(n_1, \dots, n_N)$ . All transformations can be easily generalized to the case of the vector functions.

Let us give a definition of lattice partial derivative of arbitrary positive real order  $\alpha$  in the direction  $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$  in the lattice.

**Definition 1.** A lattice fractional partial derivative is the operator  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  such that

$$\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] u = \frac{1}{a_i^\alpha} \sum_{m_i=-\infty}^{+\infty} K_\alpha^\pm(n_i - m_i) u(\mathbf{m}), \quad (i = 1, \dots, N), \tag{1}$$

where  $\alpha \in \mathbb{R}, \alpha > 0, \mathbf{m} \in \mathbb{Z}$ , and the interaction kernels  $K_\alpha^\pm(n - m)$  are defined by the equations

$$K_\alpha^+(n - m) = \frac{\pi^\alpha}{\alpha + 1} {}_1F_2 \left( \frac{\alpha + 1}{2}; \frac{1}{2}, \frac{\alpha + 3}{2}; -\frac{\pi^2(n - m)^2}{4} \right), \quad \alpha > 0, \tag{2}$$

$$K_\alpha^-(n - m) = -\frac{\pi^{\alpha+1}(n - m)}{\alpha + 2} {}_1F_2 \left( \frac{\alpha + 2}{2}; \frac{3}{2}, \frac{\alpha + 4}{2}; -\frac{\pi^2(n - m)^2}{4} \right), \quad \alpha > 0, \tag{3}$$

where  ${}_1F_2$  is the Gauss hypergeometric function [63]. The parameter  $\alpha > 0$  will be called the order of the lattice derivative (1).

Let us explain the reasons for definition the interaction kernels  $K_\alpha^\pm(n - m)$  in the forms (2), (3), and describe some properties of these kernels.

The kernels  $K_{\alpha}^{\pm}(n)$  are real-valued functions of integer variable  $n \in \mathbb{Z}$ . The kernel  $K_{\alpha}^{+}(n)$  is even (or symmetric with respect to zero) function and  $K_{\alpha}^{-}(n)$  is odd (or antisymmetric with respect to zero) function such that

$$K_{\alpha}^{+}(-n) = +K_{\alpha}^{+}(n), \quad K_{\alpha}^{-}(-n) = -K_{\alpha}^{-}(n) \tag{4}$$

hold for all  $n \in \mathbb{Z}$ .

The Fourier series transforms  $\hat{K}_{\alpha}^{+}(k)$  of the kernels  $K_{\alpha}^{+}(n)$  in the form

$$\hat{K}_{\alpha}^{+}(k) = \sum_{n=-\infty}^{+\infty} e^{-ikn} K_{\alpha}^{+}(n) = 2 \sum_{n=1}^{\infty} K_{\alpha}^{+}(n) \cos(kn) + K_{\alpha}^{+}(0) \tag{5}$$

satisfy the condition

$$\hat{K}_{\alpha}^{+}(k) = |k|^{\alpha}, \quad (\alpha > 0). \tag{6}$$

The Fourier series transforms  $\hat{K}_{\alpha}^{-}(k)$  of the kernels  $K_{\alpha}^{-}(n)$  in the form

$$\hat{K}_{\alpha}^{-}(k) = \sum_{n=-\infty}^{+\infty} e^{-ikn} K_{\alpha}^{-}(n) = -2i \sum_{n=1}^{\infty} K_{\alpha}^{-}(n) \sin(kn) \tag{7}$$

satisfy the condition

$$\hat{K}_{\alpha}^{-}(k) = i \operatorname{sgn}(k) |k|^{\alpha}, \quad (\alpha > 0). \tag{8}$$

Note that we use the minus sign in the exponents of (5) and (7) instead of plus in order to have the plus sign for plane waves and for the Fourier series.

The form (2) of the interaction term  $K_{\alpha}^{+}(n - m)$  is completely determined by the requirement (6). If we use an inverse relation to (5) with  $\hat{K}_{\alpha}^{+}(k) = |k|^{\alpha}$  that has the form

$$K_{\alpha}^{+}(n) = \frac{1}{\pi} \int_0^{\pi} k^{\alpha} \cos(nk) dk, \quad (\alpha \in \mathbb{R}, \quad \alpha > 0), \tag{9}$$

then we get Eq. (2) for the interaction kernel  $K_{\alpha}^{+}(n - m)$ . Note that

$$K_{\alpha}^{+}(0) = \frac{\pi^{\alpha}}{\alpha + 1}. \tag{10}$$

The form (3) of the interaction term  $K_{\alpha}^{-}(n - m)$  is completely determined by (6). If we use an inverse relation to (7) with  $\hat{K}_{\alpha}^{-}(k) = i \operatorname{sgn}(k) |k|^{\alpha}$  that has the form

$$K_{\alpha}^{-}(n) = -\frac{1}{\pi} \int_0^{\pi} k^{\alpha} \sin(nk) dk \quad (\alpha \in \mathbb{R}, \quad \alpha > 0), \tag{11}$$

then we get Eq. (3) for the interaction kernel  $K_{\alpha}^{-}(n - m)$ . Note that  $K_{\alpha}^{-}(0) = 0$ .

The interactions with (2) and (3) for integer and non-integer orders  $\alpha$  can be interpreted as a long-range interactions of  $n$ -particle with all other particles.

Properties of the interaction kernels (2) and (3), can be visualized by plots of the functions

$$f_{+}(x, y) = \frac{\pi^y}{y + 1} {}_1F_2\left(\frac{y + 1}{2}; \frac{1}{2}, \frac{y + 3}{2}; -\frac{\pi^2 x^2}{4}\right), \tag{12}$$

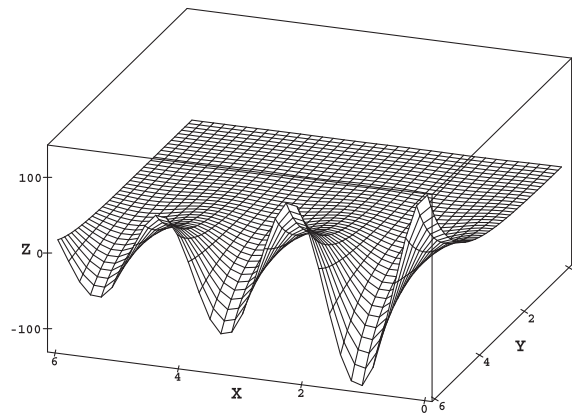
$$f_{-}(x, y) = -\frac{\pi^{y+1} x}{y + 2} {}_1F_2\left(\frac{y + 2}{2}; \frac{3}{2}, \frac{y + 4}{2}; -\frac{\pi^2 x^2}{4}\right) \tag{13}$$

for positive values of variables  $x$  and  $y$ . The function (12) is given on Figs. 1, 3 and 5, and the function (13) is presented by Figs. 2, 4 and 6.

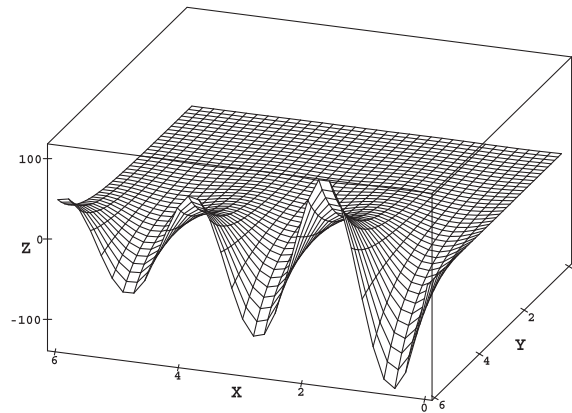
Let us give exact forms of the kernels  $\hat{K}_{\alpha}^{\pm}(k)$  for integer positive  $\alpha \in \mathbb{N}$ . Eqs. (2) and (3) for the case  $\alpha \in \mathbb{N}$  can be simplified. We can use inverse relations (9) and (11) for integer positive  $\alpha \in \mathbb{N}$ . to define exact form of the kernels  $\hat{K}_{\alpha}^{\pm}(k)$ . To obtain the simplified expressions for kernels  $\hat{K}_{\alpha}^{\pm}(k)$  with positive integer  $\alpha = m$ , we use the integrals (see Section 2.5.3.5 in [62]) of the form

$$\int_0^{\pi} x^m \cos(nx) dx = \frac{(-1)^{n+2}}{n^{m+1}} \sum_{k=0}^{[(m-1)/2]} \frac{(-1)^k m!}{(m - 2n - 1)!} (\pi n)^{m-2k-1} + \frac{(-1)^{[(m+1)/2]} m!}{n^{m+1}} (2[(m + 1)/2] - m), \quad (m \in \mathbb{N}), \tag{14}$$

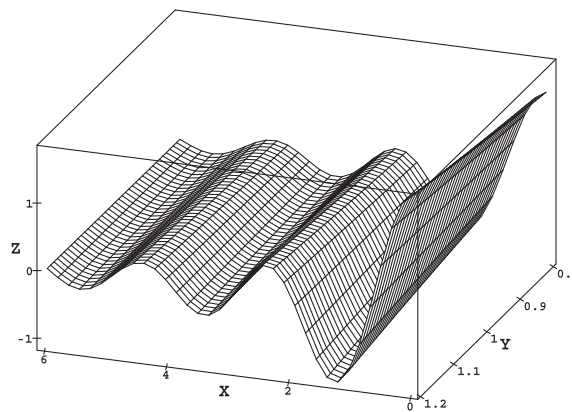
$$\int_0^{\pi} x^m \sin(nx) dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{[m/2]} \frac{(-1)^k m!}{(m - 2n)!} (\pi n)^{m-2k} + \frac{(-1)^{[m/2]} m!}{n^{m+1}} (2[m/2] - m + 1), \quad (m \in \mathbb{N}), \tag{15}$$



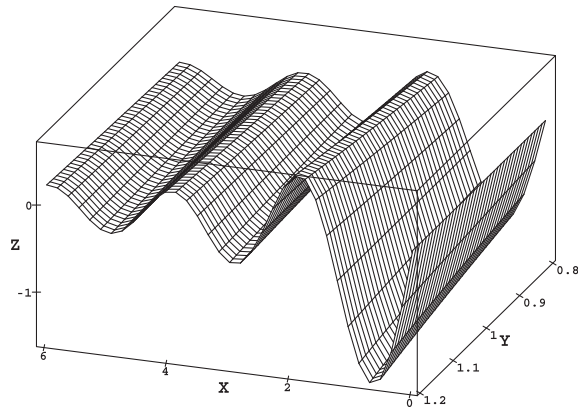
**Fig. 1.** Plot of the function  $f_+(x, y)$  (12) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 6]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .



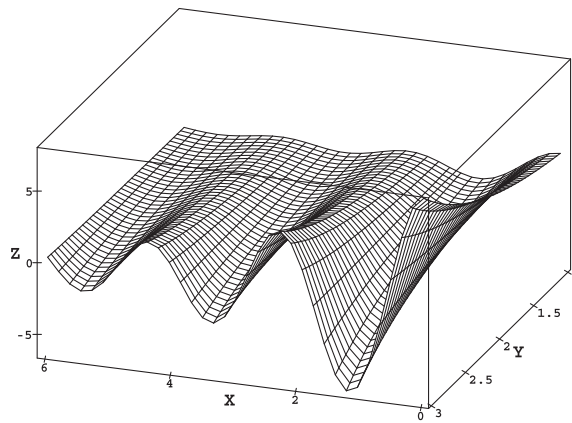
**Fig. 2.** Plot of the function  $f_-(x, y)$  (13) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 6]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .



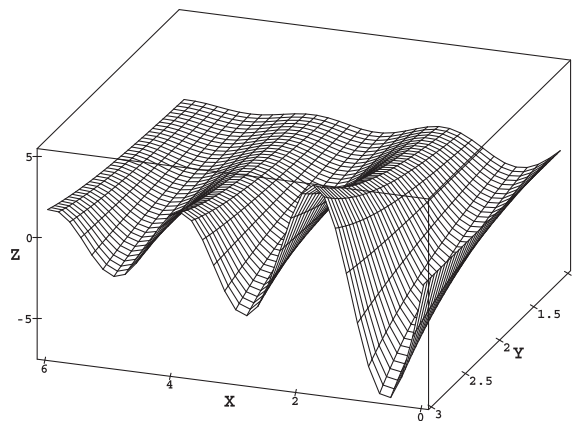
**Fig. 3.** Plot of the function  $f_+(x, y)$  (12) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0.8, 1.2]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .



**Fig. 4.** Plot of the function  $f_-(x,y)$  (13) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 0.8, 1.2]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .



**Fig. 5.** Plot of the function  $f_+(x,y)$  (12) for the range  $x \in [0, 6]$  and  $y = \alpha \in [1, 3]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .



**Fig. 6.** Plot of the function  $f_-(x,y)$  (13) for the range  $x \in [0, 6]$  and  $y = \alpha \in [1, 3]$  that represents the kernels of the lattice fractional derivatives  $\mathbb{D}_L^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$  with  $\alpha = y$ .

where  $[x]$  is the integer part of the value  $x$ , and  $n \in \mathbb{N}$ . Here  $2[(m + 1)/2] - m = 1$  for odd  $m$ , and  $2[(m + 1)/2] - m = 0$  for even  $m$ .

It is easy to see that the kernels  $K_\alpha^\pm(n)$  for integer positive  $\alpha = m \in \mathbb{N}$  are defined by the equations

$$K_\alpha^+(n) = \sum_{k=0}^{[(\alpha-1)/2]} \frac{(-1)^{n+k} s! \pi^{\alpha-2k-2}}{(\alpha-2n-1)!} \frac{1}{n^{2k+2}} + \frac{(-1)^{[(\alpha+1)/2]} s! (2[(\alpha+1)/2] - \alpha)}{\pi n^{\alpha+1}} \tag{16}$$

and

$$K_\alpha^-(n) = - \sum_{k=0}^{[\alpha/2]} \frac{(-1)^{n+k+1} s! \pi^{\alpha-2k-1}}{(\alpha-2n)!} \frac{1}{n^{2k+2}} - \frac{(-1)^{[\alpha/2]} s! (2[\alpha/2] - \alpha + 1)}{\pi n^{\alpha+1}}. \tag{17}$$

Using Eq. (16) or direct integration (9) for integer values  $\alpha = 1, 2, 3, 4$ , we get the examples of  $K_\alpha^+(n)$  with  $n \neq 0$  in the form

$$K_1^+(n) = - \frac{1 - (-1)^n}{\pi n^2}, \quad K_2^+(n) = \frac{2(-1)^n}{n^2}, \tag{18}$$

$$K_3^+(n) = \frac{3\pi(-1)^n}{n^2} + \frac{6(1 - (-1)^n)}{\pi n^4}, \quad K_4^+(n) = \frac{4\pi^2(-1)^n}{n^2} - \frac{24(-1)^n}{n^4}, \tag{19}$$

where  $n \neq 0, n \in \mathbb{Z}$ , and  $K_m^+(0) = \pi^m / (m + 1)$  for all  $m \in \mathbb{N}$ . Using Eq. (17) or direct integration (11) for  $\alpha = 1, 2, 3, 4$ , we get examples of  $K_\alpha^-(n)$  with  $n \neq 0$  in the form

$$K_1^-(n) = \frac{(-1)^n}{n}, \quad K_2^-(n) = \frac{(-1)^n \pi}{n} + \frac{2(1 - (-1)^n)}{\pi n^3}, \tag{20}$$

$$K_3^-(n) = \frac{(-1)^n \pi^2}{n} - \frac{6(-1)^n}{n^3}, \quad K_4^-(n) = \frac{(-1)^n \pi^3}{n} - \frac{12(-1)^n \pi}{n^3} - \frac{24(1 - (-1)^n)}{\pi n^5}, \tag{21}$$

where  $n \neq 0, n \in \mathbb{Z}$ , and  $K_m^-(0) = 0$  for all  $m \in \mathbb{N}$ . Note that  $(1 - (-1)^n) = 2$  for odd  $n$ , and  $(1 - (-1)^n) = 0$  for even  $n$ . In the definition of lattice fractional derivatives (1) the value  $i \in \{1, \dots, N\}$  characterizes the component  $n_i$  of the lattice vector  $\mathbf{n}$  with respect to which this derivative is taken. It is similar to the variable  $x_i$  in the usual partial derivatives for the space

$\mathbb{R}^N$ . The lattice operators  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  are analogous to the partial derivatives of order  $\alpha$  with respect to coordinates  $x_i$  for continuum model. The lattice derivative  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  is an operator along the vector  $\mathbf{e}_i = \mathbf{a}_i / |\mathbf{a}_i|$  in the lattice.

### 2.2. An extension of lattice derivatives of non-integer orders

In general, we can weaken the conditions (6) and (8) to determine a more wider class of the lattice fractional derivatives. For this aim, we replace the exact conditions (6) and (8) by the asymptotical requirements

$$\hat{K}_\alpha^+(k) = |k|^\alpha + o(|k|^\alpha), \quad \hat{K}_\alpha^-(k) = i \operatorname{sgn}(k) |k|^\alpha + o(|k|^\alpha), \quad (k \rightarrow 0),$$

where the little-o notation  $o(|k|^\alpha)$  means the terms that include higher powers of  $|k|$  than  $|k|^\alpha$ .

**Definition 2.** A lattice fractional partial derivative is the operator  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  such that

$$\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] u = \frac{1}{a_i^\alpha} \sum_{m_i=-\infty}^{+\infty} K_\alpha^\pm(n_i - m_i) u(\mathbf{m}), \quad (i = 1, \dots, N), \tag{22}$$

where the interaction kernels  $K_\alpha^\pm(n - m)$  satisfy the conditions:

- (a) The kernels  $K_\alpha^\pm(n)$  are real-valued functions of integer variable  $n \in \mathbb{Z}$ . The kernel  $K_\alpha^+(n)$  is even (or symmetric with respect to zero) function and the kernels  $K_\alpha^-(n)$  is odd (or antisymmetric with respect to zero) function such that

$$K_\alpha^+(-n) = +K_\alpha^+(n), \quad K_\alpha^-(-n) = -K_\alpha^-(n) \tag{23}$$

hold for all  $n \in \mathbb{Z}$ .

- (b) The Fourier series transforms of the kernels  $K_\alpha^+(n)$  in the form (5) satisfy the condition

$$\hat{K}_\alpha^+(k) = |k|^\alpha + o(|k|^\alpha), \quad (k \rightarrow 0), \tag{24}$$

- (c) The Fourier series transforms of the kernels  $K_\alpha^-(n)$  in the form (7) satisfy the condition

$$\hat{K}_\alpha^-(k) = i \operatorname{sgn}(k) |k|^\alpha + o(|k|^\alpha), \quad (k \rightarrow 0), \tag{25}$$

The parameter  $\alpha > 0$  will be called the order of the operator (22).

The conditions (24) and (25) also means that we can consider arbitrary functions  $K_{\alpha}^{\pm}(n - m)$  for which  $K_{\alpha}^{\pm}(k)$  are asymptotically equivalent to  $|k|^{\alpha}$  and  $i \operatorname{sgn}(k) |k|^{\alpha}$  as  $|k| \rightarrow 0$  respectively.

A simple example of the interaction kernel  $K_{\alpha}^{+}(n - m)$ , which can give the lattice fractional derivatives (22) with (24), has been suggested in [41,42] in the form

$$K_{\alpha}^{+}(n - m) = \frac{(-1)^{n-m} \Gamma(\alpha + 1)}{\Gamma(\alpha/2 + 1 + (n - m)) \Gamma(\alpha/2 + 1 - (n - m))}. \tag{26}$$

Note that for integer  $\alpha \in \mathbb{N}$   $K_{\alpha}^{+}(n - m) = 0$  for  $|n - m| \geq \alpha/2 + 1$ . For  $\alpha = 2j$ , we have  $K_{\alpha}^{+}(n - m) = 0$  for all  $|n - m| \geq j + 1$ . The function  $K_{\alpha}^{+}(n - m)$  with even value of  $\alpha = 2j$  describes an interaction of the  $n$ -particle with  $2j$  particles with numbers  $n \pm 1 \dots n \pm j$ . It is easy to see that expression (2) is more complicated than (26). Note that the long-range interaction with the kernel (26) is partially connected with fractional central differences considered in [61], and the long-range interaction of the Grünwald–Letnikov–Riesz type [56].

As an example of the interaction kernel  $K_{\alpha}^{-}(n - m)$  with (25), we can give

$$K_{\alpha}^{-}(n) = \frac{(-1)^{((n+1)/2)} (2[(n + 1)/2] - n) \Gamma(\alpha + 1)}{2^{\alpha} \Gamma((\alpha + n)/2 + 1) \Gamma((\alpha - n)/2 + 1)}. \tag{27}$$

where the brackets  $[\ ]$  is the floor function that maps a real number to the largest previous integer number. The expression  $2[(n + 1)/2] - n$  is equal to zero for even  $n = 2m$ , and it is equal to 1 for odd  $n = 2m - 1$ . It should be noted that the kernel (27) is real valued function since we have zero, when the expression  $(-1)^{(n+1)/2}$  is a complex number. This kernel  $K_{\alpha}^{-}(n)$  is the odd function  $K_{\alpha}^{-}(-n) = -K_{\alpha}^{-}(n)$ .

For  $0 < \alpha \leq 2$ , we can also use different forms of interaction kernels  $K_{\alpha}^{+}(n - m)$  that are suggested in Section 8 of [25]. For example, we have the kernel of the long-range interactions of the power-law form

$$K_{\alpha}^{+}(n - m) = \frac{1}{A(\beta) |n - m|^{\beta}}, \quad (\beta > -1), \tag{28}$$

where the function  $A(\beta)$  of the real parameter  $\beta$  is defined by the range of order  $\alpha$ . If  $1 < \beta < 2$  or  $2 < \beta < 3$ , then  $A(\beta) = -2\Gamma(1 - \beta) \sin(\pi\beta/2)$ . For non-integer  $\beta > 3$ , we have  $A_{\beta} = \zeta(\beta - 2)$ , where  $\zeta(z)$  is the Riemann zeta-function. For details see Section 8.11 and 8.12 in [25].

Note that we can define a modification of definition (22) of the derivative  $\mathbb{D}_L^{+} \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  by the condition  $m_i \neq n_i$  in the sum to exclude self-actions of the particles in the physical lattice. In this case, we should replace the condition (24) by

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e^{-ikn} K_{\alpha}^{+}(n) = 2 \sum_{n=1}^{\infty} K_{\alpha}^{+}(n) \cos(kn) = |k|^{\alpha} + o(|k|^{\alpha}), \quad (k \rightarrow 0). \tag{29}$$

In addition, differential operators of integer orders has been suggested in [48–50] for physical lattice models by using analogous to differential forms and exterior differential calculus. A fractional generalization of exterior differential calculus of differential forms is suggested [53,54,25], where non-locality is described by the Caputo fractional derivatives. We assume that this tool can be used to generalize the approach proposed in [48–50] for operators non-integer orders.

### 3. Lattice fractional integral operators

#### 3.1. Initial lattice fractional integrations

Let us give a definition lattice fractional integrations of positive real order  $\alpha$  with respect to  $n_i$ , where  $i = 1, \dots, N$ .

**Definition 3.** A lattice fractional integral is the operator  $\mathbb{I}_L^{\pm} \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  such that

$$\mathbb{I}_L^{\pm} \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] u = \frac{1}{a_i^{\alpha}} \sum_{m_i=-\infty}^{+\infty} L_{\alpha}^{\pm}(n_i - m_i) u(\mathbf{m}) \quad (i = 1, \dots, N), \tag{30}$$

where  $\alpha \in \mathbb{R}$ ,  $\alpha > 0$ ,  $\mathbf{m} \in \mathbb{Z}$ , and the interaction kernels  $L_{\alpha}^{\pm}(n - m)$  are defined by the equations

$$L_{\alpha}^{+}(n - m) = \frac{\pi^{-\alpha}}{1 - \alpha} {}_1F_2 \left( \frac{1 - \alpha}{2}; \frac{1}{2}, \frac{3 - \alpha}{2}; -\frac{\pi^2 (n - m)^2}{4} \right), \quad (0 < \alpha < 1), \tag{31}$$

$$L_{\alpha}^{-}(n - m) = \frac{\pi^{1-\alpha} (n - m)}{2 - \alpha} {}_1F_2 \left( \frac{2 - \alpha}{2}; \frac{3}{2}, \frac{4 - \alpha}{2}; -\frac{\pi^2 (n - m)^2}{4} \right), \quad (0 < \alpha < 2), \tag{32}$$

where  ${}_1F_2$  is the Gauss hypergeometric function [63]. The parameter  $\alpha > 0$  will be called the order of the lattice fractional integral (30).

Note that the kernels  $L_{\alpha}^{\pm}(n)$  of lattice fractional integrals are connected with the expressions of the kernels  $K_{\alpha}^{\pm}(n)$  of lattice derivatives by the equations

$$L_{\alpha}^{+}(n) = +K_{-\alpha}^{+}(n), \quad 0 < \alpha < 1, \tag{33}$$

$$L_{\alpha}^{-}(n) = -K_{-\alpha}^{-}(n), \quad 0 < \alpha < 2. \tag{34}$$

The minus sign in the Eq. (34), and the positive sign in the Eq. (32) are used to have the correspondence with the usual integration for  $\alpha = 1$ .

The interaction kernels  $L_{\alpha}^{\pm}(n - m)$ , which are used in the definition of lattice fractional integral, can be characterized by the following properties.

The kernels  $L_{\alpha}^{\pm}(n)$  are real-valued functions of integer variable  $n \in \mathbb{Z}$ . The kernel  $L_{\alpha}^{+}(n)$  is even function and the kernel  $L_{\alpha}^{-}(n)$  is odd function such that

$$L_{\alpha}^{+}(-n) = +L_{\alpha}^{+}(n), \quad L_{\alpha}^{-}(-n) = -L_{\alpha}^{-}(n) \tag{35}$$

hold for all  $n \in \mathbb{Z}$ .

The Fourier series transforms  $\hat{L}_{\alpha}^{+}(k)$  of the kernels  $L_{\alpha}^{+}(n)$  in the form

$$\hat{L}_{\alpha}^{+}(k) = \sum_{n=-\infty}^{+\infty} e^{-ikn} L_{\alpha}^{+}(n) = 2 \sum_{n=1}^{\infty} L_{\alpha}^{+}(n) \cos(kn) + L_{\alpha}^{+}(0), \quad (0 < \alpha < 1) \tag{36}$$

satisfy the condition

$$\hat{L}_{\alpha}^{+}(k) = \frac{1}{|k|^{\alpha}}. \tag{37}$$

The Fourier series transforms  $\hat{L}_{\alpha}^{-}(k)$  of the kernels  $L_{\alpha}^{-}(n)$  in the form

$$\hat{L}_{\alpha}^{-}(k) = \sum_{n=-\infty}^{+\infty} e^{-ikn} L_{\alpha}^{-}(n) = -2i \sum_{n=1}^{\infty} L_{\alpha}^{-}(n) \sin(kn) \tag{38}$$

satisfy the condition

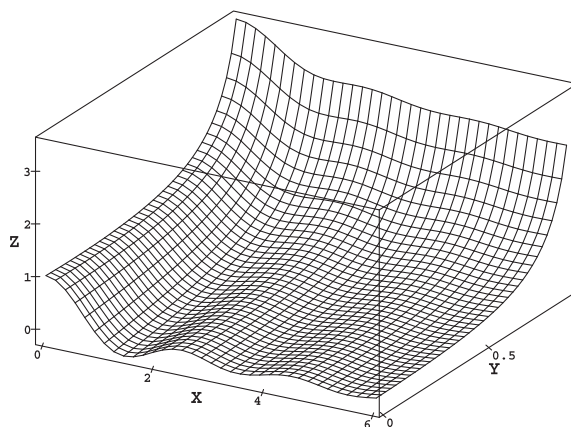
$$\hat{L}_{\alpha}^{-}(k) = -\frac{i \operatorname{sgn}(k)}{|k|^{\alpha}}, \quad (0 < \alpha < 2). \tag{39}$$

Note that the Fourier series transforms of the kernels of lattice integration are connected with the kernels of lattice differentiation by the relations

$$\hat{L}_{\alpha}^{+}(k) = +\hat{K}_{-\alpha}^{+}(k), \quad 0 < \alpha < 1, \tag{40}$$

$$\hat{L}_{\alpha}^{-}(k) = -\hat{K}_{-\alpha}^{-}(k), \quad 0 < \alpha < 2. \tag{41}$$

We also can state that  $\hat{L}_{\alpha}^{-}(k) = (ik)^{-1}$ , and the lattice integral  $\mathbb{I}_L^{\alpha} \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $\alpha = 1$  corresponds to the usual integral of first order with respect to  $x_i \in \mathbb{R}^1$  in the continuous limit. In this case, the kernel  $L_{\alpha}^{-}(n)$  can be represented by the sine-integral in the form  $L_1^{-}(n) = -\operatorname{Si}(\pi n)/\pi$ .



**Fig. 7.** Plot of the function  $g_{\alpha}(x, y)$  (42) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 0.9]$  that represents the kernels of the lattice fractional integrals  $\mathbb{I}_L^{\alpha} \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $\alpha = y$ .



To demonstrate the properties of (31) and (32), we visualize the functions

$$g_+(x,y) = f_+(x,-y) = \frac{\pi^{-y}}{1-y} {}_1F_2\left(\frac{1-y}{2}; \frac{1}{2}, \frac{3-y}{2}; -\frac{\pi^2 x^2}{4}\right), \tag{42}$$

$$g_-(x,y) = -f_-(x,-y) = \frac{\pi^{1-y} x}{2-y} {}_1F_2\left(\frac{2-y}{2}; \frac{3}{2}, \frac{4-y}{2}; -\frac{\pi^2 x^2}{4}\right) \tag{43}$$

for positive values of continuous variables  $x$  and  $y$  for the case  $0 < y < 1$ . The function (42) are presented by Figs. 7 and 9, and the function (43) are given by Figs. 8 and 10.

3.2. Lattice fractional integration for higher orders

Using (31) and (32), we can see that the kernel  $L_\alpha^+(n-m)$  of lattice fractional integral operator  $\mathbb{I}_L^+ \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  is defined for  $0 < \alpha < 1$ , and the kernel  $L_\alpha^-(n-m)$  of lattice operator  $\mathbb{I}_L^- \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  is defined for  $0 < \alpha < 2$ . Therefore these “initial” lattice fractional integral operators  $\mathbb{I}_L^+ \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  and  $\mathbb{I}_L^- \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  are defined only for  $0 < \alpha < 1$  and  $0 < \alpha < 2$  respectively. Note that there is only one “initial” lattice integral operators of integer order. It is the “odd” lattice integral operators  $\mathbb{I}_L^- \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]$ .

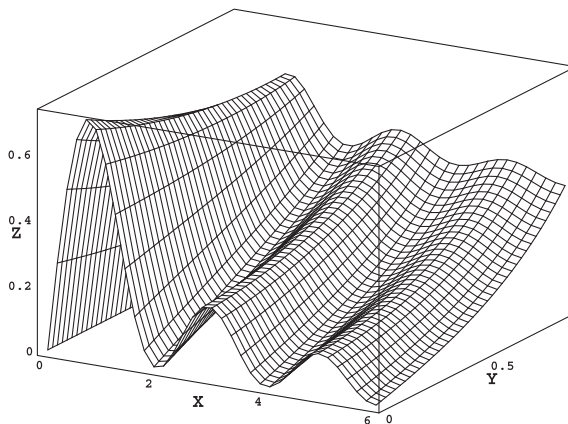


Fig. 8. Plot of the function  $g_-(x,y)$  (43) for the range  $x \in [0,6]$  and  $y = \alpha \in [0,0.9]$  that represents the kernels of the lattice fractional integrals  $\mathbb{I}_L^- \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $\alpha = y$ .

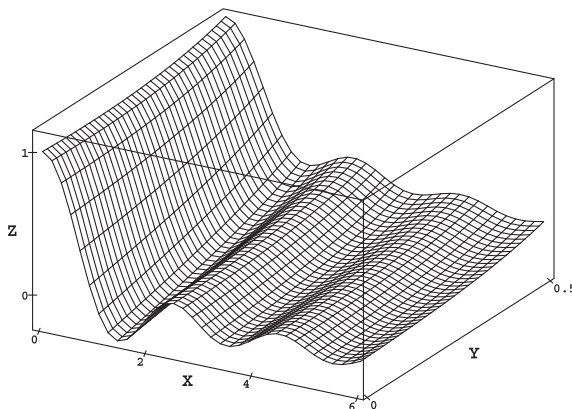
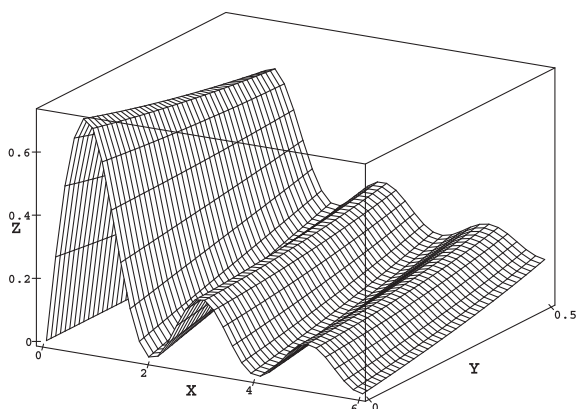


Fig. 9. Plot of the function  $g_+(x,y)$  (42) for the range  $x \in [0,6]$  and  $y = \alpha \in [0,0.5]$  that represents the kernels of the lattice fractional integrals  $\mathbb{I}_L^+ \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $\alpha = y$ .



**Fig. 10.** Plot of the function  $g_-(x, y)$  (43) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 0.5]$  that represents the kernels of the lattice fractional integrals  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $\alpha = y$ .

We can expand the definitions of lattice fractional integral operators in order to include all positive orders  $\alpha$  such that  $n - 1 < \alpha \leq n$ , where  $n \in \mathbb{N}$ . It is possible to define the lattice fractional integral operators for  $n - 1 < \alpha \leq n$  with  $n \geq 2$  by the following ways:

- (1) Using the “initial” integral operators only, we can consider two types of formulas: (a)  $I\alpha I$ -type: Composition of one operator of non-integer order and the operators of integer orders; (b)  $\alpha II$ -type: Composition of operators with non-integer orders only.
- (2) Using compositions of differential and “initial” integral operators, we can consider the following types of formulas: (a)  $\alpha DI$ -type: The integral operators act first and then we use the differential operator. (b)  $I\alpha D$ -type: The differential operator acts first and then we use the integral operators.

We can state that the  $\alpha DI$ -type operators in some sense are similar to the Riemann–Liouville fractional derivatives, and the  $I\alpha D$ -type operators are similar to the Caputo fractional derivatives [12].

We should define the lattice fractional integral operators of higher orders  $\alpha$  such that the parity of the kernels of these operators and the correspondent Fourier series transforms will be the same as for “initial” lattice integral operators (30)–(32).

Let us define the lattice fractional integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$  by using compositions of differential and “initial” integral operators.

The  $\alpha DI$ -type of lattice integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$ , is defined by the equation

$$\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] = \begin{cases} \pm (-1)^{m+1} \mathbb{D}_L^\pm \left[ \begin{smallmatrix} n - \alpha \\ i \end{smallmatrix} \right] \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right] \right)^n & n = 2m, \quad m \in \mathbb{N}, \\ \pm (-1)^m \mathbb{D}_L^\pm \left[ \begin{smallmatrix} n + 1 - \alpha \\ i \end{smallmatrix} \right] \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right] \right)^{n+1} & n = 2m + 1, \quad m \in \mathbb{N}. \end{cases} \quad (44)$$

The  $I\alpha D$ -type of lattice integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$ , is defined by the equation

$$\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] = \begin{cases} \pm (-1)^{m+1} \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right] \right)^n \mathbb{D}_L^\pm \left[ \begin{smallmatrix} n - \alpha \\ i \end{smallmatrix} \right] & n = 2m, \quad m \in \mathbb{N}, \\ \pm (-1)^m \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right] \right)^{n+1} \mathbb{D}_L^\pm \left[ \begin{smallmatrix} n + 1 - \alpha \\ i \end{smallmatrix} \right] & n = 2m + 1, \quad m \in \mathbb{N}. \end{cases} \quad (45)$$

To prove this property, we use the Fourier series transform and the relations

$$i \operatorname{sgn}(k) |k|^{n-\alpha} \left( -\frac{i \operatorname{sgn}(k)}{|k|^1} \right)^n = \frac{i(-1)^m \operatorname{sgn}(k)}{|k|^\alpha} = (-1)^{m+1} \frac{-i \operatorname{sgn}(k)}{|k|^\alpha} \quad (n = 2m, \quad m \in \mathbb{N}), \quad (46)$$

$$i \operatorname{sgn}(k) |k|^{n+1-\alpha} \left( -\frac{i \operatorname{sgn}(k)}{|k|^1} \right)^{n+1} = \frac{i(-1)^{m+1} \operatorname{sgn}(k)}{|k|^\alpha} = (-1)^m \frac{-i \operatorname{sgn}(k)}{|k|^\alpha} \quad (n = 2m + 1). \quad (47)$$

It is easy to see that the parity of the lattice fractional integral operators  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  with  $n - 1 < \alpha \leq n$  are the same as the parity of the “initial” operators (30)–(32).

Let us define the lattice fractional integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$  by using the “initial” integral operators only.

The  $\mathcal{I}I$ -type of lattice integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$ , is defined by the equations

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] = \left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha/n \\ i \end{array} \right] \right)^n \quad (48)$$

and

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] = \begin{cases} (-1)^{m+1} \left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha/(n-1) \\ i \end{array} \right] \right)^{n-1} & n = 2m, \quad m \in \mathbb{N}, \quad m \neq 1 \\ (-1)^m \left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha/n \\ i \end{array} \right] \right)^n & n = 2m + 1, \quad m \in \mathbb{N}. \end{cases} \quad (49)$$

Note that we cannot define  $\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right]$  for even  $n = 2m$  by the equations analogous to (48) since the equation will have another parity for  $n = 2m$ , and we have

$$\left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha/n \\ i \end{array} \right] \right)^n = (-1)^m \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right], \quad (n = 2m, \quad m \in \mathbb{N}). \quad (50)$$

The  $\mathcal{I}I$ -type of lattice integral operators for  $n - 1 < \alpha \leq n$ , where  $n \geq 2$ , is defined by the equations

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] = \begin{cases} \pm (-1)^m \mathbb{D}_L^{\mp} \left[ \begin{array}{c} \alpha - n + 1 \\ i \end{array} \right] \left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} 1 \\ i \end{array} \right] \right)^{n-1} & n = 2m, \quad m \in \mathbb{N}, \\ (-1)^m \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha - n + 1 \\ i \end{array} \right] \left( \mathbb{D}_L^{\pm} \left[ \begin{array}{c} 1 \\ i \end{array} \right] \right)^{n-1} & n = 2m + 1, \quad m \in \mathbb{N}. \end{cases} \quad (51)$$

Note that we use  $mp$  instead of  $\pm$  for the case  $n = 2m$ , and no signs  $\pm$  as a multiplier for the case  $n = 2m + 1$ .

The parity of suggested definitions of the lattice fractional integral operators  $\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right]$  for the case  $n - 1 < \alpha \leq n$ , where  $n \in \mathbb{N}$ , are the same as the parity of the “initial” operators.

#### 4. Properties of lattice fractional derivatives and integrals

Let us describe some properties of lattice fractional derivatives. All these properties are similar to properties of the Riesz derivatives of non-integer orders [12,60].

The lattice fractional derivatives are the linear operators

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] (a_1 u_1(\mathbf{m}) + a_2 u_2(\mathbf{m})) = a_1 \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] u_1(\mathbf{m}) + a_2 \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] u_2(\mathbf{m}), \quad (52)$$

where  $a_1, a_2 \in \mathbb{R}$ .

In general, the lattice particle derivatives for the same direction  $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$  in the lattice do not commute

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_2 \\ i \end{array} \right] \neq \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_2 \\ i \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right], \quad (\alpha_1 \neq \alpha_2). \quad (53)$$

The lattice derivatives for different direction  $\mathbf{e}_i$  and  $\mathbf{e}_j$ , where  $i \neq j$ , obviously commute

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_2 \\ j \end{array} \right] = \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_2 \\ j \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right], \quad (i \neq j). \quad (54)$$

In the general case, the semigroup property is not satisfied

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_2 \\ i \end{array} \right] \neq \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 + \alpha_2 \\ i \end{array} \right], \quad (\alpha_1, \alpha_2 > 0). \quad (55)$$

The property (55) leads to the fact that action of two repeated fractional derivatives of order  $\alpha_1$  does not equivalent to the action of fractional derivative of double order  $2\alpha_1$ ,

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right] \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha_1 \\ i \end{array} \right] \neq \mathbb{D}_L^{\pm} \left[ \begin{array}{c} 2\alpha_1 \\ i \end{array} \right], \quad (\alpha_1 > 0). \quad (56)$$

It should be noted that the Leibniz rule for lattice fractional derivative of order  $\alpha \neq 1$  does not satisfied

$$\mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] (u_1 u_2) \neq u_2 \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] u_1 + u_1 \mathbb{D}_L^{\pm} \left[ \begin{array}{c} \alpha \\ i \end{array} \right] u_2, \quad (\alpha > 0, \quad \alpha \neq 1). \quad (57)$$

This property is similar to fractional derivatives with respect to coordinates [64], and it is a characteristic property of fractional derivatives.

We can consider the value  $\alpha = 0$  that means that the correspondent lattice derivatives are the identity operator

$$\mathbb{D}_L^\pm \begin{bmatrix} 0 \\ i \end{bmatrix} = \mathbb{E}, \tag{58}$$

where  $\mathbb{E}u(\mathbf{m}) = u(\mathbf{m})$ .

The commutation relation (54) with  $\alpha_1 = \alpha_2 = 1$  is

$$\mathbb{D}_L^\pm \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbb{D}_L^\pm \begin{bmatrix} 1 \\ j \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} 1 \\ j \end{bmatrix} \mathbb{D}_L^\pm \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad (i \neq j), \tag{59}$$

that have the continuum analog in the form

$$\frac{\partial^2 u(\mathbf{r})}{\partial x_i \partial x_j} = \frac{\partial^2 u(\mathbf{r})}{\partial x_j \partial x_i}. \tag{60}$$

It is well-known that the commutation relation (60) may be broken for discontinuous functions  $u(\mathbf{r})$  and if the partial derivatives of  $u(\mathbf{r})$  are not continuous. We assume that the relation (59) can be broken for lattice models with dislocations and disclinations. However, the exact conditions for violation of this relationship remain an open question. In this paper, we consider lattices without dislocations and disclinations only.

We can define the lattice fractional mixed partial derivatives by the equations

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 \\ i & j \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}_L^\pm \begin{bmatrix} \alpha_2 \\ j \end{bmatrix}, \quad (i \neq j), \tag{61}$$

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ i & j & k \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}_L^\pm \begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \mathbb{D}_L^\pm \begin{bmatrix} \alpha_3 \\ k \end{bmatrix}, \quad (i \neq j \neq k \neq i), \tag{62}$$

where  $i, j$  and  $k$  take different values from  $\{1; \dots; N\}$  and the values of  $i, j, k$  cannot coincide. The order of the operators (61) and (62) are equal to  $\alpha = \alpha_1 + \alpha_2$  and  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$  respectively.

Using the definition (1), the mixed lattice partial derivative (61) is represented by

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 \\ i & j \end{bmatrix} u(\mathbf{m}) = \frac{1}{a_i^{\alpha_1} a_j^{\alpha_2}} \sum_{m_i=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} K_{\alpha_1}^\pm(n_i - m_i) K_{\alpha_2}^\pm(n_j - m_j) u(\mathbf{m}). \tag{63}$$

Similarly, we can define the operator (62) by the interaction kernels.

If the parameter  $\alpha_k = 0$ , then the lattice derivative (62) can be represented as an operator (61) by

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ i & j & 0 \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 \\ i & j \end{bmatrix} \tag{64}$$

and similarly we have

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & 0 \\ i & 0 \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix}. \tag{65}$$

Using (62) and the property (54), we can rearrange any pair of columns

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 \\ i & j \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_2 & \alpha_1 \\ j & i \end{bmatrix} \tag{66}$$

and

$$\mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ i & j & k \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_2 & \alpha_3 & \alpha_1 \\ j & k & i \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_3 & \alpha_1 & \alpha_2 \\ k & i & j \end{bmatrix} = \dots \tag{67}$$

It also is possible to define other lattice mixed derivatives by a combination of odd and even partial derivatives

$$\mathbb{D}_L^{\pm\mp} \begin{bmatrix} \alpha_1 & \alpha_2 \\ i & j \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}_L^\mp \begin{bmatrix} \alpha_2 \\ j \end{bmatrix}, \quad (i \neq j). \tag{68}$$

$$\mathbb{D}_L^{\pm\mp\mp} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ i & j & k \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}_L^\mp \begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \mathbb{D}_L^\mp \begin{bmatrix} \alpha_3 \\ k \end{bmatrix}, \quad (i \neq j \neq k \neq i), \tag{69}$$

$$\mathbb{D}_L^{\pm\mp\mp} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ i & j & k \end{bmatrix} = \mathbb{D}_L^\pm \begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}_L^\mp \begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \mathbb{D}_L^\mp \begin{bmatrix} \alpha_3 \\ k \end{bmatrix}, \quad (i \neq j \neq k \neq i). \tag{70}$$

The lattice fractional integral operators have properties similar to the properties that are described for lattice fractional derivatives.

Let us give some relations between lattice fractional integral operators. The even and odd lattice integral operators are connected by the following relations.

Two odd operators give the even integral operator

$$\mathbb{I}_L^- \begin{bmatrix} \alpha \\ i \end{bmatrix} \mathbb{I}_L^- \begin{bmatrix} \beta \\ i \end{bmatrix} = -\mathbb{I}_L^+ \begin{bmatrix} \alpha + \beta \\ i \end{bmatrix}, \tag{71}$$

where  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$ . To prove this property, we use the Fourier series transform and the relation

$$\hat{I}_\alpha^-(k) \hat{I}_\beta^-(k) = \left( -\frac{i \operatorname{sgn}(k)}{|k|^\alpha} \right) \left( -\frac{i \operatorname{sgn}(k)}{|k|^\beta} \right) = \left( -\frac{1}{|k|^{\alpha+\beta}} \right) = -\hat{I}_{\alpha+\beta}^-(k). \tag{72}$$

Similarly we have that the odd and even "initial" lattice integral operators give the even lattice operator

$$\mathbb{I}_L^- \begin{bmatrix} \alpha \\ i \end{bmatrix} \mathbb{I}_L^+ \begin{bmatrix} \beta \\ i \end{bmatrix} = \mathbb{I}_L^- \begin{bmatrix} \alpha + \beta \\ i \end{bmatrix}, \tag{73}$$

if  $\alpha, \beta > 0, \beta < 1$  and  $\alpha + \beta < 2$ . We also have the relation for even "initial" lattice integral operators

$$\mathbb{I}_L^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} \mathbb{I}_L^+ \begin{bmatrix} \beta \\ i \end{bmatrix} = \mathbb{I}_L^+ \begin{bmatrix} \alpha + \beta \\ i \end{bmatrix}, \tag{74}$$

if  $0 < \alpha, \beta < 1$  and  $\alpha + \beta < 1$ .

### 5. Continuum limit for lattice fractional operators

#### 5.1. Transformation of lattice fields into continuum fields

In this section, we use the methods suggested in [41,42] to define the operation that transforms a lattice field  $u(\mathbf{n})$  into a field  $u(\mathbf{r})$  of continuum, These transformations are following. We consider the lattice scalar field  $u(\mathbf{n})$  as Fourier series coefficients of some function  $\hat{u}(\mathbf{k})$  for  $k_j \in [-k_{j0}/2, k_{j0}/2]$ , where  $j = 1, \dots, N$ . As a next step we use the continuous limit  $k_0 \rightarrow \infty$  to obtain  $\tilde{u}(\mathbf{k})$ . Finally we apply the inverse Fourier integral transformation to obtain the continuum scalar field  $u(\mathbf{r})$ .

Let us give some details for these transformations of a lattice field into a continuum field [41,42].

1. The Fourier series transform  $u(\mathbf{n}) \rightarrow \mathcal{F}_\Delta\{u(\mathbf{n})\} = \hat{u}(\mathbf{k})$  of the lattice scalar field  $u(\mathbf{n})$  is defined by

$$\hat{u}(\mathbf{k}) = \mathcal{F}_\Delta\{u(\mathbf{n})\} = \sum_{n_1, \dots, n_N = -\infty}^{+\infty} u(\mathbf{n}) e^{-i(\mathbf{k}, \mathbf{r}(\mathbf{n}))}, \tag{75}$$

where the inverse Fourier series transform is

$$u(\mathbf{n}) = \mathcal{F}_\Delta^{-1}\{\hat{u}(\mathbf{k})\} = \left( \prod_{j=1}^N \frac{1}{k_{j0}} \right) \int_{-k_{10}/2}^{+k_{10}/2} dk_1 \dots \int_{-k_{N0}/2}^{+k_{N0}/2} dk_N \hat{u}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{r}(\mathbf{n}))}. \tag{76}$$

Here  $\mathbf{r}(\mathbf{n}) = n_j \mathbf{a}_j$ , and  $a_j = 2\pi/k_{j0}$  is distance between lattice particle in the direction  $\mathbf{a}_j$ . For simplicity we assume that the lattice has equal distance  $a_j$  between all particle distance in the direction  $\mathbf{a}_j$ .

2. The passage to the limit  $\hat{u}(\mathbf{k}) \rightarrow \operatorname{Lim}\{\hat{u}(\mathbf{k})\} = \tilde{u}(\mathbf{k})$ , where we use  $a_j \rightarrow 0$  (or  $k_{j0} \rightarrow \infty$ ) allows us to derive the function  $\tilde{u}(\mathbf{k})$  from  $\hat{u}(\mathbf{k})$ . By definition  $\tilde{u}(\mathbf{k})$  is the Fourier integral transform of the continuum field  $u(\mathbf{r})$ , and the function  $\hat{u}(k)$  is the Fourier series transform of the lattice field  $u(\mathbf{n})$ , where

$$u(\mathbf{n}) = \left( \prod_{j=1}^N \frac{2\pi}{k_{j0}} \right) u(\mathbf{r}(\mathbf{n}))$$

and  $\mathbf{r}(\mathbf{n}) = n_j a_j = 2\pi n_j / k_{j0} \rightarrow \mathbf{r}$ .

3. The inverse Fourier integral transform  $\tilde{u}(\mathbf{k}) \rightarrow \mathcal{F}^{-1}\{\tilde{u}(\mathbf{k})\} = u(\mathbf{r})$  is defined by

$$u(\mathbf{r}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} dk_1 \dots \int_{-\infty}^{+\infty} dk_N \tilde{u}(\mathbf{k}) e^{i\sum_{j=1}^N k_j x_j} = \mathcal{F}^{-1}\{\tilde{u}(\mathbf{k})\} \tag{77}$$

and the Fourier integral transform of the continuum scalar field  $u(\mathbf{r})$  is

$$\tilde{u}(\mathbf{k}) = \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_N u(\mathbf{r}) e^{-i\sum_{j=1}^N k_j x_j} = \mathcal{F}\{u(\mathbf{r})\}. \tag{78}$$

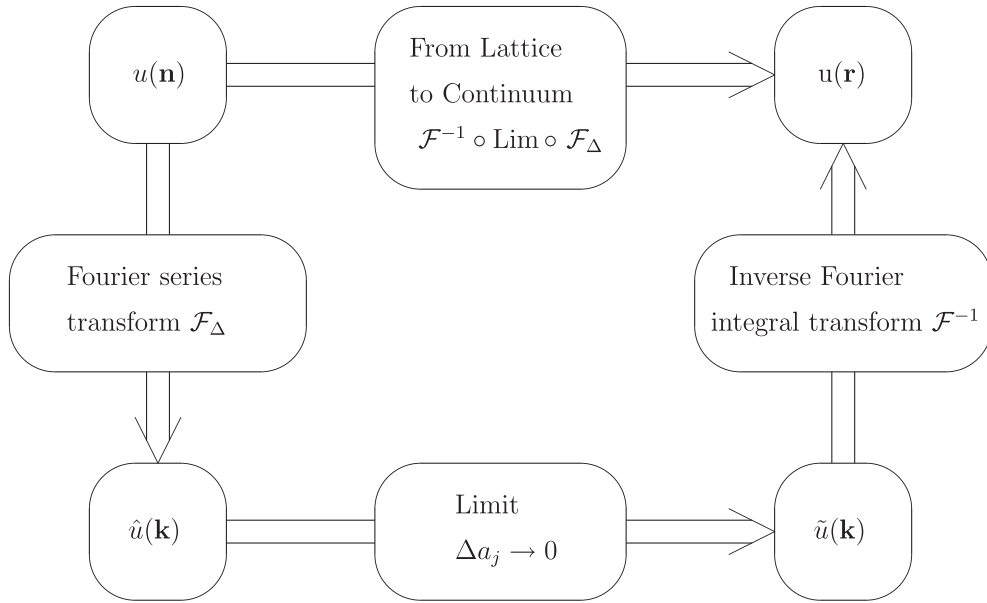


Fig. 11. Diagram of sets of operations for scalar fields.

Note that the Fourier series transforms (75) and (76) in the limit  $a_j \rightarrow 0$  ( $k_{j0} \rightarrow \infty$ ) are gives the Fourier integral transform (78) and (77), such that the sum is replaced by the integral.

These transformations can be represented by the diagram (see Fig. 11).

The combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  of the operations  $\mathcal{F}^{-1}$ , Lim, and  $\mathcal{F}_\Delta$  define the lattice-continuum transform operation

$$\mathcal{T}_{L-C} = \mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta, \tag{79}$$

that maps lattice models into the continuum models [41,42].

### 5.2. Transformation of lattice operators into continuum operators

Let us consider a transformation of lattice derivatives and integrals into the continuum fractional derivatives and integrals with respect to coordinates. The lattice-continuum transform operation  $\mathcal{T}_{L-C}$  as the combination of three operations  $\mathcal{F}^{-1} \circ \text{Limit} \circ \mathcal{F}_\Delta$  can be applied not only for lattice fields but also for lattice operators. The operation  $\mathcal{T}_{L-C}$  allows us to map of lattice derivatives  $\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  and integrals  $\mathbb{I}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  into continuum derivatives  $\mathbb{D}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  and integrals  $\mathbb{I}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  that will be defined in the next subsections.

We performed transformations  $\mathcal{F}^{-1} \circ \text{Limit} \circ \mathcal{F}_\Delta$  for differential operators to map the lattice fractional derivative into the fractional derivative for the continuum. We can represent these sets of transformations from lattice operators to operators for continuum in the form of the diagram presented by Fig. 12.

The functions  $\hat{K}_\alpha^\pm(k_j)$ ,  $\hat{L}_\alpha^\pm(k_j)$  are defined by the Fourier series transform  $\mathcal{F}_\Delta$  of the kernels of lattice operators, and the functions  $\tilde{K}_\alpha^\pm(k_j)$ ,  $\tilde{L}_\alpha^\pm(k_j)$  are defined by the Fourier integral transforms  $\mathcal{F}$  of the correspondent continuum derivatives and integrals. The equations that define  $\hat{K}_\alpha^\pm(k_j)$  and  $\hat{L}_\alpha^\pm(k_j)$  have the form

$$\mathcal{F}_\Delta \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{m}) \right) = \frac{1}{a_j^\alpha} \hat{K}_\alpha^\pm(k_j a_j) \hat{u}(\mathbf{k}), \tag{80}$$

$$\mathcal{F}_\Delta \left( \mathbb{I}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{m}) \right) = \frac{1}{a_j^\alpha} \hat{L}_\alpha^\pm(k_j a_j) \hat{u}(\mathbf{k}), \tag{81}$$

where  $\hat{u}(\mathbf{k}) = \mathcal{F}_\Delta \{ u(\mathbf{m}) \}$ , and  $\mathcal{F}_\Delta$  is an operator notation for the Fourier series transform. The equations that define  $\tilde{K}_\alpha^\pm(k_j)$  and  $\tilde{L}_\alpha^\pm(k_j)$  are

$$\mathcal{F} \left( \mathbb{D}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) = \tilde{K}_\alpha^\pm(k_j) \tilde{u}(\mathbf{k}), \tag{82}$$

$$\mathcal{F} \left( \mathbb{I}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) = \tilde{L}_\alpha^\pm(k_j) \tilde{u}(\mathbf{k}), \tag{83}$$

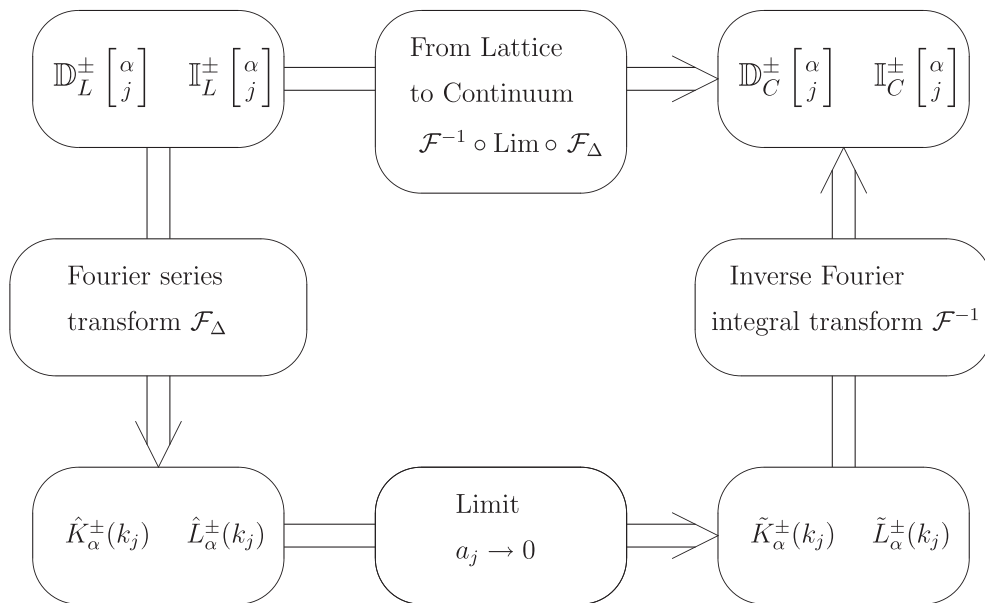


Fig. 12. Diagram of sets of operations for differential and integral operators.

where  $\tilde{u}(\mathbf{k}) = \mathcal{F}\{u(\mathbf{r})\}$ , and  $\mathcal{F}_\Delta$  is an operator notation for the Fourier transform. In general, the order of the partial derivative  $\mathbb{D}_C^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix}$  and the integrals  $\mathbb{I}_C^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix}$  is defined by the order of lattice operators  $\mathbb{D}_L^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix}$  and  $\mathbb{I}_L^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix}$ , and this order can be integer and non-integer. The continuum fractional derivatives and integrals are defined in the next subsections.

5.3. Continuum fractional derivatives  $\mathbb{D}_C^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix}$  of the Riesz type

The continuum derivative of the order  $\alpha$  is defined [10,12] by the equation

$$\mathbb{D}_C^\pm \begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{r}) = \frac{1}{d_1(m, \alpha)} \int_{\mathbb{R}^1} \frac{1}{|z_j|^{\alpha+1}} (\Delta_{z_j}^m u)(\mathbf{r}) dz_j, \quad (0 < \alpha < m), \tag{84}$$

where  $(\Delta_{z_j}^m u)(\mathbf{r})$  is a finite difference of order  $m$  of a function  $u(\mathbf{r})$  with the vector step  $\mathbf{z}_j = z_j \mathbf{e}_j \in \mathbb{R}^N$  for the point  $\mathbf{r} \in \mathbb{R}^N$ . The centered difference

$$(\Delta_{z_j}^m u)(\mathbf{r}) = \sum_{n=0}^m (-1)^n \frac{m!}{n!(m-n)!} u(\mathbf{r} - (m/2 - n)z_j \mathbf{e}_j). \tag{85}$$

The constant  $d_1(m, \alpha)$  is defined by

$$d_1(m, \alpha) = \frac{\pi^{3/2} A_m(\alpha)}{2^\alpha \Gamma(1 + \alpha/2) \Gamma((1 + \alpha)/2) \sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = 2 \sum_{s=0}^{\lfloor m/2 \rfloor} (-1)^{s-1} \frac{m!}{s!(m-s)!} (m/2 - s)^\alpha$$

for the centered difference (85). The constants  $d_1(m, \alpha)$  is different from zero for all  $\alpha > 0$  in the case of an even  $m$  and centered difference  $(\Delta_{z_j}^m u)$  (see Theorem 26.1 in [10]). Note that the integral (84) does not depend on the choice of  $m > \alpha$ . Therefore, we can always choose an even number  $m$  so that it is greater than parameter  $\alpha$ , and we can use the centered difference (85) for all positive real values of  $\alpha$ .

It should be noted that we can use the non-centered difference instead of the centered difference (85). The non-centered difference is defined by the equation

$$(\Delta_{z_j}^m u)(\mathbf{r}) = \sum_{n=0}^m (-1)^n \frac{m!}{n!(m-n)!} u(\mathbf{r} - nz_j \mathbf{e}_j) \tag{86}$$

and the correspondent coefficient  $A_m(\alpha)$  is

$$A_m(\alpha) = 2 \sum_{s=0}^m (-1)^{s-1} \frac{m!}{s!(m-s)!} s^\alpha.$$

In the case of a non-centered difference the constant  $d_1(m, \alpha)$  vanishes if and only if  $\alpha = 1, 3, 5, \dots, 2[m/2] - 1$ . Therefore the non-centered differences (86) can be used only for the non-integer positive orders  $\alpha$  and for odd integer values of  $\alpha$ . Using (84), we can see that the continuum fractional derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u$  is the Riesz derivative of the function  $u(\mathbf{r})$  with respect to one component  $x_j \in \mathbb{R}^1$  of the vector  $\mathbf{r} \in \mathbb{R}^N$ , i.e. the operator  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  can be considered as a partial fractional derivative of Riesz type.

Using that  $(-i)^{2m} = (-1)^m$ , the Riesz fractional derivatives for even  $\alpha = 2m$ , where  $m \in \mathbb{N}$ , are connected with the usual partial derivative of integer orders  $2m$  by the relation

$$\mathbb{D}_c^+ \left[ \begin{smallmatrix} 2m \\ j \end{smallmatrix} \right] u(\mathbf{r}) = (-1)^m \frac{\partial^{2m} u(\mathbf{r})}{\partial x_j^{2m}}. \tag{87}$$

For  $\alpha = 2$  the Riesz derivative is the local operator  $-\partial^2 / \partial x_j^2$ . The fractional derivatives  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} 2m \\ j \end{smallmatrix} \right]$  for even orders  $\alpha$  are local operators. Note that the Riesz derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]$  cannot be considered as a derivative of first order with respect to  $x_j$ , i.e.,

$$\mathbb{D}_c^+ \left[ \begin{smallmatrix} 1 \\ j \end{smallmatrix} \right] u(\mathbf{r}) \neq \frac{\partial u(\mathbf{r})}{\partial x_j}. \tag{88}$$

Note that the Riesz derivatives for odd orders  $\alpha = 2m + 1$ , where  $m \in \mathbb{N}$ , are non-local operators that cannot be considered as usual derivatives  $\partial^{2m+1} / \partial x_j^{2m+1}$ . For  $\alpha = 1$  the operator  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} 1 \\ j \end{smallmatrix} \right]$  is nonlocal, and it can be considered as a ‘‘square root of the 1D Laplacian’’.

An important property of the Riesz fractional derivatives is the Fourier transform  $\mathcal{F}$  of this operators in the form

$$\mathcal{F} \left( \mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) (\mathbf{k}) = |k_j|^\alpha (\mathcal{F}u)(\mathbf{k}). \tag{89}$$

The property (89) is valid for functions  $u(\mathbf{r})$  from the space  $C^\infty(\mathbb{R}^1)$  of infinitely differentiable functions on  $\mathbb{R}^1$  with compact support. It is also holds for the Lizorkin space (see Section 8.1 in [10]). Using the property (89), we can write the formula

$$\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1} (|k_j|^\alpha (\mathcal{F}u)(\mathbf{k})) (\mathbf{r}). \tag{90}$$

It is easy to see that Eq. (90) simpler than definition (84). For application we can consider formula (90) as a definition of the continuum fractional derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$ .

#### 5.4. Continuum fractional integrals $\mathbb{I}_c^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$ of the Riesz type

The Riesz fractional integrals, which are usually called the Riesz potential [10,12], can be defined by the convolution in the form

$$\mathbb{I}_r^\alpha u(\mathbf{r}) = \int_{\mathbb{R}^N} R_\alpha(\mathbf{r} - \mathbf{z}) u(\mathbf{z}) d^N \mathbf{z}, \quad (\alpha > 0), \tag{91}$$

where the function  $R_\alpha(\mathbf{r})$  is the Riesz kernel that is defined by

$$R_\alpha(\mathbf{r}) = \begin{cases} \gamma_N^{-1}(\alpha) |\mathbf{r}|^{\alpha-N} & \alpha \neq N + 2n, \quad n, N \in \mathbb{N}, \\ -\gamma_N^{-1}(\alpha) |\mathbf{r}|^{\alpha-N} \ln |\mathbf{r}| & \alpha = N + 2n, \quad n, N \in \mathbb{N}. \end{cases} \tag{92}$$

The constant  $\gamma_N(\alpha)$  has the form

$$\gamma_N(\alpha) = \begin{cases} 2^\alpha \pi^{N/2} \Gamma(\alpha/2) / \Gamma((N - \alpha)/2) & \alpha \neq N + 2n, \\ (-1)^{(N-\alpha)/2} 2^{\alpha-1} \pi^{N/2} \Gamma(\alpha/2) \Gamma(1 + [\alpha - N]/2) & \alpha = N + 2n, \end{cases} \tag{93}$$

where  $n, N \in \mathbb{N}$  and  $\alpha \in \mathbb{R}_+$ .

An important property of the Riesz fractional integrals (91) is the Fourier transform  $\mathcal{F}$  of this integrals in the form

$$\mathcal{F}(\mathbb{I}_r^\alpha u(\mathbf{r})) = |\mathbf{k}|^{-\alpha} (\mathcal{F}u)(\mathbf{k}). \tag{94}$$



Eq. (94) holds for the operator (91) if the function  $u(\mathbf{r})$  belongs to the Lizorkin space [10,12]. We can state that functions obtained by applying the Riesz integration to functions from the Lizorkin space also belong to this space, i.e., the Lizorkin space is invariant with respect to the Riesz fractional integration.

We can use the property (94) to define the Riesz fractional integrals by the equation

$$\mathbf{I}_r^\alpha u(\mathbf{r}) = \mathcal{F}^{-1}(|\mathbf{k}|^{-\alpha}(\mathcal{F}u)(\mathbf{k}))(\mathbf{r}). \tag{95}$$

This definition is somewhat simpler than the definition based on the Eq. (91), and we can use (95) as a definition of the Riesz fractional integrations for some applications.

We suggest to define the continuum fractional integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the Riesz type as the Riesz potential of order  $\alpha$  with respect to  $x_j$  by the equation

$$\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \int_{\mathbb{R}^1} R_\alpha(x_j - z_j) u(\mathbf{r} + (z_j - x_j)\mathbf{e}_j) dz_j, \quad (\alpha > 0), \tag{96}$$

where  $\mathbf{e}_j$  is the basis of the Cartesian coordinate system. This integral can also be defined by the Fourier transform

$$\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1}(|k_j|^{-\alpha}(\mathcal{F}u)(\mathbf{k}))(\mathbf{r}). \tag{97}$$

Let us note the distinction between the continuum fractional integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  and the Riesz potential  $\mathbf{I}_r^\alpha$  defined by (95) consists in the use of  $\tilde{L}_\alpha^+(k_j) = |k_j|^{-\alpha}$  instead of  $|\mathbf{k}|^{-\alpha}$ . The continuum integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the Riesz type is an integration of  $u(\mathbf{r})$  with respect to one variable  $x_j$  instead of all  $N$  variables  $x_1, \dots, x_N$  in  $\mathbf{I}_r^\alpha$ .

If  $u(\mathbf{r})$  as a function of  $x_j$  belongs to the Lizorkin space, then we have [10] the semi-group property

$$\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] \mathbb{I}_c^+ \left[ \begin{smallmatrix} \beta \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha + \beta \\ j \end{smallmatrix} \right] u(\mathbf{r}), \tag{98}$$

where  $\alpha > 0$  and  $\beta > 0$ .

The continuum fractional derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  yields an operator inverse to the continuum fractional integration  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  for a special space of functions

$$\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] \mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = u(\mathbf{r}), \quad (\alpha > 0). \tag{99}$$

Eq. (99) hold for  $u(\mathbf{r})$  belonging to the Lizorkin space of functions with respect to  $x_j \in \mathbb{R}$ . Moreover, this property is also valid for the continuum fractional integration in the frame of  $L_p$ -spaces  $L_p(\mathbb{R}^1)$  for  $1 \leq p < 1/\alpha$  (see Theorem 26.3 in [10]). Using the property (94), it is easy to see that the continuum fractional integrals  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  with  $\alpha = 1$  cannot be considered as usual integral of first order with respect to  $x_j$ . Therefore we define new continuum fractional integral  $\mathbb{I}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the Riesz type by the Fourier transforms

$$\mathbb{I}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1}(-i \operatorname{sgn}(k_j) |k_j|^{-\alpha}(\mathcal{F}u)(\mathbf{k}))(\mathbf{r}), \quad (\alpha > 0). \tag{100}$$

In this case, the Fourier integral transform  $\tilde{L}_\alpha^-(k_j)$  of this continuum fractional integral is

$$\tilde{L}_\alpha^-(k_j) = \mathcal{F} \left( \mathbb{I}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) = -i \operatorname{sgn}(k_j) |k_j|^{-\alpha}(\mathcal{F}u)(\mathbf{k}), \quad (\alpha > 0). \tag{101}$$

We can see that the fractional integral  $\mathbb{I}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the integer order  $\alpha = 1$  is the usual integral of the first order

$$\mathbb{I}_c^- \left[ \begin{smallmatrix} 1 \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \int_{\mathbb{R}^1} u(\mathbf{r}) dx_j, \tag{102}$$

since  $\tilde{L}_\alpha^-(k_j) = (ik_j)^{-1}$  in this case.

Using (96), we can see that the continuum fractional integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u$  is the Riesz integrations of the function  $u = u(\mathbf{r})$  with respect to one component  $x_j \in \mathbb{R}^1$  of the vector  $\mathbf{r} \in \mathbb{R}^N$ .

### 5.5. Continuum fractional derivative $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$ of the Riesz type

Using the property (94), we can see that the continuum fractional derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  with  $\alpha = 1$  cannot be considered as usual derivative of first order with respect to  $x_j$ . Therefore we define new continuum fractional derivative  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the Riesz type by the equation

$$\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1} (i \operatorname{sgn}(k_j) |k_j|^\alpha (\mathcal{F}u)(\mathbf{k}))(\mathbf{r}). \tag{103}$$

We can define the continuum fractional derivative  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  as a combinations of the continuum fractional derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha - 1 \\ j \end{smallmatrix} \right]$ , the derivative of the first order  $\partial/\partial x_j$ , and the continuum fractional integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} 1 - \alpha \\ j \end{smallmatrix} \right]$ . The Fourier integral transforms of the derivative  $\mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha - 1 \\ j \end{smallmatrix} \right]$ , the usual derivative  $\partial/\partial x_j$ , and the integral  $\mathbb{I}_c^+ \left[ \begin{smallmatrix} 1 - \alpha \\ j \end{smallmatrix} \right]$  have the forms

$$\mathcal{F} \left( \mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha - 1 \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) (\mathbf{k}) = |k_j|^{\alpha-1} (\mathcal{F}u)(\mathbf{k}), \quad (\alpha > 1), \tag{104}$$

$$\mathcal{F} \left( \mathbb{I}_c^+ \left[ \begin{smallmatrix} 1 - \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) (\mathbf{k}) = \frac{1}{|k_j|^{1-\alpha}} (\mathcal{F}u)(\mathbf{k}), \quad (\alpha < 1), \tag{105}$$

$$\mathcal{F} \left( \frac{\partial u(\mathbf{r})}{\partial x_j} \right) (\mathbf{k}) = i k_j (\mathcal{F}u)(\mathbf{k}). \tag{106}$$

Using Eqs. (104)–(106), and the equality  $k_j |k_j|^{\alpha-1} = \operatorname{sgn}(k_j) |k_j|^\alpha$ , we can define the fractional operator (103) as a combination of these operators in the form

$$\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] = \begin{cases} \frac{\partial}{\partial x_j} \mathbb{D}_c^+ \left[ \begin{smallmatrix} \alpha - 1 \\ j \end{smallmatrix} \right] & \alpha > 1, \\ \frac{\partial}{\partial x_j} & \alpha = 1, \\ \frac{\partial}{\partial x_j} \mathbb{I}_c^+ \left[ \begin{smallmatrix} 1 - \alpha \\ j \end{smallmatrix} \right] & 0 < \alpha < 1. \end{cases} \tag{107}$$

For  $0 < \alpha < 1$  the operator  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  is analogous to the conjugate Riesz derivative [14]. Therefore, the operator  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  for all positive values  $\alpha$  can be called a generalized conjugate derivative of the Riesz type.

The Fourier integral transform  $\mathcal{F}$  of the fractional derivative (107) is given by

$$\mathcal{F} \left( \mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) \right) (\mathbf{k}) = i k_j |k_j|^{\alpha-1} (\mathcal{F}u)(\mathbf{k}) = i \operatorname{sgn}(k_j) |k_j|^\alpha (\mathcal{F}u)(\mathbf{k}). \tag{108}$$

Using (98), (99) and (107), it is easy to prove the property

$$\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] \mathbb{I}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{r}) = \frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} u(\mathbf{r}) dx_j = u(\mathbf{r}), \quad (\alpha > 0). \tag{109}$$

For the odd values of  $\alpha$ , equations (87) and (107) gives the relation

$$\mathbb{D}_c^- \left[ \begin{smallmatrix} 2m + 1 \\ j \end{smallmatrix} \right] u(\mathbf{r}) = (-1)^m \frac{\partial^{2m+1} u(\mathbf{r})}{\partial x_j^{2m+1}}, \quad (m \in \mathbb{N}). \tag{110}$$

Eq. (110) means that the fractional derivatives  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  of the odd orders  $\alpha$  are local operators represented by the usual derivatives of integer orders.

Note that the continuum derivative  $\mathbb{D}_c^- \left[ \begin{smallmatrix} 2 \\ j \end{smallmatrix} \right]$  cannot be considered as a local derivative of second order with respect to  $x_j$ . The derivatives  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  for even orders  $\alpha = 2m$ , where  $m \in \mathbb{N}$ , are non-local operators that cannot be considered as usual derivatives  $\partial^{2m}/\partial x_j^{2m}$ .

### 5.6. Rules of fractionalization of the Riesz type

Eqs. (87) and (110) allow us to state that the partial derivatives of integer orders are obtained from the fractional derivatives of the Riesz type  $\mathbb{D}_c^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  for odd values  $\alpha = 2m + 1 > 0$  by  $\mathbb{D}_c^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  only, and for even values  $\alpha = 2m > 0$ , where  $m \in \mathbb{N}$ ,

by  $\mathbb{D}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$ . The continuum derivatives of the Riesz type  $\mathbb{D}_C^- \left[ \begin{smallmatrix} 2m \\ j \end{smallmatrix} \right]$  and  $\mathbb{D}_C^+ \left[ \begin{smallmatrix} 2m+1 \\ j \end{smallmatrix} \right]$  are nonlocal differential operators of integer orders.

In formulation of mathematical models for nonlocal continuously distributed systems, we need to generalize some well-known local model, which is described by partial differential equations of integer order. It is obvious that we would like to have a fractional generalization of partial differential equations such that to obtain the original equations in the limit case, when the orders of generalized derivatives become equal to initial integer values. This requirement is often called the correspondence principle.

In order to the correspondence principle has performed at fractional generalizations of local models, we suggest the following “rules of fractionalization” (the rule of generalization of differential equations to fractional case):

$$\frac{\partial^{2m}}{\partial x_j^{2m}} = (-1)^m \mathbb{D}_C^+ \left[ \begin{smallmatrix} 2m \\ j \end{smallmatrix} \right] \rightarrow (-1)^m \mathbb{D}_C^+ \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right], \quad (m \in \mathbb{N}, \quad 2m - 1 < \alpha < 2m + 1), \tag{111}$$

$$\frac{\partial^{2m+1}}{\partial x_j^{2m+1}} = (-1)^m \mathbb{D}_C^- \left[ \begin{smallmatrix} 2m+1 \\ j \end{smallmatrix} \right] \rightarrow (-1)^m \mathbb{D}_C^- \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right], \quad (m \in \mathbb{N}, \quad 2m < \alpha < 2m + 2). \tag{112}$$

In order to derive a fractional generalization of differential equation with partial derivatives of integer orders, we should replace the usual derivatives of odd orders with respect to  $x_i$  by the continuum fractional derivatives of the Riesz type  $\mathbb{D}_C^- \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$ , and the usual derivatives of even orders with respect to  $x_i$  by the continuum fractional derivatives of the Riesz type  $\mathbb{D}_C^+ \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$ .

5.7. Continuum limit for lattice fractional derivatives and integrals

Let us formulate and prove a proposition about the connection between the lattice fractional operators and continuum fractional operators of non-integer orders with respect to coordinates.

**Proposition.** *The combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  transforms the lattice fractional derivatives*

$$\mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{m}) = \frac{1}{a_j^\alpha} \sum_{m_j=-\infty}^{+\infty} K_\alpha^\pm(n_j - m_j) u(\mathbf{m}), \tag{113}$$

where  $K_\alpha^\pm(n - m)$  are defined by (2), (3), and the lattice fractional integrals

$$\mathbb{I}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{m}) = \frac{1}{a_j^\alpha} \sum_{m_j=-\infty}^{+\infty} L_\alpha^\pm(n_j - m_j) u(\mathbf{m}), \tag{114}$$

where  $L_\alpha^\pm(n - m)$  are defined by (31), (32), into the continuum fractional derivatives and integrals of order  $\alpha$  with respect to coordinate  $x_j$  by

$$\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta \left( \mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] \right) = \mathbb{D}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right], \tag{115}$$

$$\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta \left( \mathbb{I}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] \right) = \mathbb{I}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]. \tag{116}$$

**Proof.** For simplification, we prove the statement for the lattice fractional derivatives  $\mathbb{D}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$ . The proof for the lattice fractional integrals  $\mathbb{I}_C^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$  is realized analogously.

Let us multiply Eq. (113) by  $\exp(-ik_j n_j a_j)$ , and summing over  $n_j$  from  $-\infty$  to  $+\infty$ . Then

$$\sum_{n_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} \mathbb{D}_L^\pm \left[ \begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right] u(\mathbf{m}) = \frac{1}{a_j^\alpha} \sum_{n_j=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_\alpha^\pm(n_j - m_j) u(\mathbf{m}). \tag{117}$$

Using (75), the right-hand side of (117) gives

$$\begin{aligned} \sum_{n_j=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_{\alpha}^{\pm}(n_j - m_j) u(\mathbf{m}) &= \sum_{n_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_{\alpha}^{\pm}(n_j - m_j) \sum_{m_j=-\infty}^{+\infty} u(\mathbf{m}) \\ &= \sum_{n'_j=-\infty}^{+\infty} e^{-ik_j n'_j a_j} K_{\alpha}^{\pm}(n'_j) \sum_{m_j=-\infty}^{+\infty} u(\mathbf{m}) e^{-ik_j m_j a_j} = \hat{K}_{\alpha}^{\pm}(k_j a_j) \hat{u}(\mathbf{k}), \end{aligned} \tag{118}$$

where  $n'_j = n_j - m_j$ .

As a result, Eq. (117) has the form

$$\mathcal{F}_{\Delta} \left( \mathbb{D}_L^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{m}) \right) = \frac{1}{a_j^{\alpha}} \hat{K}_{\alpha}^{\pm}(k_j a_j) \hat{u}(\mathbf{k}), \tag{119}$$

where  $\mathcal{F}_{\Delta}$  is an operator notation for the Fourier series transform.

Then we use

$$\hat{K}_{\alpha}^{+}(a_j k_j) = |a_j k_j|^{\alpha}, \tag{120}$$

$$\hat{K}_{\alpha}^{-}(a_j k_j) = i \operatorname{sgn}(k_j) |a_j k_j|^{\alpha} \tag{121}$$

and, the limit  $a_j \rightarrow 0$  gives

$$\tilde{K}_{\alpha}^{+}(k_j) = \lim_{a_j \rightarrow 0} \frac{1}{a_j^{\alpha}} \hat{K}_{\alpha}^{+}(k_j a_j) = |k_j|^{\alpha}, \tag{122}$$

$$\tilde{K}_{\alpha}^{-}(k_j) = \lim_{a_j \rightarrow 0} \frac{1}{a_j^{\alpha}} \hat{K}_{\alpha}^{-}(k_j a_j) = i k_j |k_j|^{\alpha-1}. \tag{123}$$

As a result, Eq. (119) in the limit  $a_j \rightarrow 0$  gives

$$\operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{m}) \right) = \tilde{K}_{\alpha}^{\pm}(k_j) \tilde{u}(\mathbf{k}), \tag{124}$$

where

$$\tilde{K}_{\alpha}^{+}(k_j) = |k_j|^{\alpha}, \quad \tilde{K}_{\alpha}^{-}(k_j) = i k_j |k_j|^{\alpha-1}, \quad \tilde{u}(\mathbf{k}) = \operatorname{Lim} \hat{u}(\mathbf{k}).$$

The inverse Fourier transform of (124) is

$$\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{m}) \right) = \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{r}), \quad (\alpha > 0), \tag{125}$$

$$\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{-} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{m}) \right) = \frac{\partial}{\partial x_j} \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha - 1 \\ j \end{matrix} \right] u(\mathbf{r}), \quad (\alpha > 1), \tag{126}$$

$$\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{-} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{m}) \right) = \frac{\partial}{\partial x_j} \mathbb{D}_C^{\pm} \left[ \begin{matrix} 1 - \alpha \\ j \end{matrix} \right] u(\mathbf{r}), \quad (0 < \alpha < 1). \tag{127}$$

Here the fractional derivative and fractional integral are

$$\mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1} \{ |k_j|^{\alpha} \tilde{u}(\mathbf{k}) \}, \quad \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha \\ j \end{matrix} \right] u(\mathbf{r}) = \mathcal{F}^{-1} \{ |k_j|^{-\alpha} \tilde{u}(\mathbf{k}) \}. \tag{128}$$

where we use the connection (89) and (94) between the continuum derivative and integral of the Riesz type of the order  $\alpha$  and the correspondent Fourier integrals transforms.

As a result, we obtain that lattice fractional derivatives are transformed (115) into continuum fractional derivatives of the Riesz type.

This ends the proof.  $\square$

Using the Proposition (115), and the independence of  $n_i$  and  $n_j$  for  $i \neq j$ , it is easy to prove that the continuum limits for the lattice mixed partial derivatives (61) and (62) have the form

$$\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{\pm, \pm} \left[ \begin{matrix} \alpha_1 & \alpha_2 \\ i & j \end{matrix} \right] \right) = \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha_1 \\ i \end{matrix} \right] \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha_2 \\ j \end{matrix} \right], \quad (i \neq j), \tag{129}$$

$$\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_{\Delta} \left( \mathbb{D}_L^{\pm, \mp} \left[ \begin{matrix} \alpha_1 & \alpha_2 \\ i & j \end{matrix} \right] \right) = \mathbb{D}_C^{\pm} \left[ \begin{matrix} \alpha_1 \\ i \end{matrix} \right] \mathbb{D}_C^{\mp} \left[ \begin{matrix} \alpha_2 \\ j \end{matrix} \right], \quad (i \neq j). \tag{130}$$

We have similar relations for other mixed lattice fractional derivatives and integrals. As a result, the continuous limits of the lattice fractional derivatives and integrals give the continuum fractional derivatives and integrals of the Riesz type.

Note that the continuum fractional derivatives  $\mathbb{D}_c^+ \left[ \frac{\alpha}{j} \right]$  for  $\alpha = 1$  are non-local operators that cannot be considered as usual local derivatives  $\partial/\partial x_j$ , i.e.  $\mathbb{D}_c^+ \left[ \frac{1}{j} \right] \neq \partial/\partial x_j$ . Therefore the fractional differential operators that correspond to the even (symmetric) kernels with  $\alpha = 1$  are non-local operators also. Analogously the continuum fractional derivative  $\mathbb{D}_c^- \left[ \frac{\alpha}{j} \right]$  for  $\alpha = 2$  is nonlocal operator and it cannot be considered as usual derivatives of second order.

At the same time, the continuum fractional operators  $\mathbb{D}_c^+ \left[ \frac{\alpha}{j} \right]$  and  $\mathbb{I}_c^+ \left[ \frac{\alpha}{j} \right]$  for the even integer values of order  $\alpha$  give the usual expressions for the differential and integral operators of even integer orders up to signs. The continuum fractional operators  $\mathbb{D}_c^- \left[ \frac{\alpha}{j} \right]$  and  $\mathbb{I}_c^- \left[ \frac{\alpha}{j} \right]$  for the odd integer values of order  $\alpha$  also give the usual expressions for the differential and integral operators of odd integer orders up to signs.

## 6. Conclusion

In this paper we suggest a formulation of fractional calculus for  $N$ -dimensional lattices with long-range interactions. The main advantage of the suggested lattice fractional calculus is a possibility to consider this calculus as tools for a microstructural basis of fractional nonlocal continuum models. The lattice analogs of fractional derivatives and integrals are represented by kernels of long-range interactions of lattice particles. The Fourier series transform of these kernels have a power-law form with respect to components of wave vector. The suggested long-range interactions can be used for integer and fractional orders of lattice derivatives and integrals. The continuous limits for these lattice derivatives and integrals of non-integer order give the continuum fractional derivatives and integrals of the Riesz type with respect to coordinates. Fractional continuum dynamics can be considered as a continuous limit of lattice dynamics with long-range interactions, where the sizes of continuum elements are much larger than the distances between lattice particles. Lattice fractional calculus allows us to formulate a lot of different lattice models for wide class of media with nonlocality of power-law type. It allows us to have a microstructural basis for the fractional nonlocal continuum mechanics and physics. Lattice calculus can serve as a tools to formulate adequate lattice models in for nanomechanics [65,66]. The suggested lattice fractional calculus is formulated for discretely distributed systems with the long-range interparticles interactions. Therefore this calculus can be important to describe the non-local properties of different types of media at nano-scale and micro-scale, where the intermolecular and interatomic interactions are crucial in determining the properties of these media.

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