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Relativistic non-Hamiltonian mechanics

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ABSTRACT

Relativistic particle subjected to a general four-force is considered as a nonholonomic system. The nonholonomic constraint in four-dimensional space–time represents the relativistic invariance by the equation for four-velocity $u_\mu u^\mu + c^2 = 0$, where c is the speed of light in vacuum. In the general case, four-forces are non-potential, and the relativistic particle is a non-Hamiltonian system in four-dimensional pseudo-Euclidean space–time. We consider non-Hamiltonian and dissipative systems in relativistic mechanics. Covariant forms of the principle of stationary action and the Hamilton's principle for relativistic mechanics of non-Hamiltonian systems are discussed. The equivalence of these principles is considered for relativistic particles subjected to potential and non-potential forces. We note that the equations of motion which follow from the Hamilton's principle are not equivalent to the equations which follow from the variational principle of stationary action. The Hamilton's principle and the principle of stationary action are not compatible in the case of systems with nonholonomic constraint and the potential forces. The principle of stationary action for relativistic particle subjected to non-potential forces can be used if the Helmholtz conditions are satisfied. The Hamilton's principle and the principle of stationary action are equivalent only for a special class of relativistic non-Hamiltonian systems.

1. Introduction

The interest in non-Hamiltonian and dissipative systems has been growing continually during the last few years. Non-Hamiltonian systems have found many applications in recent studies in physics. In

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a fairly short period of time, the list of such applications has become long. It includes statistical mechanics [1–5], kinetic theory [6], plasma physics [7–9], astrophysics [10], celestial mechanics [11], quantum mechanics [12–15], and nonequilibrium molecular dynamics [16–19]. In this paper, we discuss dynamics of relativistic non-Hamiltonian systems. The relativistic particles are described as systems with nonholonomic constraint in the four-dimensional pseudo-Euclidean space–time. The principle of stationary action and the Hamilton's principle are discussed for the relativistic particles subjected to non-potential four-forces.

It is well known that components of the four-velocity $u^{\mu} = dx^{\mu}/d\tau$ ($\mu = 1,2,3,4$ and τ is a proper time) are not independent. The components of the four-velocity are connected by the equation $u_{\mu}u^{\mu} + c^2 = 0$, where c is the speed of light in vacuum. This equation allows us to consider the relativistic particle as a system with constraints in four-dimensional space–time. This constraint is a nonlinear nonholonomic (nonintegrable) constraint. The relativistic invariance for point particles can be considered as a nonholonomic constraint. Therefore any relativistic particle in the four-dimensional space–time can be described as a nonholonomic system.

We may note that only mechanics of relativistic particles can be considered as a mechanics with nonholonomic constraint. The relativistic invariance in the field theory cannot be considered as a nonholonomic constraint. At the same time, nonholonomic constraints can be used in the field theory. For example, the higher spin fields are connected with nonholonomic constraints [23] and the gauge fixing conditions for non-abelian gauge fields can be described as nonholonomic constraints [24]. The Euler–Lagrange and Hamilton equations for nonholonomic systems in classical field theory are suggested in [25].

In Ref. [20], the geometric theory of nonholonomic systems on fibred manifolds is applied to describe the motion of a particle within the relativistic mechanics. Equations of motion for relativistic particles subjected to potential forces are suggested in [20] (see also [21,22]). In Refs. [20–22], generalized non-potential four-forces and non-Hamiltonian systems are not discussed within the framework of relativistic mechanics.

In this paper, a relativistic particle subjected to a general four-force $\mathcal{F}^{\mu}$ is considered as a nonholonomic system. The nonholonomic constraint in four-dimensional space–time represents the relativistic invariance by the equation for four-velocity $u_{\mu}u^{\mu} + c^2 = 0$, where c is the speed of light in vacuum. The consideration is partially based on the results of Krupkova and Musilova [20,21] (see also [22]). The main objects of [20,21] are relativistic particles subjected to the potential forces. In the general case, the four-force $\mathcal{F}^{\mu}$ is non-potential, and the relativistic particle is a non-Hamiltonian system in four-dimensional pseudo-Euclidean space–time. We consider non-Hamiltonian and dissipative systems in relativistic mechanics to take into account an interaction between the system and the environment. Note that relativistic particle with dissipation is discussed in [33,34]. In Refs. [33,34], the Lagrangian and Hamiltonian functions for one-dimensional relativistic particles with linear dissipation are suggested. In general, non-Hamiltonian and dissipative n-dimensional systems with $n > 1$ cannot be described by the Hamiltonian or Lagrangian since the Helmholtz's conditions for these systems are not satisfied [15]. In this paper, we consider relativistic particles as n-dimensional non-Hamiltonian and dissipative systems with $n > 1$.

It is well known that holonomic variational principles cannot be used for non-Hamiltonian and dissipative systems. Covariant forms of the principle of stationary action and the Hamilton's principle for relativistic mechanics are discussed in this paper. The equivalence of these principles is considered for relativistic particles subjected to potential and non-potential forces. The analysis of these principles is based on the results of the classic papers by Rumiantsev [26,27] (see also [28–31]). We note that the equations of motion which follow from the d'Alembert–Lagrange principle are not equivalent to the equations which follow from the principle of stationary action. In Refs. [26–28,31], the authors give proofs that the solutions to the equations of motion which follow from the d'Alembert–Lagrange principle and the Hamilton's principle do not in general satisfy the equations which follow from the action principle with nonholonomic constraints. In general, the Hamilton's principle and the principle of stationary action are not equivalent in the case of systems with nonholonomic constraints. In this paper, these results are applied to a nonholonomic approach to relativistic dynamics of non-Hamiltonian systems.

In Section 2, the nonholonomic constraint in four-dimensional space–time for relativistic particle and some notations are considered. In Sections 3 and 4, we discuss the d'Alembert–Lagrange principle and the Lagrange equations for relativistic particle that is considered as a nonholonomic system. In Section 5, the equations of motion for relativistic systems with nonholonomic constraint are represented as equations for holonomic systems. In Section 6, the conditions for relativistic particle to be a non-Hamiltonian or dissipative system are considered. In Sections 7 and 8, the Hamilton's principle, the principle of stationary action and the equivalence of these principles are discussed. Finally, a short conclusion is given in Section 9.

2. Nonholonomic constraint

2.1. Four-vector representation

Let us consider a four-dimensional pseudo-Euclidean space–time of points with coordinates x^{μ}: $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$. The point coordinates in the four-dimensional space–time can be considered as components *radius four-vector* of the point particle, $\vec{R} = (x^1, x^2, x^3, x^4) = (x, y, z, ct)$. The square of the elementary radius four-vector in the four-dimensional space–time is defined by

$$(d\vec{R})^2 = \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = dx^2 + dy^2 + dz^2 - c^2\, dt^2.$$

Here and later we mean the sum on the repeated indices μ and ν from 1 to 4. The coefficients $\eta_{\mu\nu}$ define a metric of pseudo-Euclidean space–time. This metric is a diagonal tensor such that $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{44} = -1$. Note that x_{μ} is not equal to x^{μ}, since $x_{\mu} = \eta_{\mu\nu} x^{\nu}$ and $x_1 = x^1$, $x_2 = x^2$, $x_3 = x^3$, and $x_4 = -x^4$.

Assume that we have two radius four-vectors $\vec{R}$ and $\vec{R}'$ with coordinates x^{μ} and x'^{μ} of two reference frames to describe a relativistic particle. If the coordinate transformation $x'^{\mu} = a^{\mu}_{\nu} x^{\nu}$, where a^{μ}_{ν} are constant values, satisfies the invariant condition:

$$(d\vec{R}')^2 = (d\vec{R})^2 : \quad \eta_{\mu\nu}\, dx'^{\mu}\, dx'^{\nu} = \eta_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}, \tag{1}$$

then this transformation is a Lorenz transformation. The invariance under the Lorenz transformations is a main postulate of relativistic mechanics.

The coordinates of the radius four-vector in the proper reference frame are $\vec{R}_0 = (0, 0, 0, c\tau)$, where τ is a proper time. Condition (1) leads us to the relation

$$(d\vec{R})^2 = (d\vec{R}_0)^2 : \quad \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -c^2\, d\tau^2, \tag{2}$$

or $d\vec{r}^2 - c^2\, dt^2 = -c^2\, d\tau^2$. Using the definition of three-velocity $\vec{v} = d\vec{r}/dt$, we get

$$dt = \gamma\, d\tau, \quad \gamma = (1 - v^2/c^2)^{-1/2}. \tag{3}$$

Four-velocity of the point particle is defined as a derivative of the radius four-vector with respect to proper time:

$$\vec{V} = \frac{d\vec{R}}{d\tau} : \quad u^{\mu} = \frac{dx^{\mu}}{d\tau}.$$

The components of the four-velocity $\vec{V} = (\gamma\vec{v}, \gamma c)$ are

$$u^k = \frac{dx^k}{d\tau} = \frac{dt}{d\tau}\frac{dx^k}{dt} = \gamma\frac{dx^k}{dt} = \gamma v^k \quad (k = 1, 2, 3),$$

$$u^4 = \frac{dx^4}{d\tau} = c\frac{dt}{d\tau} = c\gamma.$$

Note that rest particles ($\vec{v} = 0$) have $u^4 = c$.

Eq. (2) leads to the relation

$$\left(\frac{d\vec{R}}{d\tau}\right)^2 = \left(\frac{d\vec{R}_0}{d\tau}\right)^2 : \quad \eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -c^2,$$

which means that square of the four-velocity is a constant value: $\vec{V}^2 = -c^2$. Therefore we have the constraint equation

$$\eta_{\mu\nu} u^\mu u^\nu + c^2 = 0. \tag{4}$$

As a result, a relativistic particle in the covariant formulation of relativistic mechanics is a system with the nonholonomic constraint. The constraint (4) is nonholonomic since it depends on velocity. Relativistic mechanics can be considered as nonholonomic mechanics in the four-dimensional space.

2.2. Generalized coordinate representation

If we have holonomic constraints for the relativistic system, then we should use generalized coordinates. Let us consider generalized coordinates q^k, where $k = 1,\dots,n$ ($n \leqslant 4$), to describe a relativistic particle in the four-dimensional space–time. We have $n < 4$ for the case of additional holonomic constraints. Then x^μ are the functions of the proper time τ and the generalized coordinates q^k, i.e., $x^\mu = x^\mu(q,\tau)$. Using these functions, constraint equation (4) with $u^\mu = dx^\mu/d\tau$ has the form

$$f(\dot{q}, q, \tau) = 0, \tag{5}$$

where

$$f(\dot{q}, q, \tau) = g_{kl}(q,\tau)\dot{q}^k \dot{q}^l + 2g_k(q,\tau)\dot{q}^k + g(q,\tau) + c^2.$$

Here and later we mean the sum on the repeated indices k and l from 1 to n. We use the notations

$$g_{kl}(q,\tau) = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial q^k} \frac{\partial x^\nu}{\partial q^l}, \quad g_k(q,\tau) = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial q^k} \frac{\partial x^\nu}{\partial \tau}, \quad g(q,\tau) = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}, \tag{6}$$

and $\dot{q}^k = dq^k/d\tau$. Eq. (5) is a constraint equation for generalized coordinates q^k and the velocities $\dot{q}^k$.

2.3. Constraint as simplification

It is known that constraints in mechanics are some simplifications of real particle interactions. The constraints are caused by neglect of some properties and particle interactions. (Note that this statement is not correct in the field theory. For example, the constraints in electrodynamics are not connected with some neglect of particle interactions.) For example, if we consider the pendulum then we usually neglect the forces of thread deformation. It is interesting to understand the neglected properties and interactions for constraint (4), which defines the relativistic invariance. If we use the nonholonomic constraint (4), then we neglect the gravity interaction between particles. Let us consider the deformation of Eqs. (2)–(4) in general theory of relativity [36]. In the approximation of weak gravity fields, we have

$$(d\vec{R})^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu - 2\varphi\, dt^2 = -c^2\, d\tau^2,$$

where

$$dt = \gamma'\, d\tau, \quad \gamma' = \left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2},$$

and φ is a classical (Newtonian) gravity potential. As a result, we have

$$\eta_{\mu\nu} u^\mu u^\nu + c^2 = 2\varphi\gamma'^2.$$

Therefore nonholonomic constraint (4), which defines the relativistic invariance, is connected with the neglect of the gravity interaction, $\varphi = 0$, in the framework of the general theory of relativity.

3. d'Alembert–Lagrange principle

3.1. Four-vector representation

Let m_0 be the rest mass of a point relativistic particle. The *four-momentum* of the particle is defined by

$$\vec{P} = m_0 \vec{V} = (m_0\gamma\vec{v}, m_0\gamma c).$$

The components of the four-momentum are $p^\mu = m_0 u^\mu$. Eq. (4) gives

$$\eta_{\mu\nu} p^\mu p^\nu + m_0^2 c^2 = 0, \tag{7}$$

where $m_0 \gamma \vec{v}$ is a three-momentum $\vec{p} = m_0 \gamma \vec{v}$.

The "time component" p^4 of the four-momentum has the form $p^4 = m_0 \gamma c = E/c$, where $E = m_0 \gamma c^2$ is an energy of the relativistic particle. As a result, the components of the four-momentum are $\vec{P} = (\vec{p}, E/c)$, where $\vec{p} = m_0 \gamma \vec{v}$ and $E = m_0 \gamma c^2$.

In relativistic mechanics the Newtonian equations are replaced by some generalization, which is invariant under the Lorenz transformations [35,36]. The Newtonian equations are satisfied in the proper reference frame. The four-vector analog of the Newtonian equations is

$$\frac{d\vec{P}}{d\tau} = \vec{\mathcal{F}}(\vec{R}, \vec{P}), \tag{8}$$

where $(\vec{P})^2 = -c^2 m_0^2$, which is condition (7). Eq. (8) is postulated as a main equation of relativistic dynamics. This equation describes a relativistic particle subjected to a four-force $\vec{\mathcal{F}} = \vec{\mathcal{F}}(\vec{P}, \vec{R})$. Eq. (8) can be presented in the Hamiltonian form

$$\frac{dx^\mu}{d\tau} = \frac{1}{m_0} p^\mu, \quad \frac{dp^\mu}{d\tau} = \mathcal{F}^\mu(x, p), \quad \eta_{\mu\nu} p^\mu p^\nu + m_0^2 c^2 = 0. \tag{9}$$

It is known that the general principle, which allows us to derive equations of motion with holonomic and nonholonomic constraints, is the d'Alembert–Lagrange principle. In the pseudo-Euclidean four-dimensional space–time this principle leads to the variation equation

$$\left(\frac{d\vec{P}}{d\tau} - \mathcal{F}(\vec{R}, \vec{P}), \delta\vec{R} \right) = 0 : \quad \left(\frac{d(m_0 \dot{x}^\mu)}{d\tau} - \mathcal{F}^\mu(x, p) \right) \eta_{\mu\nu}\, \delta x^\nu = 0. \tag{10}$$

Multiplying Eq. (9) with the variation $\delta x_\mu = \eta_{\mu\nu} \delta x^\nu$ and summing over μ, we obtain variational equation (10).

3.2. Generalized coordinate representation

In the case of holonomic constraints for coordinates x^μ, we should use generalized coordinates q^k, where $k = 1, \ldots, n$ $(n \leqslant 4)$, to describe a relativistic particle. We have $n = 4 - s$ for the case of s holonomic constraints. Using the generalized coordinates q^k, where $k = 1, \ldots, n$ $(n \leqslant 4)$, and the functions $x^\mu = x^\mu(q, \tau)$, we obtain

$$\delta x^\mu = \frac{\partial x^\mu}{\partial q^k} \delta q^k. \tag{11}$$

Substitution of (11) into Eq. (10) gives

$$\left(\frac{d(m_0 \dot{x}^\mu)}{d\tau} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial q^k} - \mathcal{F}^\mu \eta_{\mu\nu} \frac{\partial x^\nu}{\partial q^k} \right) \delta q^k = 0. \tag{12}$$

Here and later we mean the sum on the repeated indices k and l from 1 to n in the generalized coordinate representation. Note that we mean the sum on the repeated indices k and l from 1 to 3 for three-vector representation. We define a generalized force Q_k by the equation

$$Q_k = \mathcal{F}^\mu \eta_{\mu\nu} \frac{\partial x^\nu}{\partial q^k}.$$

By the usual transformations (see Section 6.1 in [38]) of the form

$$\frac{d(m_0 u^\mu)}{d\tau} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial q^k} = \frac{d}{d\tau} \frac{\partial}{\partial \dot{q}^k} \left(\frac{m_0}{2} \eta_{\mu\nu} u^\mu u^\nu \right) - \frac{\partial}{\partial q^k} \left(\frac{m_0}{2} \eta_{\mu\nu} u^\mu u^\nu \right),$$

we derive the variational equation

$$\left(\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k(\dot{q}, q, \tau)\right)\delta q^k = 0, \tag{13}$$

where T is a scalar of the energy [39] in the pseudo-Euclidean space–time, which is defined by

$$T = \frac{m_0}{2}\eta_{\mu\nu}u^\mu u^\nu = \frac{m_0}{2}(g_{kl}(q,\tau)\dot{q}^k\dot{q}^l + 2g_k(q,\tau)\dot{q}^k + g(q,\tau)). \tag{14}$$

Note that we cannot use constraint equation (5) for T and f in variational equation (13) before the partial derivatives are taken. The functions $g_{kl}(q,\tau)$, $g_k(q,\tau)$, and $g(q,\tau)$ are defined in Eq. (6).

4. Lagrange equations

4.1. Generalized coordinate representation

In order to construct an analytical theory we must define variations. The variations of generalized coordinates δq^k, $k = 1,\ldots,n$ ($n \leqslant 4$), are defined by the relation of the ideal constraint

$$R_k\,\delta q^k = 0, \tag{15}$$

where R_k are components of the constraint force vector. R_k can be considered as a contribution of the reaction associated with the constraint to the generalized force Q_k. Because a reaction force does no work in a virtual movement that is consistent with the corresponding kinematical restriction, we conclude that R must be perpendicular to any δq that satisfies the constraint equation. Thus, if δq satisfies constraint equation, we have $R_k\delta q^k = 0$. We now consider which condition δq must be realized in order to satisfy a constraint equation. We can derive the usual relativistic equations of motion only under the condition (15). For nonholonomic systems a definition of the variations was suggested by Tchetaev [40,41]. The variations δq^k are defined by the condition:

$$\frac{\partial f}{\partial \dot{q}^k}\delta q^k = 0. \tag{16}$$

Using (15) and (16), we have the functions R_k as linear combinations of $\partial f/\partial \dot{q}^k$, i.e., $R_k = \lambda\,\partial f/\partial \dot{q}^k$, where λ is a Lagrange multiplier.

As a result, we have the variational equation

$$\left(\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k(q,\dot{q}) - \lambda\frac{\partial f}{\partial \dot{q}^k}\right)\delta q^k = 0. \tag{17}$$

Using definition (16) of coordinate variations, we can consider $\delta\dot{q}^a$, where $a = 1,2,3$ (if $n = 4$), as independent variations. The variation $\delta\dot{q}^4$ is not an independent variable. Suppose that the Lagrange multiplier λ satisfies the following condition. The bracket of (17) with $\delta\dot{q}^4$ is equal to zero:

$$\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^4} - \frac{\partial T}{\partial q^4} - Q_4 - \lambda\frac{\partial f}{\partial \dot{q}^4} = 0.$$

We note that variations $\delta\dot{q}^a$, where $a = 1,2,3$ are independent and the sum is separated on three equations. As a result, variational equation (17) is equivalent to the Lagrange equations

$$\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k + \lambda\frac{\partial f}{\partial \dot{q}^k} \quad (k = 1,2,3,4). \tag{18}$$

We cannot use constraint equation (5) for functions T and f in variational equation (13) before the partial derivatives on q^k and $\dot{q}^k$ are taken.

The system of Eqs. (18) and (5) is a closed system of $n + 1$ equations in the same number of unknowns. Using these equations, we can find the multiplier λ as a function $\lambda = \lambda(\dot{q}, q, \tau)$. Substituting this function into (18), we get the equations for generalized coordinates q^k. Note that Eq. (18) can be derived by Jourdain's variational equation [42].

Using (5) and (14), we have

$$\frac{\partial f}{\partial \dot{q}^k} = 2g_{kl}(q,\tau)\dot{q}^l + 2g_k(q,\tau), \tag{19}$$

$$\frac{\partial T}{\partial \dot{q}^k} = m_0 g_{kl}(q,\tau)\dot{q}^l + m_0 g_k(q,\tau), \tag{20}$$

$$\frac{\partial T}{\partial q^k} = \frac{m_0}{2}\left(\frac{\partial g_{lm}}{\partial q^k}\dot{q}^l\dot{q}^m + 2\frac{\partial g_l}{\partial q^k}\dot{q}^l + \frac{\partial g(q,\tau)}{\partial q^k}\right). \tag{21}$$

Substitution of (19)–(21) into (18) gives

$$\frac{d}{d\tau}(m_0 g_{kl}\dot{q}^l + m_0 g_k) - \frac{m_0}{2}\left(\frac{\partial g_{lm}}{\partial q^k}\dot{q}^l\dot{q}^m + 2\frac{\partial g_l}{\partial q^k}\dot{q}^l + \frac{\partial g}{\partial q^k}\right) = Q_k + 2\lambda(g_{kl}\dot{q}^l + g_k), \tag{22}$$

where $g_{kl} = g_{kl}(q,\tau)$, $g_k = g_k(q,\tau)$ and $g = g(q,\tau)$. Eqs. (22) and (5) form a system of $n+1$ equations in $n+1$ unknown λ and q^k, where $k = 1,\dots,n \leqslant 4$. The solutions of these equations describe a particle motion in relativistic mechanics as a motion of system with the nonlinear nonholonomic constraint (5) which defines the relativistic invariance.

4.2. Three-vector representation

If we use coordinates x^μ and Eq. (4), then condition (16) has the form

$$\eta_{\mu\nu}u^\mu\,\delta x^\nu = 0.$$

For four-vector representation Eq. (22) is the Lagrange equation

$$\frac{d}{d\tau}(m_0 u^\mu) = \mathcal{F}^\mu + 2\lambda u^\mu, \quad u_\mu u^\mu + c^2 = 0, \tag{23}$$

where $u^\mu = dx^\mu/d\tau$. Eqs. (23) form a system of five equations in five unknowns x^μ and λ. In the three-dimensional notations Eqs. (23) have the form

$$\frac{d\vec{p}}{dt} = \vec{F} + 2\lambda\vec{v}, \tag{24}$$

$$\frac{dE}{dt} = (\vec{F},\vec{v}) + \Phi + 2\lambda c^2, \tag{25}$$

$$(\vec{p})^2 - \frac{E^2}{c^2} + m_0^2c^2 = 0. \tag{26}$$

Here we use [35] the component $\mathcal{F}^4$ of the four-force $\vec{\mathcal{F}}$ in the form $\mathcal{F}^4 = (\gamma/c)(\vec{F},\vec{v}) + (\gamma/c)\Phi$, where the function Φ describes an energy exchange with an external medium [35]. Using Eq. (25), we define the Lagrange multiplier λ by

$$\lambda = \frac{1}{2c^2}\left(\frac{dE}{dt} - (\vec{F},\vec{v}) - \Phi\right). \tag{27}$$

Substituting (27) into (24), we have

$$\frac{d\vec{p}}{dt} = \vec{F} + \vec{v}\frac{1}{c^2}\left(\frac{dE}{dt} - (\vec{F},\vec{v}) - \Phi\right). \tag{28}$$

Differentiation of (26) with respect to proper time τ gives

$$\gamma\vec{v}\frac{d\vec{p}}{d\tau} - \gamma\frac{dE}{d\tau} + c^2\frac{dm_0}{d\tau} = 0. \tag{29}$$

Substituting (28) into (29) gives

$$\frac{dE}{dt} = (\vec{F},\vec{v}) + c^2\gamma\frac{dm_0}{dt} - \frac{v^2}{c^2}\gamma^2\Phi, \tag{30}$$

where energy $E = m_0\gamma c^2$. The energy change is caused by the power of $\vec{F}$, by the change of mass m_0, and by the energy exchange with external media Φ.

If particle rest mass m_0 is a constant ($m_0 = const$) and the energy exchange with external medium is zero ($\Phi = 0$), then Eqs. (30) and (28) lead to the usual equations [35] for the relativistic particle

$$\frac{d\vec{p}}{dt} = \vec{F}, \quad \frac{dE}{dt} = (\vec{F}, \vec{v}),$$

where the energy $E = m_0\gamma c^2$ and the momentum $\vec{p} = m_0\gamma\vec{v}$ are connected by Eq. (26).

5. Nonholonomic systems as holonomic systems

5.1. Generalized coordinate representation

Using results of [37], we represent the equation of motion for a system with nonholonomic constraint as a motion of some holonomic system.

We can rewrite constraint equation (5) by using the canonical coordinates q^k and the momentum

$$p_k = \frac{\partial T}{\partial \dot{q}^k} = m_0 g_{kl}(q,\tau)\dot{q}^l + m_0 g_k(q,\tau). \tag{31}$$

Eq. (31) with $det[g_{kl}(q,\tau)] \neq 0$ can be presented as

$$\frac{dq^l}{d\tau} = g^{kl}\left(m_0^{-1}p_k - g_k(q,\tau)\right), \tag{32}$$

where g^{kl} is an inverse of the metric g_{kl}, i.e., $g^{kl}g_{lm} = \delta^k_m$. Substitution of (32) into constraint equation (5) gives

$$\tilde{f}(p,q,\tau) = 0, \tag{33}$$

where

$$\tilde{f}(p,q,\tau) = g^{kl}(q,\tau)p_k p_l + m_0^2\tilde{g}(q,\tau) + m_0^2 c^2$$

and the functions g_k, g are defined by (6),

$$\tilde{g}(q,\tau) = g - g^{kl}g_k g_l.$$

The derivative of the function (33) with respect to proper time τ is equal to zero $d\tilde{f}/d\tau = 0$, i.e.,

$$\frac{\partial\tilde{f}}{\partial p_k}\dot{p}_k + \frac{\partial\tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial\tilde{f}}{\partial\tau} = 0. \tag{34}$$

Using (31), the Lagrange equations (18) has the form

$$\frac{dp_k}{d\tau} = \frac{\partial T}{\partial q^k} + Q_k + \lambda\frac{\partial f}{\partial \dot{q}^k} \quad (k = 1,\ldots,n). \tag{35}$$

Substituting (35) into Eq. (34), we find the Lagrange multiplier

$$\lambda(\dot{q},q,\tau) = -\left(\frac{\partial\tilde{f}}{\partial p_k}\frac{\partial f}{\partial\dot{q}^k}\right)^{-1}\left(\frac{\partial\tilde{f}}{\partial p_k}\left(\frac{\partial T}{\partial q^k} + Q_k\right) + \frac{\partial\tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial\tilde{f}}{\partial\tau}\right). \tag{36}$$

The generalized force of reaction R_k of the nonholonomic constraint (33) is defined by

$$R_k = \lambda(\dot{q},q,\tau)\frac{\partial f}{\partial\dot{q}^k},$$

where $\lambda(\dot{q},q,\tau)$ is defined by (36), and $k = 1,\ldots,n$. This force is a function of $(\dot{q}^k, q^k, \tau)$. As a result, Lagrange equations (18) are

$$\frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k + R_k, \tag{37}$$

where $k = 1, \dots, n$. In general, Q_k and R_k are non-potential forces, and the system cannot be considered as Hamiltonian.

If the coordinates x^μ of the radius four-vector are not dependent on a proper time τ, i.e., $x^\mu = x^\mu(q)$, then Eq. (37) has the form

$$\frac{d}{d\tau}(m_0 g_{kl}\dot{q}^l) = Q_k + m_0 \frac{\partial g_{lm}}{\partial q^k}\dot{q}^l\dot{q}^m + \frac{1}{2c^2} g_{kl}\dot{q}^l \left(m_0 \frac{\partial g_{jm}}{\partial q^i}\dot{q}^i\dot{q}^j\dot{q}^m + 2\dot{q}^m Q_m + 2\frac{m_0}{d\tau}c^2 \right).$$

Eqs. (37) are equations of some *holonomic* system with n degrees of freedom. For any trajectory of the point particle in the four-dimensional space–time we have $f(\dot{q}, q, \tau) = const$. If the initial values of $q_k(0)$ and $\dot{q}_k(0)$ satisfy the constraint condition

$$f(q(0), \dot{q}(0), \tau_0) = 0,$$

then the solution of Eq. (37) described a motion of the nonholonomic system.

Using Eq. (37), we get the Hamilton's variational principle

$$\int_{\tau_0}^{\tau_1} (\delta T(\dot{q}, q, \tau) + (Q_k + R_k)\delta q^k) d\tau = 0. \tag{38}$$

We note that the variations δq^k, $k = 1, 2, 3, 4$, are holonomic, and condition (16) is not satisfied. Therefore condition (15), which describes an ideal constraint, is not satisfied, i.e., $R_k \delta q^k \neq 0$.

We define the generalized force

$$\Lambda_k(\dot{q}, q, \tau) = Q_k + R_k,$$

which depends on the generalized velocities $\dot{q}^k$, generalized coordinates q^k and the proper time τ. If the Helmholtz conditions

$$\frac{\partial \Lambda_k}{\partial \dot{q}^m} + \frac{\partial \Lambda_m}{\partial \dot{q}^k} = 0, \tag{39}$$

$$\frac{\partial \Lambda_k}{\partial q^m} + \frac{\partial \Lambda_m}{\partial q^k} = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial \Lambda_k}{\partial \dot{q}^m} - \frac{\partial \Lambda_m}{\partial \dot{q}^k} \right), \tag{40}$$

are satisfied, then a generalized potential $U = U(\dot{q}, q, \tau)$ exists and

$$\frac{d}{d\tau}\frac{\partial U}{\partial \dot{q}^k} - \frac{\partial U}{\partial q^k} = \Lambda_k.$$

In this case, the Hamilton's variational principle (38) has the form of the stationary action principle

$$\delta \int_{\tau_0}^{\tau_1} L(\dot{q}, q, \tau) d\tau = 0, \tag{41}$$

where $L = T - U$.

We note that nonholonomic constraint (5) and the non-potential generalized force Q_k can be compensated such that the generalized force Λ_k is a generalized potential force, and the system is a Lagrangian (and non-dissipative) system with holonomic constraints.

5.2. Four-vector representation

Let us consider the coordinates x^μ and constraint equation (33) in the form

$$\eta_{\mu\nu} p^\mu p^\nu + m_0^2 c^2 = 0. \tag{42}$$

Differentiating (42) with respect to τ, we obtain:

$$\eta_{\mu\nu} \frac{dp^\mu}{d\tau} p^\nu + m_0 \frac{dm_0}{d\tau} c^2 = 0. \tag{43}$$

Substituting (43) into Eq. (23) with $m_0 u^\mu = p^\mu$, we get

$$\eta_{\mu\nu}\mathcal{F}^\mu p^\nu + 2\lambda\eta_{\mu\nu}u^\mu p^\nu + m_0\frac{dm_0}{d\tau}c^2 = 0.$$

Using the constraint equation $\eta_{\mu\nu}u^\mu u^\nu = -c^2$ and the four-momentum $p^\mu = m_0 u^\mu$, we obtain the Lagrange multiplier

$$\lambda = \frac{1}{2c^2}(\eta_{\mu\nu}\mathcal{F}^\mu u^\nu) + \frac{1}{2}\frac{dm_0}{d\tau}.$$

As a result, we have Eq. (23) in the form

$$\frac{dp^\mu}{d\tau} = \mathcal{F}^\mu + \frac{1}{c^2}u^\mu(\mathcal{F}^\nu u_\nu) + u^\mu\frac{dm_0}{d\tau}.$$

Using $p^\mu = m_0 u^\mu$, these equations can be presented as

$$m_0\frac{du^\mu}{d\tau} = \mathcal{F}^\mu + \frac{1}{c^2}u^\mu(\mathcal{F}^\nu u_\nu). \tag{44}$$

These equations define a holonomic system subjected to the force $(\mathcal{F}+R)^\mu = \mathcal{F}^\mu + R^\mu$. If initial dates satisfy constraint equation (4), then the solution of Eq. (44) describes a motion of the relativistic point particle as a holonomic system.

Note that the four-momentum p^μ is a constant ($dp^\mu/d\tau = 0$) if the conditions

$$\mathcal{F}^\mu + \frac{1}{c^2}u^\mu(\mathcal{F}^\nu u_\nu) = 0, \quad \frac{dm_0}{d\tau} = 0 \tag{45}$$

are satisfied. For any four-force $\mathcal{F}^\mu$, which is proportional to four-velocity, $\mathcal{F}^\mu = Z(u,x)u^\mu$, stationary conditions (45) are satisfied.

Let us consider the four-force $\mathcal{F}^\mu$ as the sum

$$\mathcal{F}^\mu = G^\mu + \Pi^\mu, \tag{46}$$

where

$$(G^\mu u_\mu) = 0, \quad (\Pi^\mu u_\mu) \neq 0.$$

Substitution of (46) into (44) gives

$$m_0\frac{du^\mu}{d\tau} = G^\mu + \Pi^\mu + \frac{1}{c^2}u^\mu(\Pi^\nu u_\nu).$$

The four-force G^μ is usually called [35] a real mechanical force, which satisfies the orthogonal condition $u_\mu G^\mu = 0$. The Lorenz force $\mathcal{F}^\mu = (e/c)F^{\mu\nu}u_\nu$, where $F^{\mu\nu}$ is the electromagnetic field tensor, is an example of a real mechanical force. The four-vector Π^μ describes the energy–momentum exchange between the point particle and the medium. The components of Π^μ are

$$\Pi^\mu = (\gamma\vec{\Pi}, (\gamma/c)\Phi),$$

where $\vec{\Pi}$ and Φ are momentum and energy, respectively, which are transmitted by convection per unit time. For the heat transfer, three-momentum $\delta\vec{p}$ and energy δQ transmitted per time $d\tau$ are defined by the formulas $\delta\vec{p} = \vec{\Pi}\,dt$ and $\delta Q = \Phi\,dt$. The components of δQ^μ are

$$\delta Q^\mu = \Pi^\mu d\tau = \left(\delta\vec{p}, \frac{1}{c}\delta Q\right) = \left(\gamma\vec{\Pi}\,d\tau, \frac{\gamma}{c}\Phi\,d\tau\right),$$

where δQ^μ is a four-vector of the heat energy–momentum, which is transmitted per time $d\tau$. The value Φ_0 defined by

$$\Phi_0 = -u_\mu\Pi^\mu = -\gamma^2((\vec{\Pi},\vec{v}) - \Phi)$$

is a velocity of the convective transmission of incoming energy in the rest reference frame. In the general case, $\Phi_0 \neq (dm_0/d\tau)c^2 = dE_0/d\tau$. The four-vectors G^μ and Π^μ allow us to describe non-Hamiltonian and dissipative processes in relativistic mechanics.

6. Non-Hamiltonian and dissipative relativistic systems

6.1. Generalized coordinate representation

Eqs. (32) and (37) can be presented in the form

$$\frac{dq^k}{d\tau} = G^k(q,p), \quad \frac{dp_k}{d\tau} = F_k(q,p), \tag{47}$$

where

$$G^k(q,p) = g^{kl}\left(m_0^{-1}p_l - g_l(q,\tau)\right), \tag{48}$$

$$F_k(q,p) = \frac{\partial T}{\partial q^k} + Q_k + R_k \tag{49}$$

and $(q,p) \in \mathcal{M}$. The system is called (locally) Hamiltonian if the right-hand sides of Eqs. (47) satisfy the Helmholtz conditions [46,47]

$$\frac{\partial G^k}{\partial p_l} - \frac{\partial G^l}{\partial p_k} = 0, \quad \frac{\partial G^k}{\partial q^l} + \frac{\partial F_l}{\partial p_k} = 0, \quad \frac{\partial F_k}{\partial q^l} - \frac{\partial F_l}{\partial q^k} = 0. \tag{50}$$

If $\mathcal{M}$ is a simply connected region, then a locally Hamiltonian system is globally Hamiltonian. (A region is simply connected if it is path-connected and every path between two points can be continuously transformed into every other. A region where any two points can be joined by a path is called path-connected.) In this case, we can rewrite Eqs. (47) in the form

$$\frac{dq^k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}. \tag{51}$$

In the general case, the Helmholtz conditions are not satisfied and the system is non-Hamiltonian [15]. If

$$\Omega(q,p) = \sum_{k=1}^{n} \left(\frac{\partial F_k(q,p)}{\partial p_k} + \frac{\partial G^k(q,p)}{\partial q^k} \right) \neq 0,$$

then we have a generalized dissipative system [15]. If $\Omega(q,p) \leqslant 0$ for all points (q,p) and $\Omega(q,p) < 0$ for some points (q,p), then the system is a dissipative system.

6.2. Four-vector representation

Using the four-vectors x^μ and p_μ, the equations of motion of the relativistic particle subjected to a non-potential four-force $\mathcal{F}^\mu$ are

$$\frac{dx^\mu}{d\tau} = \frac{1}{m_0}p^\mu, \quad \frac{dp^\mu}{d\tau} = \mathcal{F}^\mu(x,p), \quad \eta_{\mu\nu}p^\mu p^\nu + m_0^2c^2 = 0. \tag{52}$$

According to the results of Section 5.2, we have that Eqs. (52) with $dm_0/d\tau = 0$ are equivalent to the equations

$$\frac{dx^\mu}{d\tau} = \frac{1}{m_0}p^\mu, \quad \frac{dp^\mu}{d\tau} = \mathcal{F}^\mu(x,p) + \mathcal{R}^\mu(x,p), \tag{53}$$

where

$$\mathcal{R}^\mu(x,p) = \frac{1}{m_0^2c^2}p^\mu(\mathcal{F}^\nu(x,p)p_\nu), \tag{54}$$

and the initial dates satisfy constraint condition (4). The solution of Eq. (53) describes the motion of the relativistic particle.

The Helmholtz conditions for Eq. (53) have the form

$$\frac{\partial \mathcal{F}^{\mu}}{\partial p^{\nu}} + \frac{\partial \mathcal{R}^{\mu}}{\partial p_{\nu}} = 0, \tag{55}$$

$$\frac{\partial \mathcal{F}^{\mu}}{\partial x^{\nu}} + \frac{\partial \mathcal{R}^{\mu}}{\partial x^{\nu}} - \frac{\partial \mathcal{F}^{\nu}}{\partial x^{\mu}} - \frac{\partial \mathcal{R}^{\nu}}{\partial x^{\mu}} = 0. \tag{56}$$

Substitution of (54) into Eqs. (55) and (56) gives

$$\frac{\partial \mathcal{F}^{\mu}}{\partial p^{\nu}} + \frac{1}{m_0^2 c^2} \frac{\partial [p^{\mu}(\mathcal{F}^{\alpha}(x,p)p_{\alpha})]}{\partial p^{\nu}} = 0, \tag{57}$$

$$\frac{\partial \mathcal{F}^{\mu}}{\partial x^{\nu}} + \frac{1}{m_0^2 c^2} p^{\mu} \left(\frac{\partial \mathcal{F}^{\alpha}}{\partial x^{\nu}} p_{\alpha} \right) - \frac{\partial \mathcal{F}^{\nu}}{\partial x^{\mu}} - \frac{1}{m_0^2 c^2} p^{\nu} \left(\frac{\partial \mathcal{F}^{\alpha}}{\partial x^{\mu}} p_{\alpha} \right) = 0. \tag{58}$$

These equations are the Helmholtz conditions in four-vector representation. If these conditions are satisfied then the particle is a relativistic Hamiltonian system. The relativistic particle subjected to a four-force $\mathcal{F}^{\mu}$ is Hamiltonian if the four-force satisfies the Helmholtz conditions (57) and (58). In general, these conditions are not satisfied and the system is non-Hamiltonian.

The example of the four-force $\mathcal{F}^{\mu}$ is a Lorenz force $\mathcal{F}^{\mu} = (e/m_0 c)F^{\mu\nu}p_{\nu}$, where $F^{\mu\nu}$ is a tensor of the electromagnetic fields. Using $\partial\mathcal{F}^{\mu}/\partial x^{\nu} = 0$, we can see that Eqs. (58) are satisfied. Using $F^{\mu\nu} = -F^{\nu\mu}$, we get $\mathcal{R}^{\mu} = 0$ and

$$\frac{\partial \mathcal{F}^{\mu}}{\partial p_{\nu}} = \frac{e}{m_0 c} F^{\mu\nu}.$$

The conditions (57) are satisfied if $F^{\mu\nu} = 0$. Using $F^{\mu\nu} = -F^{\nu\mu}$, we have

$$\Omega(x,p) = \frac{\partial \mathcal{F}^{\mu}}{\partial p^{\mu}} = \frac{e}{m_0 c} \eta_{\mu\nu} F^{\mu\nu} = 0.$$

As a result, we have that the relativistic particle subjected to the Lorenz four-force can be considered as a non-Hamiltonian non-dissipative system. The Lorenz four-force is a gyroscopic force. It is known that Lorenz-type four-forces are the only admissible four-forces for a relativistic particle, compatible with a holonomic variation principle in relativistic mechanics. Note that new forces, which are different from the usual Lorentz force, are suggested in [21]. These forces arise due to the nonholonomic constraint by using the geometric theory of nonholonomic systems on fibred manifolds. Note that a one-dimensional relativistic particle with dissipation is considered in Refs. [33,34]. The Lagrangian and Hamiltonian functions for one-dimensional relativistic particles with linear dissipation are suggested. In general, non-Hamiltonian and dissipative n-dimensional systems with $n > 1$ cannot be described by Hamiltonian or Lagrangian since the Helmholtz's conditions for these systems are not satisfied [15].

7. Hamilton's principle

We describe a relativistic particle in pseudo-Euclidean space–time by generalized coordinates q^k, $k = 1,\dots,n$, where $n \leqslant 4$, and generalized velocities $\dot{q}^k = dq^k/d\tau$. Then we have nonlinear nonholonomic constraint (5). The basic variational principle of mechanics is the d'Alembert–Lagrange principle

$$\left(\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} - Q_k \right) \delta q^k = 0, \tag{59}$$

where $L = T - U$ is a Lagrangian, T is a function defined by (14), U is a generalized potential, Q_k is a generalized force, and δq^k are variations that satisfy the Tchetaev condition (16).

It is known that the Hamilton's principle can be obtained by integrating (59) over proper time with some constant limits τ_0 and τ_1:

$$\int_{\tau_0}^{\tau_1} \left(\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} - Q_k \right) \delta q^k \, d\tau = 0, \tag{60}$$

where $\delta q^k(\tau) \in C^2[\tau_0, \tau_1]$, and $\delta q^k(\tau_0) = \delta q^k(\tau_1) = 0$, $k = 1, \dots, n$. Integration by parts gives

$$\int_{\tau_0}^{\tau_1} \left(\frac{\partial L}{\partial \dot{q}^k} \frac{d}{d\tau} \delta q^k + \frac{\partial L}{\partial q^k} \delta q^k + Q_k \delta q^k \right) d\tau = 0. \quad (61)$$

This equation contains the time derivatives of the variations $d(\delta q^k)/d\tau$. The variations δq^k are not uniquely defined by Eq. (16). There is an arbitrariness in the definition of $d\delta q^k/dt$. In analytical mechanics the two following relations of $d(\delta q^k)/d\tau$ and $\delta \dot{q}^k$ are usually used [43].

According to *Hölder definition*:

$$\frac{d}{d\tau} \delta q^k = \delta \dot{q}^k, \quad k = 1, \dots, n, \quad (62)$$

for *all* generalized coordinates q^k.

Using Tchetaev condition (16), the variation of the constraint function $f = f(\dot{q}, q, \tau)$ has the form

$$\delta f = \left(\frac{\partial f}{\partial q^k} - \frac{d}{d\tau} \frac{\partial f}{\partial \dot{q}^k} \right) \delta q^k. \quad (63)$$

Note that the right-hand side of Eq. (63) is equal to zero only for a holonomic constraint. For a non-holonomic constraint (63) the variation $\delta f \neq 0$. The condition $\delta f = 0$ and relation (5) are compatible only in the case of holonomic systems [26,27].

According to *Appel-Suslov definition*: The relations (62) are satisfied only for the independent variations δq^k ($k = 1, \dots, n-1$) and the identity $\delta f = 0$ are realized. These conditions define the variation $\delta \dot{q}^n$ of the variable q^n.

If we use Hölder definition (62), then Hamilton's principle (61) has the form

$$\int_{t_0}^{t_1} (\delta L(\dot{q}, q, \tau) + Q_k \delta q^k) d\tau = 0. \quad (64)$$

Eqs. (64) and (16) give the Lagrange equations

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = Q_k + \lambda \frac{\partial f}{\partial \dot{q}^k} \quad (k = 1, \dots, n) \quad (65)$$

with Lagrange multiplier λ. Using the results of Section 5.2, Eq. (65) can be presented in the form

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = Q_k + R_k, \quad (66)$$

where $k = 1, \dots, n$. In general, Q_k and R_k are non-potential forces, and the system is non-Hamiltonian.

We note that the Hamilton's principle is described by the nonholonomic variational equation. It allows us to use this principle to obtain equations of motion of non-Hamiltonian and dissipative systems. The principle of stationary action is defined by holonomic variational equation. Therefore the principle of stationary action cannot be used to derive equation of motion of non-Hamiltonian systems in the general case. Note that the variational Sedov's equation [48–50] (see also [52,51]) also can be used for non-Hamiltonian and dissipative systems instead of the principle of stationary action.

8. Principle of stationary action

8.1. Generalized coordinate representation

The conditions under which Hamilton's principle for nonholonomic systems has the characteristics of the principle of stationary action were derived in [26,27]. The solutions to the equations of motion which follow from the Hamilton's principle do not in general satisfy the equations which follow from the action principle with nonholonomic constraints. We derive the condition under which the Hamilton's principle for relativistic particle and the principle of stationary action are equivalent.

Let us consider a Lagrange variational problem of stationary value of the action integral

$$\delta \int_{t_0}^{t_1} L(\dot{q}, q, \tau) d\tau = 0$$

in the class of curves that satisfy constraint equation (5). The introduction of Lagrange multiplier $c(\tau)$ reduces this problem of the conditional extremum to the Lagrange problem of variation

$$\delta \int_{\tau_0}^{\tau_1} (L(\dot{q}, q, \tau) + c(\tau) f(\dot{q}, q, \tau)) d\tau = 0, \tag{67}$$

where $f(\dot{q}, q, \tau)$ is a constraint function defined by (5). The Euler's equations for (67) are

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = c(\tau) \left(\frac{\partial f}{\partial q^k} - \frac{d}{d\tau} \frac{\partial f}{\partial \dot{q}^k} \right) - \frac{dc}{d\tau} \frac{\partial f}{\partial \dot{q}^k}, \tag{68}$$

where $k = 1, \ldots, n$. Note that these equations have a derivative for the multiplier $c(\tau)$.

In general, Eqs. (65) and (5) with $Q_k = 0$ are not equivalent to Eqs. (68) and (5). There is a possibility that some solutions of these two systems of equations being the same. If a solution $q^k(\tau)$ of Eqs. (65) and (5) with $Q_k = 0$ is a solution of Eqs. (68) and (5) for the same initial conditions, then

$$c(\tau) \left(\frac{\partial f}{\partial q^k} - \frac{d}{d\tau} \frac{\partial f}{\partial \dot{q}^k} \right) = \left(\lambda + \frac{dc}{d\tau} \right) \frac{\partial f}{\partial \dot{q}^k}. \tag{69}$$

Multiplying both sides of Eq. (69) by the variations δq^k and summing over k from 1 to n, we obtain

$$\left(\frac{\partial f}{\partial q^k} - \frac{d}{d\tau} \frac{\partial f}{\partial \dot{q}^k} \right) \delta q^k = 0. \tag{70}$$

Here we use the Tchetaev condition (16). Note that the same condition is derived for the case of potential forces $Q_k \neq 0$. Eq. (70) will be called the Rumyantsev condition. This condition is necessary and sufficient for Eqs. (68), (5) and (65), (5) with a potential force Q_k to have the same solution $q^k(\tau)$. In general, Eq. (70), which allows us to use the principle of stationary action, is not realized.

As a result, we have the following statement. Hamilton's principle (64) for a relativistic particle subjected to potential forces is the principle of stationary action (67) if and only if condition (70) is satisfied.

It is easy to prove that Rumyantsev's condition (70) for nonholonomic constraint (5) is not satisfied. As a result, the principle (67) cannot be used for relativistic particles subjected to potential forces. The principle of stationary action can be used if the Helmholtz's conditions (39) and (40) for $\Lambda_k(\dot{q}, q, \tau)$ or conditions (50) for $G^k(q,p)$ and $F_k(q,p)$ are satisfied. As a result, the Hamilton's principle and the principle of stationary action (67) are equivalent only for a special class of relativistic non-Hamiltonian systems. In general, we can use nonholonomic variational equations of Hamilton's principle or Sedov's variational equation.

8.2. Four-vector representation

Let us prove that Rumyantsev's condition (70) for nonholonomic constraint (5) is not satisfied. We consider the Rumyantsev's condition (70) in the four-vector representation. Using the variables x^μ and u^μ, Eq. (70) is

$$\left(\frac{\partial f(\tau, x, u)}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial f(\tau, x, u)}{\partial u^\mu} \right) \delta x^\mu = 0. \tag{71}$$

Substitution of the function

$$f(\tau, x, u) = \eta_{\mu\nu} u^\mu u^\nu + c^2 \tag{72}$$

into Eq. (71) gives

$$\frac{du^\mu}{d\tau} \delta x_\mu = 0, \tag{73}$$

where $x_\mu = \eta_{\mu\nu} x^\nu$. Eq. (73) can be rewritten in the form

$$\frac{du^k}{d\tau} \delta x_k + \frac{du^4}{d\tau} \delta x_4 = 0. \tag{74}$$

The Tchetaevs definition (16) of covariant variations δx^{μ} has the form

$$\frac{\partial f(\tau,x,u)}{\partial u^{\mu}}\delta x^{\mu}=0. \tag{75}$$

Substitution of (72) into Eq. (75) gives

$$u^{\mu}\,\delta x_{\mu}=0, \tag{76}$$

where we mean the sum on the repeated index μ from 1 to 4. We present Eq. (76) in the form

$$u^{k}\,\delta x_{k}+u^{4}\,\delta x_{4}=0.$$

Here we mean the sum on the repeated index k from 1 to 3. This equation gives

$$\delta x_{4}=-\frac{u^{k}}{u^{4}}\delta x_{k}. \tag{77}$$

Substitution of (77) into (74) gives

$$\left(\frac{du^{k}}{d\tau}-\frac{u^{k}}{u^{4}}\frac{du^{4}}{d\tau}\right)\delta x_{k}=0.$$

In general, the variation δx_k, $k=1,2,3$, are not equal to zero. Then we have the differential equations

$$\frac{du^{k}}{d\tau}-\frac{u^{k}}{u^{4}}\frac{du^{4}}{d\tau}=0 \quad (k=1,2,3).$$

Integrating these equations, we obtain $u^k=a^k u^4$, where a^k are constants. Using $u^k=\gamma v^k$ and $u^4=\gamma c$, we get $v^k=a^k c$, i.e., the values of the velocity v^k are constants.

As a result, we have that Rumyantsev's condition (71) for the relativistic particle subjected to potential forces is satisfied only for the motion with constant velocity. If relativistic particles is a Hamiltonian system, then the Hamiltonian's principle and the principle of stationary action (67) are not equivalent.

We also note that the principle of stationary action for relativistic particle subjected to non-potential forces $\mathcal{F}^{\mu}(u,x,\tau)$ can be used if the Helmholtz's conditions (57) and (58) are satisfied. The Hamilton's principle and the principle of stationary action are equivalent only for special forms of the four-force $\mathcal{F}^{\mu}(u,x,\tau)$, and for a special class of relativistic non-Hamiltonian systems. This class is defined by the case of potential properties of the sum of non-potential force $\mathcal{F}^{\mu}(x,p)$ and the reaction force $\mathcal{R}^{\mu}(x,p)$. Note that nonholonomic constraint and non-potential force can be compensated such that the resulting force is potential, and the system is a Lagrangian system with holonomic constraints. The principle of stationary action, which uses the holonomic variational equation, can be used for non-Hamiltonian and dissipative systems if the suggested generalization of Helmholtz's conditions (57) and (58) are satisfied. In the general case, we should use nonholonomic variational equations of Hamiltons principle or Sedov's variational equation.

9. Conclusion

We formulate relativistic mechanics of point particle as mechanics of the particle with nonholonomic constraint in the four-dimensional pseudo-Euclidean space–time. The nonholonomic constraint represents the relativistic invariance by the equation for four-velocity $u_{\mu}u^{\mu}+c^2=0$, where c is the speed of light in vacuum. We consider relativistic particles subjected to generalized forces. In general, these forces are non-potential, and the particles are relativistic non-Hamiltonian systems. The conditions on the generalized forces that allow us to consider relativistic particles subjected to non-potential forces as a Hamiltonian systems are suggested. The nonholonomic constraint, which represents relativistic invariance, and the non-potential generalized force can be compensated such that the system is Hamiltonian (and non-dissipative).

The Hamilton's principle and the principle of stationary action are considered for relativistic particles subjected to non-potential forces. We prove that the principle of stationary action can be used only if the Helmholtz conditions (39), (40) or (50) are satisfied. The Hamilton's principle and the principle of stationary action are equivalent only for a special class of relativistic non-Hamiltonian systems. In general, the Hamilton's principle and nonholonomic variational equations can be used to describe relativistic non-Hamiltonian and dissipative systems. The variational Sedov's equation [48–50] (see also [51,52]), which is nonholonomic equation, can be used for relativistic non-Hamiltonian and dissipative systems instead of the principle of stationary action. Note that relativistic models of continuous media with dissipation are considered in [50,52] by using the nonholonomic variational equations.

The study of plasma systems containing ensembles of particles (dust) is a rapidly developing field of complex systems research. One of the general features of complex plasma systems is the presence of non-potential interaction forces between the dust particles due to the dynamic interaction between the dust particles and the plasma (for example, see [7–9] and references therein). In general, these systems cannot be described as Hamiltonian, since the energy is not conserved because of the openness of the systems due to plasma-particle interaction. We hope that dynamics of relativistic particle subjected to non-potential forces and models of relativistic dissipative non-Hamiltonian systems can be used to describe relativistic complex plasma systems.

Using the suggested approach to relativistic non-Hamiltonian systems, a relativistic generalization of the Liouville equations for dissipative non-Hamiltonian systems [5] can be obtained. We note that nonholonomic constraints with power-law memory [44] can be used in relativistic mechanics by using fractional derivatives [45] with respect to proper time. The covariant formulation of relativistic non-Hamiltonian mechanics as a mechanics of nonholonomic systems can be used to formulate quantum relativistic mechanics for dissipative systems by the methods suggested in [14,15].

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