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Quantum dissipation from power-law memory

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

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ABSTRACT

A new quantum dissipation model based on memory mechanism is suggested. Dynamics of open and closed quantum systems with power-law memory is considered. The processes with power-law memory are described by using integration and differentiation of non-integer orders, by methods of fractional calculus. An example of quantum oscillator with linear friction and power-law memory is considered.

1. Introduction

A new quantum dissipation model based on memory mechanism is suggested. Dynamics of open and closed quantum systems with power-law memory is considered. An example of quantum oscillator with linear friction and power-law memory is considered. The processes with power-law memory are described by using integration and differentiation of non-integer orders, by methods of fractional calculus [1,2]. Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history from 30 September 1695, when the derivatives of order $\alpha = 1/2$ has been described by Leibniz in a letter to L'Hospital [3,4]. The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Riemann, Abel, Riesz, Weyl. There are the special journals: "Fractional Calculus and Applied Analysis"; "Fractional Differential Calculus"; "Communications in Fractional Calculus". The first book dedicated specifically to the theory of fractional integrals and derivatives, is the book by Oldham and Spanier [5] published in 1974. There exists the remarkably comprehensive encyclopedic-type monograph by Samko et al. [1], which was published in Russian in 1987 and in English in 1993. In 2006 Kilbas et al. published a very important and remarkable book [2], where one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. Applications of fractional calculus in physics are described

* Tel.: +7 499 939 5989; fax: +7 499 939 0397.
E-mail address: tarasov@theory.sinp.msu.ru.

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in the books [6–14]. In general, many usual properties of the ordinary (first-order) derivative D_t are not realized for fractional derivative operators D_t^α. For example, a product rule, chain rule, semigroup property have strongly complicated analogs for the operators D_t^α.

The fractional calculus is a powerful tool to describe physical systems that have long-time memory. Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems [15–18]. The memory effects to discrete maps mean that their present state evolution depends on all past states. Note that a power-law memory has been detected for fluctuation within a single protein molecule [19]. The nonholonomic systems with generalized constraints to describe a long-time memory are considered [20]. The electrodynamics of dielectric media is described as a fractional temporal electrodynamics [21–23]. The discrete maps with memory are obtained from the fractional differential equations of classical dynamical systems [24–27].

2. Derivatives and integrals of non-integer order

There are many different definitions of fractional integrals and derivatives of non-integer orders [1,2].

2.1. A generalization of Cauchy's differentiation formula

Let G be an open subset of the complex plane $\mathbb{C}$, and $f : G \to \mathbb{C}$ is a holomorphic function:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{n+1}}\, dz. \tag{1}$$

A generalization of (1) has been suggested by Sonin and Letnikov in 1872 in the form

$$D_x^\alpha f(x) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{\alpha+1}}\, dz, \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq -1, -2, -3, \ldots$ (see Theorem 22.1 in the book by Samko et al. [1]). Expression (2) is also called Nishimoto derivative.

2.2. A generalization of finite difference

The differentiation of integer order n can be defined by

$$D_x^n f(x) = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n}, \tag{3}$$

where Δ_h^n is a finite difference of integer order n:

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh). \tag{4}$$

The difference of a fractional order $\alpha > 0$ is defined by the infinite series

$$\Delta_h^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh), \tag{5}$$

where the binomial coefficients are

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}. \tag{6}$$

The left-and right-sided Grünwald–Letnikov derivatives of order $\alpha > 0$ are defined by

$$^{GL}D_{x\pm}^\alpha f(x) = \lim_{h\to 0} \frac{\nabla_{\mp h}^\alpha f(x)}{h^\alpha}. \tag{7}$$

If

$$|f(x)| < c(1+|x|)^{-\mu}, \quad \mu > |\alpha|$$

then the series (5) can be used for $\alpha < 0$ and Eq. (7) defines Grünwald–Letnikov fractional integral. If $f(x) \in L_p(\mathbb{R})$, where $1 < p < 1/\alpha$ and $0 < \alpha < 1$, then (7) can be represented by

$${}^{GL}D^{\alpha}_{x\pm} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x \mp z)}{z^{\alpha+1}} dz.$$

2.3. A generalization by Fourier transform

If we define the Fourier transform operator $\mathcal{F}$ by

$$(\mathcal{F} f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt, \tag{8}$$

then the Fourier transform of derivative of integer order n is

$$(\mathcal{F} D^n_x f)(\omega) = (i\omega)^n (\mathcal{F} f)(\omega).$$

Therefore

$$D^n_x f(x) = \mathcal{F}^{-1}\{(i\omega)^n (\mathcal{F} f)(\omega)\}.$$

For $f(t) \in L_1(\mathbb{R})$, the left- and right-sided Liouville fractional integrals and derivatives can be defined (see Theorem 7.1 in [1] and Theorem 2.15 in [2]) by the relations

$$(I^{\alpha}_{\pm} f)(x) = \mathcal{F}^{-1}\left(\frac{1}{(\pm i\omega)^{\alpha}} (\mathcal{F} f)(\omega) \right), \tag{9}$$

$$(D^{\alpha}_{\pm} f)(x) = \mathcal{F}^{-1}\left((\pm i\omega)^{\alpha} (\mathcal{F} f)(\omega) \right), \tag{10}$$

where $0 < \alpha < 1$ and

$$(\pm i\omega)^{\alpha} = |\omega|^{\alpha} \exp\left(\pm \mathrm{sgn}(\omega) \frac{i\alpha\pi}{2} \right).$$

The Liouville fractional integrals (9) can be represented by

$$(I^{\alpha}_{\pm} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} z^{\alpha-1} f(x \mp z) dz. \tag{11}$$

The Liouville fractional derivatives (10) are

$$(D^{\alpha}_{\pm} f)(x) = D^n_x (I^{n-\alpha}_{\pm} f)(x).$$

Therefore

$$(D^{\alpha}_{\pm} f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^{\infty} z^{n-\alpha-1} f(x \mp z) dz, \tag{12}$$

where $n = [\alpha] + 1$.

2.4. Caputo derivative

We can define the derivative of fractional order α by

$${}^{C}D^{\alpha}_{\pm} f(t) = I^{n-\alpha}_{\pm} (D^n_t f)(t).$$

For $x \in [a, b]$ the left-sided Caputo fractional derivative of order $\alpha > 0$ is defined by

$$ {}_a^C D_t^\alpha f(t) = {}_a I_t^{n-\alpha} D_t^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{d\tau \, D_\tau^n f(\tau)}{(t-\tau)^{\alpha-n+1}}, \tag{13} $$

where $n-1 < \alpha < n$, and ${}_a I_t^\alpha$ is the left-sided Riemann–Liouville fractional integral of order $\alpha > 0$ that is defined by

$$ {}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}}, \quad (a < t). $$

The Riemann–Liouville fractional derivative has some notable disadvantages in applications such as nonzero of the fractional derivative of constants,

$$ {}_0 D_t^\alpha C = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C, $$

which means that dissipation does not vanish for a system in equilibrium. The Caputo fractional differentiation of a constant results in zero

$$ {}_0^C D_t^\alpha C = 0. $$

The desire to use the usual initial value problems

$$ f(t_0) = C_0, \qquad (D_t^1 f)(t_0) = C_1, \qquad (D_t^2 f)(t_0) = C_2, \ldots $$

lead to the application Caputo fractional derivatives instead of the Riemann–Liouville derivative.

If $f(t)$ be a function for which the Caputo derivatives of order α exist together with the Riemann–Liouville derivatives, then these fractional derivatives are connected by the relation

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a). \tag{14} $$

The second term of the right hand side of Eq. (14) regularizes the Caputo fractional derivative to avoid the potentially divergence from singular integration at $t = 0$.

3. Power-law memory and fractional derivatives

A physical interpretation of equations with derivatives and integrals of non-integer order with respect to time is connected with the memory effects.

Let us consider the evolution of a dynamical system in which some quantity $A(t)$ is related to another quantity $B(t)$ through a memory function $M(t)$ by

$$ A(t) = \int_0^t M(t-\tau) B(\tau) d\tau. \tag{15} $$

This operation is a particular case of composition products suggested by Vito Volterra. In mathematics, Eq. (15) means that the value $A(t)$ is related with $B(t)$ by the convolution operation

$$ A(t) = M(t) * B(t). $$

Eq. (15) is a typical equation obtained for the systems coupled to an environment, where environmental degrees of freedom being averaged.

Let us consider the limiting cases widely used in physics: (1) the absence of the memory; (2) the complete memory; (3) the power-like memory. As a result, we have the following special cases of Eq. (15).

(1) *The absence of the memory:* For a system without memory, the time dependence of the memory function is

$$ M(t-\tau) = M(t) \, \delta(t-\tau), \tag{16} $$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $A(t)$ is defined by $B(t)$ at the only instant t. In this case, the system loses all its values of quantity except for one. Using (15) and (16), we have

$$A(t) = \int_0^t M(t)\delta(t-\tau)B(\tau)d\tau = M(t)B(t). \tag{17}$$

Expression (17) corresponds to the well-known physical process with complete absence of memory. This process relates all subsequent values to previous values through the single current value at each time t.

(2) *Complete memory:* If memory effects are introduced into the system, then the delta-function turns into some function with the time interval during which $B(t)$ affects on the function $A(t)$. Let $M(t)$ be the step function

$$M(t-\tau) = t^{-1}[\theta(\tau) - \theta(t-\tau)], \tag{18}$$

where $\theta(t)$ is the Heaviside function, also called the unit step function. The Heaviside function $\theta(t)$ is a discontinuous function whose value is zero for negative argument and one for positive argument. In Eq. (18), the factor t^{-1} is chosen to get normalization of the memory function to unity:

$$\int_0^t M(\tau)d\tau = 1.$$

Then in the evolution process the system passes through all states continuously without any loss. In this case,

$$A(t) = \frac{1}{t}\int_0^t B(\tau)d\tau,$$

and this corresponds to a complete memory.

(3) *Power-law memory:* The power-like memory function is defined by

$$M(t-\tau) = M_0\,(t-\tau)^{\varepsilon-1}, \tag{19}$$

where M_0 is a real parameter. This function indicates the presence of the fractional derivative or integral. Substitution of (19) into (15) gives the temporal fractional integral of order ε:

$$A(t) = \lambda I_t^{\varepsilon} B(t) = \frac{\lambda}{\Gamma(\varepsilon)}\int_0^t (t-\tau)^{\varepsilon-1}B(\tau)d\tau, \quad 0<\varepsilon<1, \tag{20}$$

where $\lambda = \Gamma(\varepsilon)M_0$. The parameter λ can be regarded as the strength of the perturbation induced by the environment of the system. The physical interpretation of the fractional integration is an existence of a memory effect with power-like memory function. The memory determines an interval $[0, t]$ during which $B(\tau)$ affects $A(t)$.

Eq. (20) is a special case of relation for $A(t)$ and $B(t)$, where $A(t)$ is directly proportional to $M(t) * B(t)$. In a more general case, the values $A(t)$ and $B(t)$ can be related by the equation

$$f(A(t), M(t) * D_t^n B(t)) = 0, \tag{21}$$

where f is a smooth function. In this case Eq. (21) gives the relation $f(A(t),\ {}_0^C D_t^{\alpha} B(t)) = 0$ with Caputo fractional derivatives.

Note that Leibnitz rule for fractional derivatives has the form of infinite series

$$(D_{a+}^{\alpha}(fg))(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}(D_x^{\alpha-k}f)(x)\,D_x^k g$$

for analytic functions on (a, b) (see Theorem 15.1 in [1, p. 216]). This sum has the fractional integral of fractional order for $k > [\alpha] + 1$.

4. Quantum dynamics with power-law memory

4.1. Time fractional Lindblad equation

The dynamics of open quantum systems can be described in terms of the infinitesimal change of the system. This change is defined by some form of infinitesimal generator. The most general explicit form of the infinitesimal superoperator was suggested by Gorini et al. in [28–30]. There exists a one-to-one correspondence between the completely positive norm continuous semigroups and the completely dissipative generating superoperators.

In order to take into account a power-law memory, we can consider a generalization of Lindblad equation for quantum observables in the form

$$ {}_0^C D_t^{\alpha} A_t = -\mathcal{L}_V A_t, \tag{22}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative with respect to time t (dimensionless variable), and $\mathcal{L}_V$ is defined by

$$\mathcal{L}_V A_t = \frac{1}{i\hbar}[H, A_t] - \frac{1}{2\hbar}\sum_{k=1}^{\infty}\Big(V_k^*[A_t, V_k] + [V_k^*, A_t]V_k\Big). \tag{23}$$

For $\alpha = 1$ we have the usual Lindblad equation [28–30]. If α is non-integer, then Eq. (22) defines the quantum processes with power-law memory.

If all operators V_k are equal to zero ($V_k = 0$), then we have a generalization of the Heisenberg equation for Hamiltonian system with memory

$$ {}_0^C D_t^{\alpha} A_t = -\frac{1}{i\hbar}[H, A_t]. \tag{24}$$

Note that the form of $\mathcal{L}_V$ is not uniquely defined. The transformations

$$V_k \to V_k + a_k I, \qquad H \to H + \frac{1}{2i\hbar}\sum_{k=1}^{\infty}(a_k^* V_k - a_k V_k^*),$$

where a_k are arbitrary complex numbers, preserve the form of Eq. (22).

For the density operator, we have

$$ {}_0^C D_t^{\alpha} \rho_t = \frac{1}{i\hbar}[H, \rho_t] - \frac{1}{\hbar}\sum_{k=1}^{\infty}\Big(V_k \rho_t V_k^* - \rho_t V_k^* V_k - V_k^* V_k \rho_t\Big). \tag{25}$$

Eq. (25) describes dynamics of quantum state of open systems with memory. The memory means that the present state evolution depends on all past states.

4.2. Cauchy-type problem

If we consider the Cauchy-type problem for Eq. (22) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be represented [31] in the form

$$A_t = \Phi_t(\alpha) A_0, \quad (t \geq 0),$$

where

$$\Phi_t(\alpha) = E_{\alpha}[-t^{\alpha}\mathcal{L}_V]. \tag{26}$$

Here $E_{\alpha}[\mathcal{L}]$ is the Mittag-Leffler function with the superoperator argument

$$E_{\alpha}[\mathcal{L}] = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k + 1)}\mathcal{L}^k.$$

Note that the relation

$$ {}_a^C D_t^{\alpha} E_{\alpha}[\lambda(t-a)^{\alpha}] = \lambda E_{\alpha}[\lambda(t-a)^{\alpha}]$$

holds for $\lambda \in \mathbb{C}$, $t > a$, $a \in \mathbb{R}$, and $\alpha > 0$ (see Lemma 2.23 in [2]).

4.3. Quantum dynamical groupoid

The superoperators $\Phi_t(\alpha)$, $t \geqslant 0$, describe dynamics of open quantum systems with power-law memory. The superoperator $\mathcal{L}_V$ can be considered as a generator of the one-parameter groupoid $\Phi_t(\alpha)$ on operator algebra of quantum observables:

$$ {}_0^C D_t^\alpha \Phi_t(\alpha) = -\mathcal{L}_V \Phi_t(\alpha). $$

The set $\{\Phi_t(\alpha) | t \geqslant 0\}$, will be called a quantum dynamical groupoid. Note that the following properties are realized

$$ \Phi_t(\alpha) I = I, $$
$$ (\Phi_t(\alpha) A)^* = \Phi_t(\alpha) A $$

for self-adjoint operators A ($A^* = A$), and

$$ \lim_{t \to 0+} \Phi_t(\alpha) = L_I, $$

where L_I is an identity superoperator ($L_I A = A$). As a result the superoperators $\Phi_t(\alpha)$, $t \geqslant 0$, are real and unit preserving maps on operator algebra of quantum observables.

For $\alpha = 1$, we have

$$ \Phi_t(1) = E_1[-t\mathcal{L}_V] = \exp\{-t\mathcal{L}_V\}. $$

The superoperators $\Phi_t = \Phi_t(1)$ form a semigroup such that

$$ \Phi_t \Phi_s = \Phi_{t+s}, \quad (t, s > 0), \qquad \Phi_0 = L_I. $$

This property holds since

$$ \exp\{-t\mathcal{L}_V\} \exp\{-s\mathcal{L}_V\} = \exp\{-(t+s)\mathcal{L}_V\}. $$

For $\alpha \notin \mathbb{N}$ we have

$$ E_\alpha[-t^\alpha \mathcal{L}_V] E_\alpha[-s^\alpha \mathcal{L}_V] \neq E_\alpha[-(t+s)^\alpha \mathcal{L}_V]. $$

Therefore the semigroup property is not satisfied for non-integer values of α:

$$ \Phi_t(\alpha) \Phi_s(\alpha) \neq \Phi_{t+s}(\alpha), \quad (t, s > 0). $$

As a result, the superoperators $\Phi_t(\alpha)$ with $\alpha \notin \mathbb{N}$ cannot form a semigroup. This property means that we have a quantum processes with memory. The superoperators $\Phi_t(\alpha)$ describe quantum dynamics of open systems with memory. The memory effects to dynamical maps mean that their present evolution of $A(t) = \Phi_t(\alpha) A_0$ depends on all past values of $A(\tau)$ for $\tau < t$.

5. Linear oscillator with friction and memory

Let us consider an oscillator with linear friction and power-like memory. In this example, the basic assumption is that the general form of a bounded completely dissipative superoperator holds for an unbounded superoperator $\mathcal{L}_V$. We assume that the operators H, and V_k are functions of the operators Q and P such that the obtained model is exactly solvable [32,33] (see also [34,35]). Therefore we consider $V_k = V_k(Q, P)$ as the first-degree polynomials in Q and P, and the Hamiltonian $H = H(Q, P)$ as a second degree polynomial in Q and P:

$$ H = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2 + \frac{\mu}{2}(PQ + QP), \tag{27} $$
$$ V_k = a_k P + b_k Q, $$

where a_k, and b_k, $k = 1, 2$, are complex numbers. These assumptions mean that the friction force is proportional to the velocity.

Using the definition of $\mathcal{L}_V$ and the canonical commutation relations for operators Q and P, we obtain

$$-\mathcal{L}_V Q = \frac{1}{m}P + \mu Q - \lambda Q,$$
$$-\mathcal{L}_V P = -m\omega^2 Q - \mu P - \lambda P,$$

where $\lambda = \mathrm{Im}\left(a_1 b_1^* + a_1 b_1^*\right)$. Let us consider the generalized Lindblad equation (22) for Q_t and P_t in the form

$$ {}_0^C D_t^\alpha Q_t = -\mathcal{L}_V Q_t, \qquad {}_0^C D_t^\alpha P_t = -\mathcal{L}_V P_t, \tag{28}$$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative with respect to time t, which is dimensionless variable.

We define the matrices

$$A = \begin{pmatrix} Q \\ P \end{pmatrix}, \qquad M = \begin{pmatrix} \mu - \lambda & m^{-1} \\ -m\omega^2 & -\mu - \lambda \end{pmatrix}. \tag{29}$$

Then Eqs. (28) for quantum observables have the matrix representation

$$ {}_0^C D_t^\alpha A_t = M A_t, \tag{30}$$

where $-\mathcal{L}_V A_t = M A_t$.

If we consider the Cauchy-type problem for Eq. (30) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be represented [31] in the form

$$A_t = \Phi_t(\alpha) A_0,$$

where (see Eq. (26))

$$\Phi_t(\alpha) = E_\alpha[t^\alpha M].$$

The Mittag-Leffler function with the matrix argument is defined by

$$E_\alpha[t^\alpha M] = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n + 1)} M^n.$$

For $\alpha = 1$, we obtain

$$\Phi_t(1) = \Phi_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n = e^{tM}. \tag{31}$$

Let the matrix M be represented as

$$M = N^{-1} F N, \tag{32}$$

where F is a diagonal matrix, and

$$N = \begin{pmatrix} m\omega^2 & \mu + \nu \\ m\omega^2 & \mu - \nu \end{pmatrix}, \tag{33}$$

$$F = \begin{pmatrix} -(\lambda + \nu) & 0 \\ 0 & -(\lambda - \nu) \end{pmatrix}. \tag{34}$$

Here we use the complex parameter ν, such that $\nu^2 = \mu^2 - \omega^2$.

Using (32), the one-parameter superoperators $\Phi_t(\alpha)$ are represented by

$$\Phi_t(\alpha) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n + 1)} M^n = N^{-1} \left(\sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(\alpha n + 1)} F^n \right) N.$$

As a result, we have

$$\Phi_t(\alpha) = N^{-1} E_\alpha[t^\alpha F]\, N. \tag{35}$$

For $\alpha = 1$, we have

$$\Phi_t(1) = N^{-1} e^{tF} N.$$

Substitution of (33) and (34) into (35) gives

$$\Phi_t(\alpha) = \begin{pmatrix} C_\alpha[\lambda,\nu,t] + (\mu/\nu)\, S_\alpha[\lambda,\nu,t] & (1/m\nu)\, S_\alpha[\lambda,\nu,t] \\ -(m\omega^2/\nu)\, S_\alpha[\lambda,\nu,t] & C_\alpha[\lambda,\nu,t] - (\mu/\nu)\, S_\alpha[\lambda,\nu,t] \end{pmatrix},$$

where we use the notations

$$S_\alpha[\lambda,\nu,t] = \frac{1}{2}\Big(E_\alpha[(-\lambda+\nu)t^\alpha] - E_\alpha[(-\lambda-\nu)t^\alpha]\Big),$$

$$C_\alpha[\lambda,\nu,t] = \frac{1}{2}\Big(E_\alpha[(-\lambda+\nu)t^\alpha] + E_\alpha[(-\lambda-\nu)t^\alpha]\Big).$$

As a result, we obtain $A_t(\alpha) = \Phi_t(\alpha) A_0$ in the form

$$Q_t = \left(C_\alpha[\lambda,\nu,t] + \frac{\mu}{\nu} S_\alpha[\lambda,\nu,t]\right) Q_0 + \frac{1}{m\nu} S_\alpha[\lambda,\nu,t] P_0, \tag{36}$$

$$P_t = -\frac{m\omega^2}{\nu} S_\alpha[\lambda,\nu,t] Q_0 + \left(C_\alpha[\lambda,\nu,t] - \frac{\mu}{\nu} S_\alpha[\lambda,\nu,t]\right) P_0. \tag{37}$$

For $\alpha = 1$, we get

$$S_\alpha[\lambda,\nu,t] = e^{-\lambda t}\sinh(\nu t), \qquad C_\alpha[\lambda,\nu,t] = e^{-\lambda t}\cosh(\nu t),$$

and Eqs. (36) and (37) give the well-known solutions [33]:

$$Q_t = e^{-\lambda t}\left(\cosh(\nu t) + \frac{\mu}{\nu}\sinh(\nu t)\right) Q_0 + \frac{1}{m\nu} e^{-\lambda t}\sinh(\nu t) P_0, \tag{38}$$

$$P_t = -\frac{m\omega^2}{\nu} e^{-\lambda t}\sinh(\nu t) Q_0 + e^{-\lambda t}\left(\cosh(\nu t) - \frac{\mu}{\nu}\sinh(\nu t)\right) P_0, \tag{39}$$

where sinh and cosh are hyperbolic sine and cosine.

Note that for $V_k = 0$ $(k = 1, 2)$, we have closed Hamiltonian system with memory, that is described by Heisenberg equation (24). The solution of the equation for linear oscillator with Hamiltonian (27) and a power-law memory is given by (36) and (37), where $\lambda = 0$.

For non-integer α, the Mittag-Leffler function in Eqs. (36) and (37) can be represented in the form

$$E_{\alpha,1}(-zt^\alpha) = f_\alpha(z^{1/\alpha} t) + g_\alpha(z^{1/\alpha} t), \tag{40}$$

where

$$f_\alpha(t) = \frac{1}{\pi}\int_0^\infty e^{-rt} \frac{r^{\alpha-1}\sin(\pi\alpha)}{r^{2\alpha} + 2r^\alpha\cos(\pi\alpha) + 1} dr,$$

$$g_\alpha(t) = \frac{2}{\alpha} e^{t\cos(\pi/\alpha)} \cos\left[t\sin(\pi/\alpha)\right]. \tag{41}$$

The function $g_\alpha(t)$ exhibits oscillations with circular frequency $\Omega(\alpha) = \sin(\pi/\alpha)$, and exponentially decaying amplitude with rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$. The functions $f_\alpha(t)$ exhibit an algebraic decay as $t \to \infty$. Therefore the linear oscillator with memory demonstrates power-law decay. Note that we have power-law decay for open and closed Hamiltonian quantum systems with memory. As a result, the power-law memory leads to dissipation.

As a result, generalized Lindblad equation with time fractional derivative describes evolution of quantum observables of open quantum systems with memory. The quantum processes with power-law memory ($\alpha \notin \mathbb{N}$) cannot be described by a semigroup. It can be described only as a quantum dynamical groupoid.

As a result, the long-term memory for open and closed quantum systems can lead to dissipation with power-law decay.

6. Conclusion

In this paper, we consider simple open and closed quantum systems with the power-law memory. These systems can be described by differential equations with non-integer derivative with respect to time. The fractional differential equations describe quantum processes with memory. The fractional calculus is a powerful instrument to describe a wide class of open classical and quantum systems [13,36–38]. An open quantum system is a quantum system which is found to be in interaction with an external quantum system, the environment. The open quantum system can be viewed as a distinguished part of a larger closed quantum system, the other part being the environment. We think that fractional differential equations can have found many applications in the theory quantum processes with power-law memory for closed and open (non-Hamiltonian) systems [39–44].

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