FRACTIONAL POWERS OF DERIVATIVES IN CLASSICAL MECHANICS

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ABSTRACT. Fractional analysis is applied to describe classical dynamical systems. Fractional derivative can be defined as a fractional power of derivative. The infinitesimal generators $\{H, \cdot\}$ and $\mathcal{L} = G(q, p)\partial_q + F(q, p)\partial_p$, which are used in equations of motion, are derivative operators. We consider fractional derivatives on a set of classical observables as fractional powers of derivative operators. As a result, we obtain a fractional generalization of the equation of motion. This fractional equation is exactly solved for the simple classical systems. The suggested fractional equations generalize a notion of classical systems to describe dissipative processes.

Key words. Fractional derivative, Fractional equation, Classical dynamics

1. INTRODUCTION

The analysis of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. There are many books about fractional calculus and fractional differential equations [1, 2, 3]. Derivatives of fractional order, and fractional differential equations have found many applications in recent studies in physics (see for example [4, 5, 6] and references therein).

The classical variables, which are also called observables, are defined as functions on the phase space. The dynamical description of system is given by an operator. The natural description of the motion is in terms of the infinitesimal change of the system. The infinitesimal operator of equation of motion is defined by some form of derivation of functions.

Fractional derivative can be defined as a fractional power of derivative (see for example [7, 8]). It is known that the infinitesimal generator $\{H, \cdot\}$ and $\mathcal{L} = G(q, p)\partial_p + F(q, p)\partial_p$, which are used in the equation of motion, are derivations on an algebra of classical observables. A derivation of an algebra $\mathcal{M}$ is a linear map $\mathcal{L}$, which satisfies $\mathcal{L}(AB) = (\mathcal{L}A)B + A(\mathcal{L}B)$ for all $A, B \in \mathcal{M}$. In this paper, we consider a fractional derivative as a fractional power of derivative. As a result, we obtain a fractional generalization of the equation of motion. It allows us to generalized a notion of classical Hamiltonian systems. Note that some fractional generalization

of gradient systems has been suggested in [9], and a generalization of Hamiltonian systems is considered in [10]. The suggested fractional equation is exactly solved for a free particle, harmonic oscillator and damped oscillator. A classical systems that is presented by fractional equation can be considered as a dissipative system. Fractional derivatives can be used as a possible approach to describe an interaction between the system and an environment. Note that fractional dynamics can be considered with low-level fractionality by some generalization of method suggested in [11, 12, 13].

In section 2, the fractional power of derivative and the fractional equation are suggested. In section 3, the Cauchy problem for the fractional equation and the properties of time evolution described by it are considered. In section 4, the solution for simple examples of the fractional equation are derived.

2. FRACTIONAL DERIVATIVE AND FRACTIONAL EQUATIONS

Let us consider the classical systems

$$\frac{d}{dt}q_k = G_k(q,p), \quad \frac{d}{dt}p_k = F_k(q,p) \quad (k=1,\ldots,n). \tag{1}$$

A linear algebra $\mathcal{M}$ of classical observables is described by functions $A = A(q,p)$ on the phase space $\mathbb{R}^{2n}$. Let $\mathcal{L}$ be a differential operator on $\mathcal{M}$ given by

$$\mathcal{L} = -\left(G_k(q,p)\frac{\partial}{\partial q_k} + F_k(q,p)\frac{\partial}{\partial p_k}\right). \tag{2}$$

Here and later we mean the sum on the repeated index k from 1 to n. The equation of motion for the classical observable has the form

$$\frac{d}{dt}A_t = -\mathcal{L}A_t. \tag{3}$$

Equations (1) are special cases of (3).

If the functions $G_k(q,p)$ and $F_k(q,p)$ satisfy the Helmholtz conditions

$$\frac{\partial G_k}{\partial p_l} - \frac{\partial G_l}{\partial p_k} = 0, \tag{4}$$

$$\frac{\partial G_l}{\partial q_k} + \frac{\partial F_k}{\partial p_l} = 0, \tag{5}$$

$$\frac{\partial F_k}{\partial q_l} - \frac{\partial F_l}{\partial q_k} = 0, \tag{6}$$

then the classical system (3) is a Hamiltonian system. In this case, G_k and F_k can be represented in the form

$$G_k(q,p) = \frac{\partial H}{\partial p_k}, \quad F_k(q,p) = -\frac{\partial H}{\partial q_k}.$$

If $H = H(q, p)$ is a continuous differentiable function, then the condition (4), (5) and (6) are satisfied. The equations of motion (3) for Hamiltonian system can be written in the form

$$\frac{d}{dt} A_t = -\{H, A_t\}, \tag{7}$$

where $\{ , \}$ is the Poisson brackets

$$\{A, B\} = \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k}.$$

The time evolution of the Hamiltonian system is induced by the Hamiltonian H.

It is interesting to obtain fractional generalizations of equations (3) and (7). We will consider here concept of fractional power for $\mathcal{L}$. If $\mathcal{L}$ is a closed linear operator with an everywhere dense domain $D(\mathcal{L})$, having a resolvent $R(z, \mathcal{L}) = (zL_I - \mathcal{L})^{-1}$ on the negative half-axis, then there exists [16, 17, 18] the operator

$$-(\mathcal{L})^\alpha = \frac{\sin \pi \alpha}{\pi} \int_0^\infty dz \, z^{\alpha-1} R(-z, \mathcal{L}) \mathcal{L} \tag{8}$$

defined on $D(\mathcal{L})$ for $0 < \alpha < 1$. The operator $(\mathcal{L})^\alpha$ is a *fractional power of the operator* $\mathcal{L}$. It is known that the linear differential operator (2) is a closable operator [17]. Note that

$$(\mathcal{L})^\alpha (\mathcal{L})^\beta = (\mathcal{L})^{\alpha+\beta}$$

for $\alpha, \beta > 0$, $\alpha + \beta < 1$.

As a result, we obtain the equation

$$\frac{d}{dt} A_t = -(\mathcal{L})^\alpha A_t, \tag{9}$$

where t is dimensionless variable. This is the *fractional equation of motion*. We can define a fractional generalization of equation (7) by

$$\frac{d}{dt} A_t = -(\{H, \cdot\})^\alpha A_t. \tag{10}$$

Note that $(\{H, \cdot\})^\alpha$ cannot be presented in the form $\{H', \cdot\}$ with a function H'. Therefore, the fractional system described by (9) with $\mathcal{L} = \{H, \cdot\}$ are not Hamiltonian systems. The systems will be called the fractional Hamiltonian systems (FHS). Usual Hamiltonian systems can be considered as a special case of FHS. Note that another fractional generalization of Hamiltonian systems has been suggested in [10].

3. SOLUTIONS OF FRACTIONAL EQUATIONS OF MOTION

3.1. **Cauchy problem for fractional equations.** If we consider the Cauchy problem for equation (3) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be written in the form

$$A_t = \Phi_t A_0.$$

The operator Φ_t is called the evolution operator. It is not hard to prove that the following properties are satisfied:

$$\Phi_t \Phi_s = \Phi_{t+s}, \quad (t, s > 0), \quad \Phi_0 = I,$$

where $IA = A$ for all $A \in \mathcal{M}$. As a result, the operators Φ_t form a semi-group. Then the operator $\mathcal{L}$ is called the generating operator, or infinitesimal generator, of the semi-group $\{\Phi_t, t \geq 0\}$.

Let us consider the Cauchy problem for fractional equation (9) in which the initial condition is given by A_0. Then its solution can be presented [17, 18] as

$$A_t(\alpha) = \Phi_t^{(\alpha)} A_0,$$

where the operators $\Phi_t^{(\alpha)}$, $t > 0$, form a semi-group which will be called the *fractional semi-group*. The operator $(\mathcal{L})^\alpha$ is infinitesimal generator of the semi-group $\{\Phi_t^{(\alpha)}, t \geq 0\}$ that can be presented by

$$(\mathcal{L})^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty ds\, s^{-\alpha-1} (\Phi_t^{(\alpha)} - I).$$

This is the Balakrishnan equation [16, 17].

3.2. Properties of fractional evolution operator.
Let us consider some properties of temporal evolution described by a fractional semi-group $\{\Phi_t^{(\alpha)}, t \geq 0\}$.

(1) The operators $\Phi_t^{(\alpha)}$ can be constructed in terms of the operators Φ_t by the Bochner-Phillips formula [14, 15, 17]:

$$\Phi_t^{(\alpha)} = \int_0^\infty ds f_\alpha(t, s) \Phi_s \quad (t > 0). \tag{11}$$

Here $f_\alpha(t, s)$ is defined by

$$f_\alpha(t, s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dz\, \exp(sz - tz^\alpha), \tag{12}$$

where $a, t > 0$, $s \geq 0$, and $0 < \alpha < 1$. The branch of z^α is so taken that $Re(z^\alpha) > 0$ for $Re(z) > 0$. This branch is a one-valued function in the z-plane cut along the negative real axis. The convergence of this integral is obviously in virtue of the convergence factor $\exp(-tz^\alpha)$. By denoting the path of integration in (12) to the union of two paths $r \exp(-i\theta)$, and $r \exp(+i\theta)$, where $r \in (0, \infty)$, and $\pi/2 \leq \theta \leq \pi$, we can obtain

$$f_\alpha(t, s) = \frac{1}{\pi} \int_0^\infty dr\, \exp(sr \cos\theta - tr^\alpha \cos(\alpha\theta)) \cdot$$

$$\cdot \sin(sr \sin\theta - tr^\alpha \sin(\alpha\theta) + \theta). \tag{13}$$

If we have a solution A_t of equation (3), then formula (11) gives the solution

$$A_t(\alpha) = \int_0^\infty ds\, f_\alpha(t, s) A_s, \quad (t > 0) \tag{14}$$

of fractional equation (9). As a result, we can obtain solution of fractional equation by using well-known solutions of usual equations.

(2) In classical mechanics, the most important is the class of real operators. Let $*$ be a complex conjugation. If Φ_t is a real operator on $\mathcal{M}$, then

$$(\Phi_t A)^* = \Phi_t(A^*)$$

for all $A \in D(\Phi_t) \subset \mathcal{M}$. A classical observable is a real-valued function. If Φ_t is a real operator and A is a real-valued function $A^* = A$, then the function $A_t = \Phi_t A$ is real-valued, i.e., $(\Phi_t A)^* = \Phi_t A$. An operator, which is a map from a set of observables into itself, should be real. All possible dynamics, i.e., temporal evolutions of classical observables, should be described by real operators. Therefore the following statement is very important. If Φ_t is a real operator, then $\Phi_t^{(\alpha)}$ is real. The proof will follows from the Bochner-Phillips formula, which gives

$$(\Phi_t^{(\alpha)} A)^* = \int_0^\infty ds\, f_\alpha^*(t, s)(\Phi_s A)^*, \quad (t > 0).$$

Using (13), it is easy to see that $f_\alpha^*(t, s) = f_\alpha(t, s)$ is a real-valued function. Then $(\Phi_t A)^* = \Phi_t A^*$ leads to

$$(\Phi_t A)^* = \Phi_t(A^*)$$

for all $A \in D(\Phi_t^{(\alpha)}) \subset \mathcal{M}$.

(3) Let Φ_t be a operator on $\mathcal{M}$. An adjoint operator of Φ_t is a operator $\bar{\Phi}_t$ on $\mathcal{M}^*$, such that

$$(\bar{\Phi}_t(A)|B) = (A|\Phi_t(B))$$

for all $B \in D(\Phi_t) \subset \mathcal{M}$ and some $A \in \mathcal{M}^*$. The scalar product on $\mathcal{M}$ can be defined by

$$(A|B) = \int_{\mathbb{R}^{2n}} dq dp\, [A(q, p)]^* B(q, p)$$

Then an operator $\bar{\Phi}_t$ is called adjoint if

$$\int_{\mathbb{R}^{2n}} dq dp\, [(\bar{\Phi}_t A)(q, p)]^* B(q, p) = \int_{\mathbb{R}^{2n}} dq dp\, [A(q, p)]^* (\Phi_t B)(q, p).$$

Let us give the basic statement regarding the adjoint operator. If $\bar{\Phi}_t$ is an adjoint operator of Φ_t, then the operator

$$\bar{\Phi}_t^{(\alpha)} = \int_0^\infty ds f_\alpha(t, s)\bar{\Phi}_s, \quad (t > 0),$$

is an adjoint operator of $\Phi_t^{(\alpha)}$. We prove this statement by using the Bochner-Phillips formula:

$$\int_{\mathbb{R}^{2n}} dq dp\, (\bar{\Phi}_t^{(\alpha)} A)^* B = \int_0^\infty ds\, f_\alpha(t, s) \int_{\mathbb{R}^{2n}} dq dp\, (\bar{\Phi}_s A)^* B =$$

$$= \int_0^\infty ds\, f_\alpha(t, s) \int_{\mathbb{R}^{2n}} dq dp\, A^*(\Phi_s B) = \int_{\mathbb{R}^{2n}} dq dp\, A^*(\Phi_t^{(\alpha)} B).$$

The semi-group $\{\bar{\Phi}_t, t > 0\}$ describes a temporal evolution of the distribution function $\rho_t(q, p) = \bar{\Phi}_t \rho_0(q, p)$ by the Liouville equation

$$\frac{d}{dt} \rho_t(q, p) = -\bar{\mathcal{L}} \rho_t(q, p),$$

where

$$\bar{\mathcal{L}} = \mathcal{L} + \Omega(q, p), \quad \Omega(q, p) = \sum_{k=1}^{n} \left(\frac{\partial G_k}{\partial q_k} + \frac{\partial F_k}{\partial p_k} \right).$$

If $\Omega < 0$, then the system is called dissipative. The semi-group $\{\bar{\Phi}_t^{(\alpha)}, t > 0\}$ describes the evolution of the density function

$$\rho_t(\alpha, q, p) = \bar{\Phi}_t^{(\alpha)} \rho_0(q, p)$$

by the fractional equation

$$\frac{d}{dt} \rho_t = -(\bar{\mathcal{L}})^{\alpha} \rho_t.$$

This is the *fractional Liouville equation*.

(4) It is known that $\bar{\Phi}_t$ is a real operator if Φ_t is real. Analogously, if $\Phi_t^{(\alpha)}$ is a real operator, then $\bar{\Phi}_t^{(\alpha)}$ is real.

(5) Let Φ_t, $t > 0$, be a positive one-parameter operator, i.e.,

$$\Phi_t A \geq 0$$

for $A \geq 0$. Using the Bochner-Phillips formula and the property

$$f_\alpha(t, s) \geq 0 \quad (s > 0),$$

it is easy to prove that

$$\Phi_t^{(\alpha)} A \geq 0 \quad (A \geq 0),$$

i.e. the operators $\Phi_t^{(\alpha)}$ are also positive.

4. EXAMPLES OF FRACTIONAL EQUATIONS OF MOTION

4.1. **Fractional free motion of particle.** Let us consider equation (3) for free particle. Then

$$H = \frac{1}{2m} p^2, \quad \mathcal{L} = \{H, \cdot\} = \frac{p}{m} \partial_q,$$

where p is dimensionless variable and m^{-1} has the action dimension. For $A = q$, and $A = p$, equation (3) gives

$$\frac{d}{dt} q_t = \frac{1}{m} p_t, \quad \frac{d}{dt} p_t = 0.$$

The well-known solutions of these equations are

$$q_t = q_0 + \frac{t}{m} p_0, \quad p_t = p_0. \tag{15}$$

Using these solutions and the Bochner-Phillips formula, we cab obtain solutions of the fractional equations

$$\frac{d}{dt} q_t = -\frac{1}{m^\alpha} (p\partial_q)^\alpha q_t, \quad \frac{d}{dt} p_t = 0. \tag{16}$$

in the form

$$q_t(\alpha) = \Phi_t^{(\alpha)} q_0 = \int_0^\infty ds f_\alpha(t, s) q_s, \quad p_t(\alpha) = p_0,$$

where q_s is given by (15). Then

$$q_t(\alpha) = q_0 + \frac{1}{m} b_\alpha(t) p_0, \quad p_t = p_0,$$

where

$$b_\alpha(t) = \int_0^\infty ds f_\alpha(t, s)\, s.$$

If $\alpha = 1/2$, then we have

$$b_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds \frac{1}{\sqrt{s}} e^{-t^2/4s} = \frac{t^2}{2},$$

and

$$q_t(1/2) = q_0 - \frac{t^2}{2m^2} p_0, \quad p_t = p_0. \tag{17}$$

These equations describe a fractional free motion for $\alpha = 1/2$.

4.2. Fractional equation for harmonic oscillator.

Let us consider equation (3) for harmonic oscillator. Then $\mathcal{L} = \{H, \cdot\}$, where

$$H = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2, \tag{18}$$

where t and p are dimensionless variables. For $A = q$, and $A = p$, equation (3) gives

$$\frac{d}{dt} q_t = \frac{1}{m} p_t, \quad \frac{d}{dt} p_t = -m\omega^2 q_t. \tag{19}$$

The well-known solutions of these equations are

$$q_t = q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t),$$

$$p_t = p_0 \cos(\omega t) - m\omega q_0 \sin(\omega t). \tag{20}$$

Using (20) and the Bochner-Phillips formula, we can obtain solution of the fractional equations

$$\frac{d}{dt} q_t = -(\{H, \cdot\})^\alpha q_t, \quad \frac{d}{dt} p_t = -(\{H, \cdot\})^\alpha p_t, \tag{21}$$

where H is defined by (18). It can be written in the form

$$\frac{d}{dt} q_t = -\frac{1}{m^\alpha} \left(m^2\omega^2 q\partial_p - p\partial_q \right)^\alpha q_t,$$

$$\frac{d}{dt} p_t = -\frac{1}{m^\alpha} \left(m^2\omega^2 q\partial_p - p\partial_q \right)^\alpha p_t.$$

It is easy to see that these equations with $\alpha = 1$ give Eqs. (19). The solutions of fractional equations (21) have the forms

$$q_t(\alpha) = \Phi_t^{(\alpha)} q_0 = \int_0^\infty ds\, f_\alpha(t, s) q_s,$$

$$p_t(\alpha) = \Phi_t^{(\alpha)} p_0 = \int_0^\infty ds\, f_\alpha(t, s) p_s. \tag{22}$$

Substitution of (20) into (22) gives

$$q_t(\alpha) = q_0 C_\alpha(t) + \frac{p_0}{m\omega} S_\alpha(t), \tag{23}$$

$$p_t(\alpha) = p_0 C_\alpha(t) - m\omega q_0 S_\alpha(t), \tag{24}$$

where

$$C_\alpha(t) = \int_0^\infty ds\, f_\alpha(t, s)\, \cos(\omega s),$$

$$S_\alpha(t) = \int_0^\infty ds\, f_\alpha(t, s)\, \sin(\omega s).$$

Equations (23) and (24) describe solutions of fractional equations (21) for classical harmonic oscillator.

If $\alpha = 1/2$, then

$$C_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds\, \frac{\cos(\omega s)}{s^{3/2}}\, e^{-t^2/4s},$$

$$S_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds\, \frac{\sin(\omega s)}{s^{3/2}}\, e^{-t^2/4s}.$$

These functions can be presented through the Macdonald function (see [19], Sec. 2.5.37.1.) such that

$$C_{1/2}(t) = \left(\frac{\omega t^2}{4\pi}\right)^{1/4} \left[e^{+\pi i/8} K_{-1/2}\left(2e^{+\pi i/4}\sqrt{\frac{\omega t^2}{4}}\right) + e^{-\pi i/8} K_{-1/2}\left(2e^{-\pi i/4}\sqrt{\frac{\omega t^2}{4}}\right) \right],$$

$$S_{1/2}(t) = i\left(\frac{\omega t^2}{4\pi}\right)^{1/4} \left[e^{+\pi i/8} K_{-1/2}\left(2e^{+\pi i/4}\sqrt{\frac{\omega t^2}{4}}\right) - e^{-\pi i/8} K_{-1/2}\left(2e^{-\pi i/4}\sqrt{\frac{\omega t^2}{4}}\right) \right],$$

where $\omega > 0$, and $K_\alpha(z)$ is the Macdonald function [1], which is also called the modified Bessel function of the third kind.

Note that fractional oscillators are an object of numerous investigations (see for example [20, 21, 22, 23, 24, 25]) because of different applications.

4.3. Fractional equation for damped oscillator.

Let us consider oscillator with linear friction

$$\frac{d}{dt} q_t = \frac{1}{m} p_t, \quad \frac{d}{dt} p_t = -m\omega^2 q_t - 2\beta p_t, \tag{25}$$

where $\beta < \omega$. The solution of (25) has the form

$$q_t = e^{-\beta t} \left[q_0 \cos\left(\sqrt{\omega^2 - \beta^2} t\right) + \frac{1}{m\omega} p_0 \sin\left(\sqrt{\omega^2 - \beta^2} t\right) \right],$$

$$p_t = e^{-\beta t} \left[p_0 \cos\left(\sqrt{\omega^2 - \beta^2} t\right) - m\omega q_0 \sin\left(\sqrt{\omega^2 - \beta^2} t\right) \right]. \tag{26}$$

The fractional equations has the form

$$\frac{d}{dt} q_t = - \left((m\omega^2 q + 2\beta p)\partial_p - \frac{p}{m}\partial_q \right)^\alpha q_t,$$

$$\frac{d}{dt} p_t = - \left((m\omega^2 q + 2\beta p)\partial_p - \frac{p}{m}\partial_q \right)^\alpha p_t.$$

It is easy to see that these equations with $\alpha = 1$ give Eqs. (25). Using (26) and the Bochner-Phillips formula, we obtain the solutions

$$q_t(\alpha) = q_0 C_{\alpha,\beta}(t) + \frac{1}{m\omega} p_0 S_{\alpha,\beta}(t),$$

$$p_t(\alpha) = p_0 C_{\alpha,\beta}(t) - m\omega q_0 S_{\alpha,\beta}(t), \tag{27}$$

where

$$C_{\alpha,\beta}(t) = \int_0^\infty ds\, f_\alpha(t,s)\, e^{-\beta t} \cos(\sqrt{\omega^2 - \beta^2} s),$$

$$S_{\alpha,\beta}(t) = \int_0^\infty ds\, f_\alpha(t,s)\, e^{-\beta t} \sin(\sqrt{\omega^2 - \beta^2} s). \tag{28}$$

These equations describe solutions of the fractional damped motion of classical harmonic oscillator.

REFERENCES

[1] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications* (Gordon and Breach, New York, 1993).

[2] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Application of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).

[4] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).

[5] A. Carpinteri, F. Mainardi, (Eds), *Fractals and Fractional Calculus in Continuum Mechanics* (Springer, Wien, 1997).

[6] G. M. Zaslavsky, "Chaos, fractional kinetics, and anomalous transport," Phys. Rep. **371** (2002) 461–580.

[7] C. Martinez, M. Sanz, *The Theory of Fractional Powers of Operators* (Elsevier, New York, 2000).

[8] V. E. Tarasov, "Fractional derivative as fractional power of derivative," Int. J. Math. **18**(3) (2007) 281–299.

[9] V. E. Tarasov, "Fractional generalization of gradient systems," Lett. Math. Phys. **73**(1) (2005) 49–58.

[10] V. E. Tarasov, "Fractional generalization of gradient and Hamiltonian systems," J. Phys. A **38** (2005) 5929–5943.

[11] V. E. Tarasov, G. M. Zaslavsky "Dynamics with low-level fractionality," Physica A **368**(2) (2006) 399–415.

[12] A. Tofighi, H. N. Pour, "Epsilon-expansion and the fractional oscillator," Physica A **374**) (2007) 41–45.

[13] A. Tofighi, A. Golestani, "A perturbative study of fractional relaxation phenomena," Physica A **387** (2008) 1807–1817.

[14] S. Bochner, "Diffusion equations and stochastic processes," Proc. Nat. Acad. Sci USA **35** (1949) 369–370.

[15] R. S. Phillips, "On the generation of semi-groups of linear operators," Pacific J. Math. **2** (1952) 343–396.

[16] V. Balakrishnan, "Fractional power of closed operator and the semi-group generated by them," Pacific J. Math. **10** (1960) 419–437.

[17] K. Yosida, *Functional analysis* (Springer, Berlin, 1965).

[18] S. G. Krein, *Linear Differential Equations in Banach Space*, Transl. Math. Monogr., Vol.29, Amer. Math. Soc., 1971 (Translated from Russian) Nauka, Moscow, 1967.

[19] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Elementary Functions, Integrals and Series* Vol. 1 (Gordon and Breach, New York, 1986).

[20] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," Chaos, Solitons and Fractals, **7** (1996) 1461–1477.

[21] G. M. Zaslavsky, A. A. Stanislavsky, M. Edelman, "Chaotic and pseudochaotic attractors of perturbed fractional oscillator," Chaos **16** (2006) 013102.

[22] A. A. Stanislavsky, "Fractional oscillator," Phys. Rev. E **70** (2004) 051103.

[23] A. A. Stanislavsky, "Twist of fractional oscillations," Physica A **354** (2005) 101–110.

[24] A. Tofighi, "The intrinsic damping of the fractional oscillator," Physica A **329** (2003) 29–34.

[25] Y. E. Ryabov, A. Puzenko, "Damped oscillations in view of the fractional oscillator equation," Phys. Rev. B **66** (2002) 184201.