Cent. Eur. J. Phys. • 10(2) • 2012 • 382-389
DOI: 10.2478/s11534-012-0008-0

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Central European Journal of **Physics**

The fractional oscillator as an open system

Research Article

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Received 07 December 2011; accepted 17 January 2012

Abstract: A dynamical system governed by equations with derivatives of non-integer order, such as the fractional oscillator, can be considered as an open (non-isolated) system with memory. Fractional equations of motion are obtained from the interaction between the system and the environment with power-law spectral density.

PACS (2008): 45.10.Hj,45.05.+x, 03.65.Yz

Keywords: fractional derivatives • open systems • fractional oscillator

1. Introduction

Fractional calculus [1] and fractional equations [2, 3] have found many applications in recent studies in mechanics and physics (for example, see books [4–14]) The interest in fractional equations has been growing continually during the last few years. More often fractional equations for dynamics or kinetics appear as some phenomenological models. Recently, the method to obtain fractional analogues of equations of motion was considered for problems related to sets of coupled particles that interact by a long-range power-law [15–19].

In this paper, we describe the interaction between simple classical systems and some environments. These systems can be considered as open systems with memory. Understanding dissipative dynamics of open systems remains a challenge in mathematical physics. This problem is relevant in various areas of fundamental and applied physics. As a model of the environment, we consider an infinite set of harmonic oscillators coupled to the system. This is an independent-oscillator model, since the oscillators are not interacting with each other. Power-law spectral densities (PLSD) for the environment lead us to power-laws for memory functions. Note that a power-law memory has been detected for fluctuation within a single protein molecule [20]. We obtain fractional equations from the interaction between the system and the PLSD environment. It allows us to use fractional differential equations for non-Markovian dynamics of a wide class of open and non-Hamiltonian systems. The linear fractional oscillator (LFO) is considered as an open system, i.e. a system that interacts with the PLSD environment. Note that the LFO has been the subject of numerous investigations [35–51, 61] because of its different applications. The LFO can be considered as a simple model of fractional generalization of classical mechanics [52–60].

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2. Interaction between system and environment

As a model of environment, we consider an infinite set of harmonic oscillators coupled to a system. This model is called the independent-oscillator model, since the oscillators do not interact with each other. Let Q and P be the coordinate and momentum of the system respectively, and q_k and p_k describe those of the environment. The usual Poisson brackets are

$$\{Q, P\} = 1, \quad \{q_n, p_k\} = \delta_{nk}, \tag{1}$$

and all other brackets vanish.

The total Hamiltonian H of the system and the environment is composed of the system part

$$H_s = \frac{P^2}{2M} + V(Q), \tag{2}$$

the environment Hamiltonian

$$H_e = \sum_{n=1}^{N} \left(\frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2 q_n^2}{2} \right), \tag{3}$$

and the interaction part

$$H_i = -Q \sum_{n=1}^{N} C_n q_n + Q^2 \sum_{n=1}^{N} \frac{C_n^2}{2m_n \omega_n^2}, \tag{4}$$

where C_n are the coupling constants between the system and the environment. As a result, the total Hamiltonian $H = H_s + H_e + H_i$ has the form

$$H = \frac{P^2}{2M} + V(Q) + \sum_{n=1}^{N} \left[\frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} \left(q_n - \frac{C_n}{m_n \omega_n^2} Q \right)^2 \right]. \tag{5}$$

Note that the function (5) with $V(Q) = (M\Omega^2/2)Q^2$ is called the Caldeira-Legget Hamiltonian. The infinity of choices for m_n and ω_n give this model its great generality.

3. Equations of motion for open systems

Using (5) we can derive Hamiltonian equations of motion for the system and the environment. For the system these are

$$\frac{dQ}{dt} = \{Q, H\} = M^{-1} P,$$

$$\frac{dP}{dt} = \{P, H\} = -V'(Q) + \sum_{n=1}^{N} \left(C_n q_n - \frac{C_n^2}{m_n \omega_n^2} Q \right). \tag{6}$$

The equations for the environment are

$$\frac{dq_n}{dt} = \{q_n, H\} = m_n^{-1} p_n,$$

$$\frac{dp_n}{dt} = \{p_n, H\} = -m_n \omega_n^2 q_n + C_n Q. \tag{7}$$

Eliminating the momenta variables P and p_n, $n = 1, ..., N$, we can write Eqs. (6) and (7) in the form

$$M \frac{d^2 Q}{dt^2} + V'(Q) = \sum_{n=1}^{N} \left(C_n q_n - \frac{C_n^2}{m_n \omega_n^2} Q \right), \tag{8}$$

$$m_n \frac{d^2 q_n}{dt^2} + m_n \omega_n^2 q_n = C_n Q. \tag{9}$$

The solution of Eq. (9) has the form

$$q_n(t) = q_n(0)\cos(\omega_n t) + \frac{p_n(0)}{m_n \omega_n} \sin(\omega_n t) + \frac{C_n}{m_n \omega_n} \int_0^t Q(\tau) \sin \omega_n (t-\tau)\, d\tau, \tag{10}$$

where $q_n(0)$ and $p_n(0)$ are the initial coordinate and momentum of the environment n-th oscillators. We suppose that for $t = 0$ the environment is not perturbed, i.e., $q_n(0) = 0$ and $p_n(0) = 0$. Integration by parts of the last term in Eq. (10) gives

$$\int_0^t Q(\tau) \sin \omega_n (t-\tau) d\tau = \frac{1}{\omega_n} \int_0^t Q(\tau) d \cos \omega_n (t-\tau) =$$

$$= \frac{1}{\omega_n} Q(t) - \frac{1}{\omega_n} Q(0) \cos(\omega_n t) - \int_0^t \frac{dQ(\tau)}{d\tau} \cos \omega_n (t-\tau) d\tau.$$

Setting $Q(0) = 0$ for simplicity, we obtain

$$q_n(t) = \frac{C_n}{m_n \omega_n^2} Q(t) - \frac{C_n}{m_n \omega_n^2} \int_0^t \frac{dQ(\tau)}{d\tau} \cos \omega_n (t-\tau) d\tau. \tag{11}$$

Substituting (11) into (8) gives the equation

$$M \frac{d^2 Q}{dt^2} + \int_0^t \mathcal{M}(t-\tau) \frac{dQ(\tau)}{d\tau} d\tau + V'(Q) = 0, \tag{12}$$

where

$$\mathcal{M}(t) = \sum_{n=1}^{N} \frac{C_n^2}{m_n \omega_n^2} \cos(\omega_n t) \tag{13}$$

is a memory kernel. As a result, we have the non-Markovian equation of motion for the system. The interaction between the system and the independent-oscillator environment allows us to describe the dynamics with a memory.

In general ($q_n(0) \neq 0$, $p_n(0) \neq 0$, $Q(0) \neq 0$) the right-hand side of equation (12) is not zero. It is equal to the function $F(t)$ of the form

$$F(t) = \sum_{n=1}^{N} \left(C_n q_n(0) \cos(\omega_n t) + \frac{C_n p_n(0)}{m_n \omega_n} \sin(\omega_n t) - \frac{C_n^2}{m_n \omega_n^2} Q(0) \cos(\omega_n t) \right). \tag{14}$$

The function (14) can be interpreted as a stochastic force. The stochastic interpretation of the function can be applied since the initial states of the environment are uncertain and it can be determined by a distribution of initial states $q_n(0)$ and $p_n(0)$. For example, if the initial state of the environment is an equilibrium state, then the distribution of environment oscillators, described by $F(t)$, is Gaussian. In this paper, we assume that the initial states are equal to zero for simplicity.

4. Memory and fractional calculus

Let us consider the evolution of a dynamical system in which some quantity $K(t)$ is related to $Q'(t)$ through a memory function $\mathcal{M}(t)$:

$$K(t) = \int_0^t \mathcal{M}(t-\tau) Q'(\tau) d\tau, \tag{15}$$

where $Q'(\tau) = dq(\tau)/d\tau$. Equation (15) means that the value $K(t)$ is related with $Q'(t)$ by the convolution operation

$$K(t) = \mathcal{M}(t) * Q'(t).$$

Equation (15) is a typical non-Markovian equation obtained in the study of systems coupled to an environment, with environmental degrees of freedom being averaged. We consider special cases of Eq. (15). We can consider the memory effects and limiting cases widely used in physics: (a) absence of the memory, (b) complete memory, and (c) power-like memory.

(A) For a system without memory, we have the Markov processes, and the time dependence of the memory function is

$$\mathcal{M}(t-\tau) = \delta(t-\tau), \tag{16}$$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $K(t)$ is defined by $Q'(t)$ only at instant t. For this limiting case, the system loses all its states except for one with infinitely high density. Using (15) and (16), we have

$$F(t) = \int_0^t \delta(t-\tau) Q'(\tau) d\tau = Q'(t). \tag{17}$$

Expression (17) corresponds to the process with complete absence of memory. This process relates all subsequent states to previous states through the single current state at each time t.

(B) If memory effects are introduced into the system the delta-function turns into some function, with the time interval during which $Q'(t)$ influences the function $K(t)$. Let $\mathcal{M}(t)$ be the step function

$$\mathcal{M}(t-\tau) = \begin{cases} t^{-1}, & 0 < \tau < t; \\ 0, & \tau > t. \end{cases} \tag{18}$$

The factor t^{-1} is chosen to get normalization of the memory function to unity:

$$\int_0^t \mathcal{M}(\tau) d\tau = 1.$$

Then in the evolution process the system passes through all states continuously without any loss. In this case,

$$K(t) = \frac{1}{t} \int_0^t Q'(\tau) d\tau,$$

and this corresponds to complete memory.

(C) Let us consider the power-like memory function

$$\mathcal{M}(t-\tau) = A(t-\tau)^{-\beta}. \tag{19}$$

The function indicates the presence of the fractional derivative or integral. Substitution of (19) into (15) gives the temporal fractional derivative of order β:

$$K(t) = \frac{\lambda}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} Q'(\tau) d\tau, \quad (0 < \beta < 1), \tag{20}$$

where $\Gamma(1-\beta)$ is the Gamma function, and $\lambda = \Gamma(1-\beta)A$. The parameter λ can be regarded as the strength of the perturbation induced by the environment of the system. The physical interpretation of the fractional derivative is an existence of a memory effect with power-like memory function. The memory determines an interval t during which the function $Q'(\tau)$ affects the function $K(t)$.

Equation (15) is a special type of equations for $K(t)$ and $Q'(t)$, where K is directly proportional to $\mathcal{M}(t) * Q'(t)$. In our case, we have equation (12) of the form

$$MQ''(t) + \mathcal{M}(t) * Q'(t) + V'(Q) = 0. \tag{21}$$

Equation (21) is a fractional equation for the dynamical system. This equation describes the processes with memory. As a result, we can use the fractional calculus [2] to describe dynamics of classical and quantum systems with memory.

5. Memory function for open systems

To describe dissipative systems by Eq. (12), we must define the conditions for the frequencies ω_n, the number of environment oscillators N, and the coupling constants C_n. The memory function $\mathcal{M}(t)$ describes dissipation if $\mathcal{M}(t)$ is positive definite and decreases monotonically. These conditions are achieved if $N \to \infty$ and if $C_n^2/(m_n\omega_n^2)$ and ω_n are sufficiently smooth functions of the index n. For $N \to \infty$, we replace the sum in Eq. (13) by the integral

$$\mathcal{M}(t) = \int_0^\infty g(\omega)C(\omega)\cos(\omega t)\, d\omega, \tag{22}$$

where $g(\omega)$ is a density of states. We assume that the oscillator environment contains an infinite number of oscillators with a continuous spectrum.
The spectral density $J(\omega)$ is related with the memory function $\mathcal{M}(t)$ by

$$J(\omega) = \omega \int_0^\infty \mathcal{M}(t) cos(\omega t) dt \tag{23}$$

or by the inverse equation

$$\mathcal{M}(t) = \frac{2}{\pi} \int_{-\infty}^\infty \frac{J(\omega)}{\omega} \cos(\omega t) d\omega. \tag{24}$$

Using equations (22) and (23), we have

$$J(\omega) = \frac{\pi\omega}{2} g(\omega)C(\omega). \tag{25}$$

For the spectral density

$$J(\omega) = \frac{\pi\omega}{2} \sum_{n=1}^N \frac{C_n^2}{m_n\omega_n^2} \delta(\omega - \omega_n), \tag{26}$$

equation (24) gives the memory function (13). If we consider the Cauchy distribution

$$J(\omega) = \frac{a}{\omega^2 + \lambda^2}, \tag{27}$$

then Eq. (24) gives the exponential memory kernel

$$\mathcal{M}(t) = \frac{a}{\lambda} e^{-\lambda t}. \tag{28}$$

For a wide class of environments, we can consider a power-law for the spectral density.

6. Fractional equations for open systems

To derive a fractional differential equation for the open system, we consider a power-law for the spectral density (PLSD):

$$J(\omega) = A\omega^\beta, \quad 0 < \beta < 1, \tag{29}$$

where $A > 0$. Equation (29) can be achieved by a variety of combinations of coupling coefficients $C(\omega)$ and density of states $g(\omega)$ in Eq. (25). The density (29) leads to the power-law for the memory function $\mathcal{M}(t) \sim t^{-\beta}$.
Let us obtain an explicit form of equation (12) for the spectral density (29). Using the Fourier cosine-transform (see Sec. 1.3.1. in [62])

$$\int_0^\infty x^{-\alpha} \cos(xy) dx = \frac{\pi}{2\Gamma(\alpha)\cos(\pi\alpha/2)} y^{\alpha-1}, \quad (0 < \alpha < 1), \tag{30}$$

and equations (29), (24), we have

$$\mathcal{M}(t) = \frac{A}{\Gamma(1-\beta)\cos(\pi(1-\beta)/2)} t^{-\beta}. \tag{31}$$

Substitution of (31) into (12) gives

$$M\frac{d^2Q}{dt^2} + \frac{A}{\cos(\pi(1-\beta)/2)}\, {}_0D_t^\beta Q + V'(Q) = 0. \tag{32}$$

This equation is a fractional equation with the Caputo fractional derivative

$${}_0D_t^\beta Q(t) = {}_0I_t^{1-\beta} D_t^1 Q(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{Q'(\tau)d\tau}{(t-\tau)^\beta}, \quad (0 < \beta < 1), \tag{33}$$

where $D_t^1 Q(t) = Q'(t) = dQ(t)/dt$. As a result, we obtain a fractional differential equation from the interaction between the system and the environment with power-law spectral density. Note that fractional equations of motion can be derived from the variational principle that has been suggested by Agrawal in [64, 65].

7. Fractional friction for open systems

We can consider an environment with an interaction with system different to that considered in (4). The simplest Hamiltonian for the interaction between the system and the environment has the bilinear form

$$H'_i = -Q \sum_{n=1}^{N} C_n q_n. \tag{34}$$

This is Hamiltonian (4) without the quadratic term. We consider the total Hamiltonian of the system and the environment

$$H' = H_s + H_e + H'_i,$$

where H_s is defined by (2) and the environment Hamiltonian H_e is (3). In this case, the equations of motion for the system are

$$\frac{dQ}{dt} = \{Q, H\} = M^{-1} P,$$

$$\frac{dP}{dt} = \{P, H\} = -V'(Q) + \sum_{n=1}^{N} C_n q_n. \tag{35}$$

Eliminating the momentum P, we have

$$M \frac{d^2 Q}{dt^2} + V'(Q) = \sum_{n=1}^{N} C_n q_n. \tag{36}$$

The equations for the environment are (7). The corresponding solutions are presented by (10). If we integrate by parts the last term in Eq. (10), then

$$q_n(t) = \frac{C_n}{m_n \omega_n^2} Q(t) - \frac{C_n}{m_n \omega_n^2} \int_0^t \frac{dQ(\tau)}{d\tau} \cos \omega_n (t-\tau) \quad d\tau + D_n(t), \tag{37}$$

where

$$D_n(t) = q_n(0) \cos(\omega_n t) + \frac{p_n(0)}{m_n \omega_n} \sin(\omega_n t) - \frac{C_n}{m_n \omega_n^2} Q(0) \cos(\omega_n t). \tag{38}$$

If we assume that for $t = 0$ the environment is not perturbed, i.e., $q_n(0) = 0$, $p_n(0) = 0$, and $Q(0) = 0$ for simplicity, then $D_n(t) = 0$. Substituting (37) into (36) gives

$$M \frac{d^2 Q}{dt^2} + \int_0^t \mathcal{M}(t-\tau) \frac{dQ(\tau)}{d\tau} d\tau + U'_(Q) = F(t), \tag{39}$$

where

$$U(Q) = V(Q) - \frac{M \Omega_0^2}{2} Q^2, \tag{40}$$

$$\Omega_0^2 = \frac{1}{M} \sum_{n=1}^{N} \frac{C_n^2}{m_n \omega_n^2}, \tag{41}$$

$$F(t) = \sum_{n=1}^{N} C_n D_n(t). \tag{42}$$

The memory kernel $\mathcal{M}(t-\tau)$ in Eq. (39) is defined by Eq. (13). Equation (39) is the integro-differential equation of the non-Markovian dynamics of the system. We note that this equation depends on the environment variables only through their initial values $q_n(0)$ and $p_n(0)$ that occur in the function $F(t)$. If we assume that for $t = 0$ the environment is not perturbed, i.e., $q_n(0) = 0$, and $p_n(0) = 0$, then equation (39) has no dependence on the environment variables. The memory kernel $\mathcal{M}(t)$ and the potential $U(Q)$ depend on the strength of the environment coupling parameters C_n and the spectrum of frequencies ω_n of the environment oscillators. If $F(t) \neq 0$, then Eq. (39) can be considered as a generalized Langevin equation which is suggested by Mori [21] and Kubo [22].
For $N \to \infty$, we replace the sum in Eq. (13) by the integral (22). Using the spectral density (29) equations (39) and (31) give

$$M \frac{d^2 Q}{dt^2} + \frac{A}{\cos(\pi(1-\beta)/2)} \, {}_0D_t^{\beta} Q + U'(Q) = F(t). \tag{43}$$

This equation defines systems with a fractional friction term of order β. It is used to describe damping processes with a fractional damping term [23, 24]. If $F(t) \neq 0$, then Eq. (43) can be considered as a fractional Langevin equation [25–33] which is a generalized Langevin equation (39) with a power-law memory kernel. Equation (43) is fractional differential equation of the open dynamical system. As a result, fractional equations can be obtained from the interaction between the system and the environment with power-law spectral density. Fractional calculus is a powerful instrument for the description of open classical and quantum systems.

8. The linear fractional oscillator as an open system

The equations of motion for the linear fractional oscillator can be derived from equations for open system (32) and (43). If $V(Q) = 0$, then Eq. (32) gives

$$D_t^2 Q + \Omega^2 \, {}_0D_t^{\beta} Q = 0, \quad 0 < \beta < 1, \tag{44}$$

where

$$\Omega^2 = \frac{A}{M\cos(\pi(1-\beta)/2)}.$$

In general, the operations D^1 and $I^{1-\beta}$ in (33) do not commute:

$$D^1_t\, {}_0I^{1-\beta}_t Q(t) = {}_0I^{1-\beta}_t D^1_t Q(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} Q(0). \qquad (45)$$

For $Q(0)=0$, we get

$${}_0D^{\beta}_t Q(t) = D^1_t\, {}_0I^{1-\beta}_t Q(t) = {}_0I^{1-\beta}_t D^1_t Q(t). \qquad (46)$$

Performing the fractional integration of order β in Eq. (44), and using $I^{\alpha_1}_t I^{\alpha_2}_t = I^{\alpha_1+\alpha_2}_t$, we obtain

$${}_0D^{2-\beta}_t Q + \Omega^2\, Q = 0, \quad (0<\beta<1). \qquad (47)$$

This equation defines the linear fractional oscillator (LFO). As a result, the fractional oscillator can be considered as a free system that interacts with the PLSD environment. Note that the LFO has been the subject of numerous investigations [35–39, 42, 43, 45–51, 61] because of its different applications. As a result, the fractional oscillator can be considered as an open system.
Equation (43) with $V(Q)=0$ gives

$$D^2_t Q + \Omega^2_c\, {}_0D^{\beta}_t Q = \frac{1}{M} F(t), \quad (0<\beta<1), \qquad (48)$$

where

$$\Omega^2_c = \Omega^2 - \Omega^2_0.$$

and Ω_0 is defined by (41). Performing the fractional integration of order β in Eq. (48), we obtain

$${}_0D^{2-\beta}_t Q + \Omega^2_c\, Q = f(t), \quad (0<\beta<1), \qquad (49)$$

where

$$f(t) = \frac{1}{M}\, {}_0I^{1-\beta}_t F(t) + \Omega^2_c Q(0).$$

We note that the function $f(t)$ depends on the strength of the environment coupling constants C_n, the spectrum of frequencies ω_n of the environment oscillators, and initial values $q_n(0)$, $p_n(0)$, $Q(0)$. Equation (49) describes a fractional oscillator driven by the force $f(t)$.
The linear fractional oscillator (47) can be defined by the equation

$${}_0D^{\alpha}_t Q(t) + \Omega^2 Q(t) = 0, \quad 1<\alpha<2, \qquad (50)$$

where $\alpha = 2-\beta$, and $0<\beta<1$. The fractional Caputo derivative ${}_0D^{\alpha}_t$ allows us to use the usual initial conditions [3].
The exact solution of Eq. (50) is

$$Q(t) = Q(0) E_{\alpha,1}(-\Omega^2 t^{\alpha}) + t Q'(0) E_{\alpha,2}(-\Omega^2 t^{\alpha}), \qquad (51)$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (52)$$

is the generalized two-parameter Mittag-Leffler function. The decomposition of (51) is [35]:

$$\begin{aligned} Q(t) = & Q(0)\left[f_{\alpha,0}(\Omega^{2/\alpha} t) + g_{\alpha,0}(\Omega^{2/\alpha} t)\right] + \\ & \Omega^{2/\alpha} t Q'(0)\left[f_{\alpha,1}(\Omega^{2/\alpha} t) + g_{\alpha,1}(\Omega^{2/\alpha} t)\right]. \end{aligned} \qquad (53)$$

Here

$$f_{\alpha,k}(t) = \frac{(-1)^k}{\pi}\int_0^{\infty} e^{-rt} \frac{r^{\alpha-1-k}\sin(\pi\alpha)}{r^{2\alpha} + 2r^{\alpha}\cos(\pi\alpha)+1} dr,$$

$$g_{\alpha,k}(t) = \frac{2}{\alpha} e^{t\cos(\pi/\alpha)} \cos\left[t\sin(\pi/\alpha) - \pi k/\alpha\right], \qquad (54)$$

where $k=0,1$. We note that this function exhibits oscillations with angular frequency $\omega(\alpha) = \sin(\pi/\alpha)$ and an exponentially decaying amplitude with rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$. The functions $f_{\alpha,k}(t)$ exhibit an algebraic decay as $t\to\infty$.
To describe this algebraic decay, we consider the integral representation for the generalized Mittag-Leffler function of the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{Ha} \frac{\xi^{\alpha-\beta} e^{\xi}}{\xi^{\alpha} - z} d\xi, \qquad (55)$$

where Ha denotes the Hankel contour, a loop, which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|\xi| \le |z|^{1/\alpha}$ in the positive direction, and ends at $-\infty$ along the upper side of the negative real axis. By the replacement $\xi^{\alpha} \to \xi$ Eq. (55) transforms into [3, 66]:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i\alpha}\int_{\gamma(a,\delta)} \frac{e^{\xi^{(1-\beta)/\alpha}}}{\xi - z} d\xi, \quad (1<\alpha<2), \qquad (56)$$

where $\pi\alpha/2 < \delta < \min\{\pi, \pi\alpha\}$. The contour $\gamma(a,\delta)$ is defined by two rays $S_{\pm\delta} = \{\arg(\xi) = \pm\delta, |\xi| \ge a\}$, and a circular arc $C_{\delta} = \{|\xi| = 1, -\delta \le \arg(\xi) \le \delta\}$. Let us denote the region on the left from $\gamma(a,\delta)$ as $G^-(a,\delta)$. Then [66]:

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(\beta-\alpha n)}, \quad z \in G^-(a,\delta), \quad (|z| \to \infty), \tag{57}$$

and $\delta \leq |\arg(z)| \leq \pi$. In our case, $z = -\Omega^2 t^\alpha$, $\arg(z) = \pi$, and

$$E_{\alpha,1}(-\Omega^2 t^\alpha) \approx \frac{t^{-\alpha}}{\Omega^2 \Gamma(1-\alpha)} \approx -\beta \frac{1}{\Omega^2} t^{-2+\beta}. \tag{58}$$

In a similar way, we get

$$E_{\alpha,2}(-\Omega^2 t^\alpha) \approx \frac{t^{-\alpha}}{\Omega^2 \Gamma(2-\alpha)} \approx \beta \frac{1}{\Omega^2} t^{-2+\beta}. \tag{59}$$

We can use the asymptotic behavior of the generalized Mittag-Leffler function (52) for the exact solution of (51). Then, substitution of (58) and (59) into (51) gives

$$Q(t) \approx -\beta \frac{Q(0)}{\Omega^2} t^{-2+\beta} + \beta \frac{Q'(0)}{\Omega^2} t^{-1+\beta}, \quad (\beta t \gg 1). \tag{60}$$

As the result, we arrive to the asymptotic (60), which exhibits an algebraic decay for $t \to \infty$. This algebraic decay is the most important effect of the non-integer derivative in the considered fractional equations, contrary to the exponential decay of the usual damped-oscillation and relaxation phenomena.

9. Conclusion

In this paper, we consider simple open systems in an environment with power-law spectral density (PLSD). These systems can be described by fractional differential equations. The environment is defined as an infinite set of independent harmonic oscillators coupled to a system. These oscillators do not interact with each other. The power-law spectral density of the environment leads us to a power-law for the memory function. Equations of motion with this memory are differential equations with fractional time derivatives. The fractional differential equations describe non-Markovian motion of open systems. The fractional derivatives in the equations describe the interaction with the PLSD environment. As a result, fractional differential equations allow us to consider open and non-Hamiltonian systems with memory.

Fractional calculus can be a powerful instrument to describe a wide class of open classical and quantum systems. An open quantum system is a quantum system which is found to be in interaction with an external quantum system, the environment. The open quantum system can be viewed as a distinguished part of a larger closed quantum system, the other part being the environment. We hope that fractional differential equations can find many applications in the non-Markovian dynamics of quantum open and non-Hamiltonian systems [67–72].

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