Commun Nonlinear Sci Numer Simulat 18 (2013) 2945–2948

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

No violation of the Leibniz rule. No fractional derivative

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

ARTICLE INFO

Article history:
Received 20 March 2013
Accepted 1 April 2013
Available online 13 April 2013

Keywords:
Fractional derivative
Leibniz rule

ABSTRACT

We demonstrate that a violation of the Leibniz rule is a characteristic property of derivatives of non-integer orders. We prove that all fractional derivatives $\mathcal{D}^{\alpha}$, which satisfy the Leibniz rule $\mathcal{D}^{\alpha}(fg) = (\mathcal{D}^{\alpha}f)g + f(\mathcal{D}^{\alpha}g)$, should have the integer order $\alpha = 1$, i.e. fractional derivatives of non-integer orders cannot satisfy the Leibniz rule.

1. Introduction

Fractional derivatives of non-integer orders [1,2] have wide applications in physics and mechanics [4–13]. The tools of fractional derivatives and integrals allow us to investigate the behavior of objects and systems that are characterized by power-law non-locality, power-law long-term memory or fractal properties.

There are different definitions of fractional derivatives such as Riemann–Liouville, Riesz, Caputo, Grünwald-Letnikov, Marchaud, Weyl, Sonin-Letnikov and others [1,2]. Unfortunately all these fractional derivatives have a lot of unusual properties. The well-known Leibniz rule $\mathcal{D}^{\alpha}(fg) = (\mathcal{D}^{\alpha}f)g + f(\mathcal{D}^{\alpha}g)$ is not satisfied for differentiation of non-integer orders [1]. For example, we have the infinite series

$$\mathcal{D}^{\alpha}(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} (\mathcal{D}^{\alpha-k}f)(D^{k}g) \tag{1}$$

for analytic functions on $[a,b]$ (see Theorem 15.1 in [1]), where $\mathcal{D}^{\alpha}$ is the Riemann–Liouville derivative, D^{k} is derivative of integer order k. Note that the sum is infinite and contains integrals of fractional order for $k > [\alpha] + 1$. Formula (1) first appeared in the paper by Liouville [3] in 1832.

The unusual properties lead to some difficulties in application of fractional derivatives in physics and mechanics. There are some attempts to define new type of fractional derivative such that the Leibniz rule holds (for example, see [15–17]).

In this paper we proof that a violation of the Leibniz rule is one of the main characteristic properties of fractional derivatives. We state that linear operator $\mathcal{D}^{\alpha}$ that can be defined on $C^{2}(U)$, where $U \subset \mathbb{R}^{1}$, such that it satisfied the Leibniz rule cannot have a non-integer order α. In other words, a fractional derivative that satisfies the Leibniz rule is not fractional. It should have integer order.

2. Hadamard's theorem

We denote by $C^{m}(U)$ a space of functions $f(x)$, which are m times continuously differentiable on $U \subset \mathbb{R}^{1}$. Let $D_{x}^{1} = d/dx : C^{m}(U) \to C^{m-1}(U)$ be a usual derivative of first order with respect to coordinate x.

* Tel.: +7 495 939 5989; fax: +7 495 939 0397.
E-mail address: tarasov@theory.sinp.msu.ru

1007-5704/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cnsns.2013.04.001

It is well-known the following Hadamard's theorem [14].

Hadamard's Theorem. *Any function $f(x) \in C^1(U)$ in a neighborhood U of a point x_0 can be represented in the form*

$$f(x) = f(x_0) + (x - x_0)g(x), \tag{2}$$

where $g(x) \in C^m(U)$.

Proof. Let us consider the function

$$F(t) = f(x_0 + (x - x_0)t). \tag{3}$$

Then $F(0) = f(x_0)$ and $F(1) = f(x)$. The Newton–Leibniz formula gives

$$F(1) - F(0) = \int_0^1 dt\,(D_t^1 F)(t) = \int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t)\,(x - x_0) = (x - x_0)\int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t). \tag{4}$$

We define the function

$$g(x) = \int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t). \tag{5}$$

As the result, we have proved representation (2). □

3. Algebraic approach to fractional derivatives

We consider fractional derivatives $\mathcal{D}^\alpha$ of non-integer orders α by using an algebraic approach. Special forms of fractional derivatives are not important for our consideration. We take into account the property of linearity and the Leibniz rule only.

For the operator $\mathcal{D}^\alpha$ we will consider the following conditions.

(1) $\mathbb{R}$-linearity:

$$\mathcal{D}_x^\alpha(c_1 f(x) + c_2 g(x)) = c_1(\mathcal{D}_x^\alpha f(x)) + c_2\,(\mathcal{D}_x^\alpha g(x)), \tag{6}$$

where c_1 and c_2 are real numbers. Note that all known fractional derivatives are linear [1,2].

(2) The Leibniz rule:

$$\mathcal{D}_x^\alpha(f(x)g(x)) = (\mathcal{D}_x^\alpha f(x))\,g(x) + f(x)\,(\mathcal{D}_x^\alpha g(x)). \tag{7}$$

(3) If the linear operator satisfies the Leibniz rule, then the action on the unit (and on a constant function) is equal to zero:

$$\mathcal{D}_x^\alpha 1 = 0. \tag{8}$$

Let us proof the following theorem.

Theorem ("No violation of the Leibniz rule. No fractional derivative"). *If an operator $\mathcal{D}_x^\alpha$ can be applied to functions from $C^2(U)$, where $U \subset \mathbb{R}^1$ be a neighborhood of the point x_0, and conditions (6) and (7) are satisfied, then the operator $\mathcal{D}_x^\alpha$ is the derivative D_x^1 of integer (first) order, i.e. it can be represented in the form*

$$\mathcal{D}_x^\alpha = a(x)\,D_x^1, \tag{9}$$

where $a(x)$ are functions on $\mathbb{R}^1$.

Proof

(1) Using Hadamard's theorem for the function $g(x)$ in the decomposition (2), the function $f(x)$ for $x \in U$ can be represented in the form

$$f(x) = f(x_0) + (x - x_0)g(x_0) + (x - x_0)^2 g_2(x), \tag{10}$$

where $g_2(x) \in C^2(U)$, and $U \subset \mathbb{R}^1$ is a neighborhood of the point x_0.

Applying to equality (10) the operator D_x^1 and use $D_x^1 f(x_0) = 0$, we get

$$(D_x^1 f)(x) = g(x_0) + 2(x - x_0)\,g_2(x) + (x - x_0)^2\,(D_x^1 g_2)(x). \tag{11}$$

Then

$$(D_x^1 f)(x_0) = g(x_0).$$

As a result, we have

$$f(x) = f(x_0) + (x - x_0)(D_x^1 f)(x_0) + (x - x_0)^2 g_2(x). \tag{12}$$

(2) Applying to equality (12) the operator $\mathcal{D}_x^\alpha$, we get

$$(\mathcal{D}_x^\alpha f)(x) = \mathcal{D}_x^\alpha f(x_0) + \mathcal{D}_x^\alpha \Big((x - x_0)(D_x^1 f)(x_0) \Big) + \mathcal{D}_x^\alpha \Big((x - x_0)^2 g_2(x) \Big). \tag{13}$$

The Leibniz rule gives

$$(\mathcal{D}_x^\alpha f)(x) = \mathcal{D}_x^\alpha f(x_0) + a(x)\,(D_x^1 f)(x_0) + (x - x_0)\,(\mathcal{D}_x^\alpha (D_x^1 f)(x_0) + 2a(x)\,(x - x_0)\,g_2(x) + (x - x_0)^2\,(\mathcal{D}_x^\alpha g_2)(x), \tag{14}$$

where we use the notation

$$a(x) = (\mathcal{D}_x^\alpha (x - x_0))(x). \tag{15}$$

Then

$$(\mathcal{D}_x^\alpha f)(x_0) = \mathcal{D}_x^\alpha f(x_0) + a(x_0)\,(D_x^1 f)(x_0). \tag{16}$$

As a result, we have

$$\mathcal{D}_x^\alpha = a(x) D_x^1 + b(x), \tag{17}$$

where we define the function

$$b(x) = \mathcal{D}_x^\alpha 1 \tag{18}$$

and we use the $\mathbb{R}$-linearity in the form

$$\mathcal{D}_x^\alpha f(x_0) = f(x_0)\,(\mathcal{D}_x^\alpha 1).$$

(3) Using that $\mathcal{D}_x^\alpha 1 = 0$ for linear operator, which satisfies the Leibniz rule, we get $b(x) = 0$, $\mathcal{D}_x^\alpha x_0 = x_0 \mathcal{D}_x^\alpha 1 = 0$ and

$$\mathcal{D}_x^\alpha = a(x) D_x^1, \tag{19}$$

where $a(x) = \mathcal{D}_x^\alpha x$.
As the result, we prove (9). □

Remark. In general, the property (8) is not satisfied for all type of fractional derivatives. For example, we have

$$\mathcal{D}_x^\alpha 1 = \frac{1}{\Gamma(1 - \alpha)} x^{-\alpha}$$

for Riemann–Liouville fractional derivative [2]. Note that $\mathcal{D}_x^\alpha x$ is not equal to one in general. For example,

$$\mathcal{D}_x^\alpha x = \frac{1}{\Gamma(2 - \alpha)} x^{1-\alpha}$$

for Riemann–Liouville fractional derivative [2].

Note that this theorem can be proved for multivariable case.

The theorem state that fractional derivative that satisfies the Leibniz rule coincides with differentiation of the order equal to one, i.e. fractional derivatives of non-integer orders cannot satisfy the Leibniz rule. Unfortunately the Leibniz rule is suggested for some new fractional derivatives (the modified Riemann–Liouville derivative that is suggested by Jumarie [15,16], and local fractional derivative in the form that is suggested by Yang [17] and some other derivatives).

Linear operators $\mathcal{D}_x^\alpha$ that satisfy the Leibniz rule cannot be considered as fractional derivatives of non-integer orders. Fractional derivatives should be subject to a rule that is a generalization of the classical Leibniz rule

$$D_x^n (fg) = \sum_{k=0}^{n} (D^{n-k} f)\,(D^k g)$$

to the case of differentiation and integration of fractional order (see Section 15. "The generalized Leibniz rule" of [1] and references in it). It can be assumed with a high degree of reliability that the generalization of the Leibniz rule for all types of fractional derivatives should be represented by an infinite series in general. The history of the generalizations of the Leibniz rule for fractional derivatives, which is began from the paper [3] in 1868, is described in Section 17 (Bibliographical Remarks and Additional Information to Chapter 3.) in [2].

References

[1] Samko SG, Kilbas AA, Marichev OI. Integrals and derivatives of fractional order and applications. Minsk: Nauka i Tehnika; 1987;
Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives theory and applications. New York: Gordon and Breach; 1993.
[2] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier; 2006.
[3] Liouville J. Memoire sur le calcul des differentielles a indices quelconques. J l'Ecole R Polytech 1832 [Extraits (complet) du Tome 13. Section 21, p. 71–162. (see p. 118)].
[4] Carpinteri A, Mainardi F, editors. Fractals and fractional calculus in continuum mechanics. New York: Springer; 1997.
[5] Hilfer R, editor. Applications of fractional calculus in physics. Singapore: World Scientific; 2000.
[6] Sabatier J, Agrawal OP, Tenreiro Machado JA, editors. Advances in fractional calculus. Theoretical developments and applications in physics and engineering. Dordrecht: Springer; 2007.
[7] Luo ACJ, Afraimovich VS, editors. Long-range interaction, stochasticity and fractional dynamics. Berlin: Springer; 2010.
[8] Klafter J, Lim SC, Metzler R, editors. Fractional dynamics. Recent advances. Singapore: World Scientific; 2011.
[9] Mainardi F. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. Singapore: World Scientific; 2010.
[10] Uchaikin VV. Fractional derivatives for physicists and engineers. Vol. 1. Background and theory. Vol 2. Application. Springer; 2013.
[11] Tarasov VE. Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media. New York: Springer; 2011.
[12] Uchaikin V, Sibatov R. Fractional kinetics in solids: anomalous charge transport in semiconductors, dielectrics and nanosystems. World Science; 2013. p. 276.
[13] Tarasov VE. Review of some promising fractional physical models. Int. J. Mod. Phys. B 2013;27:1330005.
[14] Nestruev Jet. Smooth manifolds and observables. Springer; 2003 [chapter 1].
[15] Jumarie G. Table of some basic fractional calculus formulae derived from a modified Riemann–Liouvillie derivative for nondifferentiable functions. Appl Math Lett 2009;22:378–85 [see p. 382, Eq. (4.3)].
[16] Jumarie G. Modified Riemann–Liouville derivative and fractional Taylor series of non-differentiable functions further results. Math Comput Appl 2006;51:1367–76 [see p. 1371, Eq. (3.11)].
[17] Yang XJ. Advanced local fractional calculus and its applications. New York: World Scientific; 2012 [see p. 39, Eq. (2.23)].