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Vector calculus in non-integer dimensional space and its applications to fractal media

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ABSTRACT

We suggest a generalization of vector calculus for the case of non-integer dimensional space. The first and second orders operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space are defined. For simplification we consider scalar and vector fields that are independent of angles. We formulate a generalization of vector calculus for rotationally covariant scalar and vector functions. This generalization allows us to describe fractal media and materials in the framework of continuum models with non-integer dimensional space. As examples of application of the suggested calculus, we consider elasticity of fractal materials (fractal hollow ball and fractal cylindrical pipe with pressure inside and outside), steady distribution of heat in fractal media, electric field of fractal charged cylinder. We solve the correspondent equations for non-integer dimensional space models.

1. Introduction

In general we can assume that space and space–time dimensions are D, which need not be an integer. Non-integer dimensional spaces and method of dimensional continuation are initially emerged in statistical mechanics and quantum field theory. Non-integer dimension $D = 4 - \varepsilon$ of space–time and ε-expansion are actively used in the theory of critical phenomena and phase transitions in statistical physics (for example, see [1,2]). Integration over non-integer dimensional spaces is used in the dimensional regularization method as a powerful tool to obtain exact results without ultraviolet divergences in quantum field theory [3–5]. In quantum theory, the divergences are parameterized as quantities with coefficients $\varepsilon^{-1} = (4 - D)^{-1}$, and then these divergences can be removed by renormalization to obtain finite physical values.

The axioms for integrals in non-integer dimensional space are suggested by Wilson in [6]. These properties are natural and necessary in applications in different areas [5]. Theory of integration in non-integer dimensional spaces has been suggested in [7,5,8]. Stillinger introduces [7] a mathematical basis of integration on spaces with non-integer dimensions. In [7] has been suggested a generalization of the Laplace operator for non-integer dimensional spaces also. In the book by Collins [5] the integration in non-integer dimensional spaces is formulated for rotationally covariant functions. The product measure method, which is suggested in [9], and the Stillinger's approach [7] are extended by Palmer and Stavrinou [8] to multiple variables and different degrees of confinement in orthogonal directions. In the paper [8] extensions of integration and scalar Laplace operator for non-integer dimensional spaces are suggested.

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The scalar Laplace operators, which are suggested in [7,8] for non-integer dimensional spaces, have a wide application in physics and mechanics. Non-integer dimensional space has successfully been used as an effective physical description. The Stillinger's form of Laplacian first applied by He [10–13], where the Schrödinger equation in non-integer dimensional space is used and the real confining structure is replaced by an effective space, such that the measure of the anisotropy or confinement is given by the non-integer dimension. Non-integer dimensions is used by Thilagam to describe stark shifts of excitonic complexes in quantum wells [14], exciton–phonon interaction in fractional dimensional space [15], and blocking effects in quantum wells [16]. The non-integer dimensional space approach is used by Matos-Abiague [17–23] to describe momentum operators for quantum systems and Bose-like oscillator in non-integer dimensional space, the polaron effect in quantum wells. Quantum mechanical models with non-integer (fractional) dimensional space has been suggested by Palmer and Stavrinou [8], Lohe and Thilagam [24]. The non-integer dimensional space approach is used to describe algebraic properties of Weyl-ordered polynomials for the momentum and position operators [25,26] and the correspondent coherent states [27]. The Stillinger's form of Laplacian has been applied to the Schrödinger equation in non-integer dimensional space by Eid, Muslih, Baleanu, Rabei in [28,29], Muslih and Agrawal [30,31], by Calcagni, Nardelli, Scalisi in [32]. The fractional Schrödinger equation with non-integer dimensions is considered by Martins, Ribeiro, Evangelista, Silva, Lenzi in [33] and by Sandev, Petreska, Lenzi [34]. Recent progress in non-integer dimensional space approach includes the description of the scalar field on non-integer dimensional spaces by Trinchero [35], the fractional diffusion equation in non-integer dimensional space and its solutions are suggested in [36]. The gravity in fractional dimensional space is described by Sadallah, Muslih, Baleanu in [37,38], and by Calcagni in [39–41]. The electromagnetic fields in non-integer dimensional space are considered in [42–49].

Unfortunately, the basic articles [7,8] proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian are not considered in [7,8]. In the book [49] (see also [45–48]), the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Consideration only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional space for fractal media and material. For example, we cannot use the Stillinger's form of Laplacian for displacement vector field $\mathbf{u}(\mathbf{r},t)$ in elasticity and thermoelasticity theories. We cannot consider equations for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic fields $\mathbf{B}(\mathbf{r},t)$ for electromagnetic theory of fractal media by using continuum models with non-integer dimensional space.

In this paper, we propose a vector calculus for non-integer dimensional space and we define the first and second orders differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space. For simplification we consider rotationally covariant scalar and vector functions that are independent of angles. In order to derive the vector differential operators in non-integer dimensional space we use the method of analytic continuation in dimension. For this aim we get equations for these differential operators for rotationally covariant functions in $\mathbb{R}^n$ for arbitrary integer n to highlight the explicit relations with dimension n. Then the vector differential operators for non-integer dimension D are defined by analytic continuation from integer dimensions n to non-integer D. These proposed equations allows us to reduce D-dimensional vector differentiations to usual derivatives with respect to one variable $r = |\mathbf{r}|$. It allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to r. The proposed operators allows us to describe fractal materials and media in the framework of continuum models with non-integer dimensional spaces. In order to give examples of the possible applications, we consider continuum models of fractal media and materials in the elasticity theory in the heat theory, and in the theory of electric fields. The correspondent equations for non-integer dimensional space are solved.

2. Fractal media

The cornerstone of fractal media is the non-integer dimension [50] such as mass or charge dimensions [51,52]. In general, fractal media and materials can be treated with three different approaches: (1) Using the methods of "Analysis on fractals" [53–58] it is possible to describe fractal materials; (2) To describe fractal media we can apply fractional–integral continuous models suggested in [59–62,52] (see also [63–76]). In this case we use integrations of non-integer orders and two different notions such as density of states and distribution function [52]; (3) Fractal materials can be described by using the theory of integration and differentiation for a non-integer dimensional space [5,7,8].

The first approach, which is based on the use of analysis on fractal sets, is the most stringent possible method to describe idealized fractal media. Unfortunately, it has two lacks. Firstly, a possibility of application of the analysis on fractals to solve differential equations for real problems of fractal material is very limited due to weak development of this area of mathematics at this moment. Secondly, fractal materials and media cannot be described as fractals. The main property of the fractal is non-integer Hausdorff dimension [77] that should be observed on all scales. The fractal structure of real media cannot be observed on all scales from the infinitely small to the infinitely large sizes. Materials may have a fractal structure only for scales from the characteristic size of atoms or molecules of fractal media up to size of investigated sample of material.

The second approach, which is based on the use of fractional integration in integer dimensional spaces, can give adequate models to describe fractal media. The main disadvantage of the fractional–integral continuum models is the existence of various types of fractional integrals, which led to the arbitrariness in the choice of the correspondent densities of states.

In this paper, we consider the third approach used the non-integer dimensional spaces. One of the advantages of this approach is a possibility to avoid the arbitrary choice of densities of states. In addition, we also suggest a generalization of the vector calculus to the case of non-integer dimensional space. It allows us to use different continuum models of fractal media and materials in the framework of non-integer dimensional space approach.

Real fractal materials can be characterized by the relation between the mass $M_D(W)$ of a ball region W of fractal medium, and the radius R of this ball in the form $M_D(W) = M_0(R/R_0)^D,\ R/R_0 \gg 1$, where R_0 is the characteristic size of fractal medium such as a minimal scale of self-similarity of a considered fractal medium. The number D is called the mass dimension. The parameter D, does not depend on the shape of the region W, or on whether the packing of sphere of radius R_0 is close packing, a random packing or a porous packing with a uniform distribution of holes. As a result, fractal materials can be considered as media with non-integer mass dimensions. Although, the non-integer dimension does not reflect completely the geometric and dynamic properties of the fractal media, it nevertheless allows us to get important conclusions about the behavior of these media. As it will be shown in the next section, the power law $M_D(W) \sim R^D$ can be naturally derived by using the integration over non-integer dimensional space, where the space dimension is equal to the mass dimension of fractal media.

In order to describe fractal media by continuum models with non-integer dimensional space, we should use the concepts of density of states $c_n(D, \mathbf{r})$ and distribution function $\rho(\mathbf{r})$. The density of states describes how closely packed permitted places (states) in the space $\mathbb{R}^n$, where the fractal medium is distributed. The expression $c_n(D, \mathbf{r})dV_n$ is equal to the number of permitted places (states) between V_n and $V_n + dV_n$ in $\mathbb{R}^n$. The distribution function describes a distribution of physical values such as mass, electric charge, number of particles on a set of permitted places (possible states). In general, the concepts of density of states and distribution function are different. We cannot reduce all properties of the fractal media to the distribution function only, and we should use the concepts of density of states to characterize how closely packet permitted states places in the media.

The most important property of fractal medium is the fractality, which means that the mass $M_D(W)$ of this medium in any region $W \subset \mathbb{R}^n$ increases more slowly than the n-dimensional volume $V_n(W)$ of this region. For the ball region of the fractal medium, this property can be described by the power law $M_D(W) \sim R^D$, where R is the radius of the ball, and D is the mass dimension.

Another important property of some fractal media is homogeneity. Fractal medium is called homogeneous if the power law $M_D(W) \sim R^D$ does not depend on the translation of the region. The homogeneity property of the fractal medium means that two regions W_1 and W_2 with the equal volumes $V_n(W_1) = V_n(W_2)$ have equal masses $M_D(W_1) = M_D(W_2)$.

To adequately describe the fractal media by continuum models with non-integer dimensional spaces, the following two requirements must be satisfied.

- In the continuum models the mass density of homogeneous fractal medium should be described by the constant distribution function $\rho(\mathbf{r}) = \rho_0 = const$. Then equations with constant density should describe the homogeneous media, i.e., the conditions $\rho(\mathbf{r}) = const$ and $V_n(W_1) = V_n(W_2)$ should lead to the relation $M_D(W_1) = M_D(W_2)$.
- In the continuum models the mass of the ball region W of fractal homogeneous medium should be described by a power law relation $M \sim R^D$, where $0 < D < 3$, and R is the radius of the ball. Then the conditions $V_n(W_1) = \lambda^n V_n(W_2)$ and $\rho(\mathbf{r}) = const$, should lead to the relation $M_D(W_1) = \lambda^D M_D(W_2)$.

These requirements cannot be realized if the mass of fractal medium is described by integration of integer order over the integer dimensional space without using the concept of density of states $c_n(D, \mathbf{r})$. In order to realize these requirements we propose to use the integration and differentiation in non-integer dimensional spaces. In this case we can use the equation

$$dM_D(W) = \rho(\mathbf{r})\, dV_D(\mathbf{r}, n), \tag{1}$$

where $\rho(\mathbf{r})$ is a distribution function, and the density of states $c_n(D, \mathbf{r})$ in $\mathbb{R}^n$ is chosen such that

$$dV_D(\mathbf{r}, n) = c_n(D, \mathbf{r})\, dV_n,$$

describes the number of permitted states in dV_n. For different values of $n \in \{1, 2, 3\}$ we can use the notations

$$dV_D = c_3(D, \mathbf{r})\, dV_3, \quad dS_d = c_2(d, \mathbf{r})\, dS_2, \quad dl_\beta = c_1(\beta, \mathbf{r})\, dl_1, \tag{2}$$

to describe fractal media in the spaces $\mathbb{R}^n$, where these media are distributed. For simplification, we also use the notation $d^D\mathbf{r}$ instead of $dV_D(\mathbf{r}, n)$. The form of function $c_n(D, \mathbf{r})$ is defined by the properties of considered fractal medium. The symmetry of the density of states $c_n(D, \mathbf{r})$ is dictated by the symmetry properties of the described fractal medium, but in any cases it should be a function of power-law type to adequately reflect a scaling property (fractality) of described fractal medium. To simplify the analysis in this article we will consider only isotropic fractal media with densities of states that are independent of angles.

In the continuum models of fractal media, it is convenient to work in the dimensionless space variables $x/R_0 \to x,\ y/R_0 \to x,\ z/R_0 \to x,\ \mathbf{r}/R_0 \to \mathbf{r}$, that yields dimensionless integration and dimensionless differentiation in non-integer dimensional space. In this case the physical and mechanical quantities of fractal media have correct physical dimensions.

3. Integration over non-integer dimensional space

The integral for all non-integer values of D is defined by continuation in D [4,5]. Let us give properties must we impose on a functional of $f(\mathbf{r})$ in order to regard it as D-dimensional integration. The following properties or axioms [6] for integrals in D-dimensional space are natural and necessary in applications [5]:

1. Linearity:

$$\int (af_1(\mathbf{r}) + bf_2(\mathbf{r}))\, d^D\mathbf{r} = a \int f_1(\mathbf{r})\, d^D r + b \int f_2(\mathbf{r})\, d^D\mathbf{r}, \tag{3}$$

where a and b are arbitrary real numbers, and $d^D\mathbf{r} = dV_D(\mathbf{r}, n)$ represents the volume element in the non-integer dimensional space.

2. Translational invariance:

$$\int f(\mathbf{r} + \mathbf{r}_0)\, d^D\mathbf{r} = \int f(\mathbf{r})\, d^D\mathbf{r} \tag{4}$$

for any vector $\mathbf{r}_0$.

3. Scaling property:

$$\int f(\lambda\mathbf{r})\, d^D\mathbf{r} = \lambda^{-D} \int f(\mathbf{r})\, d^D\mathbf{r} \tag{5}$$

for any positive λ.

Linearity is true of any integration, while translation and rotation invariance are basic properties of an Euclidean space. The scaling property embodies the D-dimensionality. Not only are the above three axioms necessary, but they also ensure that integration is unique, aside from an overall normalization [6].

These properties must be imposed on a functional of $f(\mathbf{r})$ in order to regard it as D-dimensional integrations [5]. These properties are natural and are necessary in application of dimensional regularization to quantum field theory (see Section 4 in [5]).

A function $f(\mathbf{r})$ that we integrate could in principle be any function of the components of its vector argument $\mathbf{r}$. However, we do not, a priori, know the meaning of the components of, say, a vector in non-integer dimensions. In this paper, we will work with rotationally covariant functions. So we will assume that f is a scalar or vector function only of scalar products of vectors or of length of vectors. For example, in the elasticity theory, we consider the case, where the displacement vector $\mathbf{u}(\mathbf{r})$, is independent of the angles $\mathbf{u}(\mathbf{r}) = \mathbf{u}(r)$, where $r = |\mathbf{r}|$.

The integral defined in Eq. (6) satisfies the properties (3)–(5).

The D-dimensional integration for scalar functions $f(\mathbf{r}) = f(|\mathbf{r}|)$ can be defined in terms of ordinary integration by the expression

$$\int d^D\mathbf{r}\, f(\mathbf{r}) = \int_{\Omega_{D-1}} d\Omega_{D-1} \int_0^\infty dr\, r^{D-1} f(r), \tag{6}$$

where we can use

$$\int_{\Omega_{D-1}} d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = S_{D-1}. \tag{7}$$

For integer $D = n$, Eq. (7) gives the well-known area S_{n-1} of $(n-1)$-sphere with unit radius.

As a result, we have [5] the explicit definition of the continuation of integration from integer n to arbitrary fractional D in the form

$$\int d^D\mathbf{r}\, f(|\mathbf{r}|) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dr\, r^{D-1} f(r). \tag{8}$$

This equation reduced D-dimensional integration to ordinary integration. Therefore the linearity and translation invariance follow from linearity and translation invariance of ordinary integration. The scaling and rotation covariance are explicit properties of the definition.

Let us give some examples of an application of Eq. (8). For the function

$$f(\mathbf{r}^2) = \frac{\mathbf{r}^2 + a}{\mathbf{r}^2 + b}, \tag{9}$$

where a and b are real numbers. The integral for (9) can be explicitly computed

$$\int d^D\mathbf{r}\,\frac{r^2+a}{r^2+b} = (\pi b)^{D/2}\,(a/b-1)\,\Gamma(1-D/2). \tag{10}$$

The other example is the integral

$$\int d^D\mathbf{r}\,\frac{r^{2\alpha}}{(r^2+a^2)^{\beta}} = \frac{\Gamma(\alpha+D/2)\,\Gamma(\beta-\alpha-D/2)}{\Gamma(D/2)\Gamma(\beta)}\,\pi^{D/2}\,a^{D+2\alpha-2\beta}, \tag{11}$$

where $r=|\mathbf{r}|$.

The mass of material in W is described by the integral

$$M_D(W) = \int_W \rho(\mathbf{r})\,d^D\mathbf{r}, \tag{12}$$

where $\mathbf{r}$ is dimensionless vector variable. For a ball with radius R and the density $\rho(\mathbf{r}) = \rho_0 = \text{const}$, we get the mass is defined by

$$M_D(W) = \rho_0\,V_D = \frac{\pi^{D/2}\,\rho_0}{\Gamma(D/2+1)}\,R^D. \tag{13}$$

This equation defines the mass of the fractal homogeneous ball with volume V_D. For $D=3$, Eq. (13) gives the well-known equation for mass of non-fractal ball $M_3 = (4\rho_0\pi/3)R^3$ because $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(z+1) = z\,\Gamma(z)$.

4. Divergence for non-integer dimensional space

Let us consider hollow ball $B_D(R_1;R_2)$ with internal radius R_1 and external radius R_2 in non-integer dimensional space. The boundary $\partial B_D(R_1;R_2)$ of this ball consists of two $(D-1)$-dimensional spheres $S_{D-1}(R_1)$ and $S_{D-1}(R_2)$.

We assume that the vector field $\mathbf{u}(\mathbf{r})$ is radially directed and $\mathbf{u}(\mathbf{r})$ is not dependent on the angles, i.e.,

$$\mathbf{u}(\mathbf{r}) = u_r(r)\,\mathbf{e}_r, \tag{14}$$

where $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$ and $r = |\mathbf{r}|$. We can defined a flux of the vector field $\mathbf{u}(r)$ across a surface $\partial B_D(R_1;R_2)$ by using the integration in non-integer dimensional space suggested in [5]. A flux of the vector field $\mathbf{u}(r)$ across a $(D-1)$-dimensional surface S_{D-1} is the integral

$$\Phi_{\mathbf{u}}(S_{D-1}) = \int_{S_{D-1}} (\mathbf{u}, d\mathbf{S}_{D-1}), \tag{15}$$

where $d\mathbf{S}_{D-1} = \mathbf{e}_r\,dS_{D-1}$.

The volume of $B_D(R_1;R_2)$ is equal to

$$V(B_D(R_1;R_2)) = \frac{\pi^{D/2}}{\Gamma(D/2+1)}\,(R_2^D - R_1^D). \tag{16}$$

An exact expression for dependence of the divergence operator on the non-integer dimension D and the vector field $\mathbf{u}(\mathbf{r}) = u_r(r)\,\mathbf{e}_r$ can be derived by the equation

$$\mathrm{Div}_r^D\mathbf{u} = \lim_{V(B_D(R_1;R_2))\to 0} \frac{\Phi_{\mathbf{u}}(\partial B_D(R_1;R_2))}{V(B_D(R_1;R_2))}. \tag{17}$$

Here the flux of the vector field can be represented by

$$\Phi_{\mathbf{u}}(\partial B_D(R_1;R_2)) = \int_{S_{D-1}(R_2)} (\mathbf{u}, d\mathbf{S}_{D-1}) - \int_{S_{D-1}(R_1)} (\mathbf{u}, d\mathbf{S}_{D-1}). \tag{18}$$

Using (14), we get

$$\begin{aligned}\int_{\partial B_D(R_1;R_2)} (\mathbf{u}, d\mathbf{S}_{D-1}) &= \int_{S_{D-1}(R_2)} (\mathbf{u}, d\mathbf{S}_{D-1}) - \int_{S_{D-1}(R_1)} (\mathbf{u}, d\mathbf{S}_{D-1}) = u_r(R_2)\int_{S_{D-1}(R_2)} dS_{D-1} - u_r(R_1)\int_{S_{D-1}(R_1)} dS_{D-1} \\ &= \frac{2\,\pi^{D/2}}{\Gamma(D/2)}\left(u_r(R_2)\,R_2^{D-1} - u_r(R_1)\,R_1^{D-1}\right),\end{aligned} \tag{19}$$

where $S_{D-1} = (\mathbf{e}_r, \mathbf{S}_{D-1})$.

For $D=3$, we have $\Gamma(3/2) = (1/2)\,\Gamma(1/2) = \sqrt{\pi}/2$, and

$$\int_{\partial B_D(R_1;R_2)} (\mathbf{u}, d\mathbf{S}_{D-1}) = 4\pi\left(u_r(R_2)\,R_2^2 - u_r(R_1)\,R_1^2\right). \tag{20}$$

For an infinitely thin hollow ball (thin spherical shell) with $R_1 = r$ and $R_2 = r + \Delta r$, we can use

$$\mathrm{Div}_r^D \mathbf{u} = \lim_{\Delta r \to 0} \frac{\Phi_{\mathbf{u}}(\partial B_D(r; r+\Delta r))}{V(B_D(r; r+\Delta r))}, \tag{21}$$

to derive an expression for the divergence for our case.

The volume of the ball $B_D(r; r+\Delta r)$ is

$$V(B_D(r; r+\Delta r)) = \frac{\pi^{D/2}}{\Gamma(D/2+1)} \left((r+\Delta r)^D - r^D \right) = \frac{\pi^{D/2}}{\Gamma(D/2+1)} \left(D\, r^{D-1} \Delta r + O((\Delta r)^2) \right)$$
$$= \frac{2\pi^{D/2}}{\Gamma(D/2)} \left(r^{D-1} \Delta r + O((\Delta r)^2) \right). \tag{22}$$

The flux of the vector field $\mathbf{u} = u_r(r)\, \mathbf{e}_r$ across a surface $\partial B_D(R_1; R_2)$ is given by

$$\Phi_{\mathbf{u}}(\partial B_D(r; r+\Delta r)) = \int_{S_{D-1}(r+\Delta r)} (\mathbf{u}, d\mathbf{S}_{D-1}) - \int_{S_{D-1}(r)} (\mathbf{u}, d\mathbf{S}_{D-1}) = \frac{2\,\pi^{D/2}}{\Gamma(D/2)} \left(u_r(r+\Delta r)\, (r+\Delta r)^{D-1} - u_r(r)\, r^{D-1} \right)$$
$$= \frac{2\,\pi^{D/2}}{\Gamma(D/2)} \left(\left[u_r(r) + \frac{\partial u_r(r)}{\partial r} \Delta r + O((\Delta r)^2) \right] \left[r^{D-1} + (D-1)\, r^{D-2} \Delta r + O((\Delta r)^2) \right] - u_r(r)\, r^{D-1} \right)$$
$$= \frac{2\,\pi^{D/2}}{\Gamma(D/2)} \left(r^{D-1} \frac{\partial u_r(r)}{\partial r} + (D-1)\, r^{D-2}\, u_r(r) + O(\Delta r) \right) \Delta r. \tag{23}$$

Using (22) and (23), we get

$$\frac{\Phi_{\mathbf{u}}(\partial B_D(r; r+\Delta r))}{V(B_D(r; r+\Delta r))} = \frac{\partial u_r(r)}{\partial r} + \frac{D-1}{r}\, u_r(r) + O(\Delta r). \tag{24}$$

As a result, the divergence for the vector field $\mathbf{u} = u_r(r)\, \mathbf{e}_r$ in non-integer dimensional space has the form

$$\mathrm{Div}_r^D \mathbf{u} = \lim_{\Delta r \to 0} \frac{\Phi_{\mathbf{u}}(\partial B_D(r; r+\Delta r))}{V(B_D(r; r+\Delta r))} = \frac{\partial u_r(r)}{\partial r} + \frac{D-1}{r}\, u_r(r). \tag{25}$$

The Gauss theorem in non-integer dimensional space can be written in the form

$$\int_{B_D(R_1; R_2)} \mathrm{Div}_r^D \mathbf{u}\; d^D \mathbf{r} = \int_{\partial B_D(R_1; R_2)} (\mathbf{u}, d\mathbf{S}_{D-1}), \tag{26}$$

where we assume that the dimension D of the region of fractal materials and the dimension d of boundary of this region are related by the equation $d = D - 1$.

5. Vector differential operators in non-integer dimensional space

We would like to derive equations for vector differential operators in non-integer dimensional space. For this aim we should have equations for these differential operators for rotationally covariant functions in the spherical coordinates in $\mathbb{R}^n$ for arbitrary n to highlight the explicit relations with dimension n. Then the vector differential operators for non-integer dimension D can be defined by analytic continuation in dimension from integer n to non-integer D.

To simplify we will consider only scalar fields φ and vector fields $\mathbf{u}$ that are independent of angles

$$\varphi(\mathbf{r}) = \varphi(r), \quad \mathbf{u}(\mathbf{r}) = \mathbf{u}(r) = u_r\, \mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r,\ r = |\mathbf{r}|$ and $u_r = u_r(r)$ is the radial component of $\mathbf{u}$. We will work with rotationally covariant functions only. This simplification is analogous to the simplification for definition of integration over non-integer dimensional space suggested in [5].

Let us give equations for differential operators for functions $\mathbf{u} = u_r(r)\, \mathbf{e}_r$ and $\varphi = \varphi(r)$ in the spherical coordinates in $\mathbb{R}^n$ for arbitrary n.

The divergence in integer dimensional space $\mathbb{R}^n$ for the vector field $\mathbf{u} = u(r)\, \mathbf{e}_r$ is

$$\mathrm{div}\, \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{n-1}{r}\, u_r. \tag{27}$$

The gradient in integer dimensional space $\mathbb{R}^n$ for the scalar field $\varphi = \varphi(r)$ is

$$\mathrm{grad}\, \varphi = \frac{\partial \varphi}{\partial r}\, \mathbf{e}_r. \tag{28}$$

The scalar Laplacian in integer dimensional space $\mathbb{R}^n$ for the scalar field $\varphi = \varphi(r)$ is

$$\Delta \varphi(r) = \mathrm{div\, grad}\, \varphi(r) = \frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r}. \tag{29}$$

The vector Laplacian [78] in integer dimensional space $\mathbb{R}^n$ for the vector field $\mathbf{u} = u(r)\,\mathbf{e}_r$ is

$$\Delta\mathbf{u}(r) = \text{grad div}\,\mathbf{u}(r) = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{n-1}{r}\frac{\partial u_r}{\partial r} - \frac{n-1}{r^2}\,u_r\right)\mathbf{e}_r. \tag{30}$$

As a result, we have equations of differential operators in $\mathbb{R}^n$ for continuation from integer n to arbitrary non-integer D in the following forms.

The divergence in non-integer dimensional space for the vector field $\mathbf{u} = \mathbf{u}(r)$ is

$$\text{Div}_r^D\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-1}{r}\,u_r. \tag{31}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$\text{Grad}_r^D\varphi = \frac{\partial\varphi}{\partial r}\,\mathbf{e}_r. \tag{32}$$

The curl operator for the vector field $\mathbf{u} = \mathbf{u}(r)$ is equal to zero

$$\text{Curl}_r^D\mathbf{u} = 0. \tag{33}$$

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$${}^S\Delta_r^D\varphi = \text{Div}_r^D\text{Grad}_r^D\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{D-1}{r}\frac{\partial\varphi}{\partial r}. \tag{34}$$

The vector Laplacian in non-integer dimensional space for the vector field $\mathbf{u} = u(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^D\mathbf{u} = \text{Grad}_r^D\text{Div}_r^D\mathbf{u} = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r}{\partial r} - \frac{D-1}{r^2}\,u_r\right)\mathbf{e}_r. \tag{35}$$

Let us consider a case of axial symmetry of the fractal material, where the fields $\varphi(r)$ and $\mathbf{u}(r) = u_r(r)\,\mathbf{e}_r$ are also axially symmetric. Let Z-axis be directed along the axis of symmetry. Therefore it is convenient to use a cylindrical coordinate system. Equations for differential vector operations for cylindrical symmetry case have the following forms.

The divergence in non-integer dimensional space for the vector field $\mathbf{u} = \mathbf{u}(r)$ is

$$\text{Div}_r^D\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r}\,u_r. \tag{36}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$\text{Grad}_r^D\varphi = \frac{\partial\varphi}{\partial r}\,\mathbf{e}_r. \tag{37}$$

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$${}^S\Delta_r^D\varphi = \text{Div}_r^D\text{Grad}_r^D\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{D-2}{r}\frac{\partial\varphi}{\partial r}. \tag{38}$$

The vector Laplacian in non-integer dimensional space for the vector field $\mathbf{u} = u(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^D\mathbf{u} = \text{Grad}_r^D\text{Div}_r^D\mathbf{u} = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2}\,u_r\right)\mathbf{e}_r. \tag{39}$$

For $D = 3$ Eqs. (31)–(39) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\varphi = \varphi(r)$ and $\mathbf{u}(r) = u_r(r)\,\mathbf{e}_r$.

It is easy to generalize these equations for the case $\varphi = \varphi(r,z)$ and $\mathbf{u}(r,z) = u_r(r,z)\,\mathbf{e}_r + u_r(r,z)\,\mathbf{e}_z$. In this case the curl operator for $\mathbf{u}(r,z)$ is different from zero, and

$$\text{Curl}_{r,z}^D\mathbf{u} = \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_r. \tag{40}$$

The suggested operators allow us to reduce D-dimensional vector differentiations (31)–(39) to derivatives with respect to $r = |\mathbf{r}|$. It allows us to reduce partial differential equations for fields in non-integer dimensional space to ordinary differential equations with respect to r.

6. Stillinger's Laplacian for non-integer dimensional space

In the paper [7], the integration in a non-integer dimensional space is described by using the equation

$$\int_{R^D} d^D\mathbf{r}\,\varphi(\mathbf{r}) = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \int_0^\infty dr\, r^{D-1} \int_0^\pi d\theta\, \varphi(r,\theta) \sin^{D-2}\theta, \tag{41}$$

where $d^D\mathbf{r} = dV_D(\mathbf{r}, n)$ represents the volume element in the non-integer dimensional space. Using (41) with $\varphi(r,\theta) = 1$, and

$$\int_0^\pi d\theta \sin^{D-2}\theta = \frac{\pi^{1/2}\,\Gamma(D/2-1)}{\Gamma(D/2)}, \tag{42}$$

we get

$$V_D = \frac{\pi^{D/2}}{\Gamma(D/2+1)} R^D, \tag{43}$$

which is the volume of a D-dimensional ball with radius R.

Using the analytic continuation of Gaussian integrals the scalar Laplace operator for non-integer dimensional space has been suggested in [7]. For a function $\varphi = \varphi(r,\theta)$ of radial distance r and related angle θ measured relative to an axis passing through the origin, the Laplacian in non-integer dimensional space proposed by Stillinger [7] is

$$^{St}\Delta^D = \frac{1}{r^{D-1}} \frac{\partial}{\partial r}\left(r^{D-1} \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^{D-2}\theta} \frac{\partial}{\partial \theta}\left(\sin^{D-2}\theta \frac{\partial}{\partial \theta}\right), \tag{44}$$

where D is the dimension of space ($0 < D < 3$), and the variables $r \geqslant 0,\ 0 \leqslant \theta \leqslant \pi$. Note that $\left({}^{St}\Delta^D\right)^2 \neq {}^{St}\Delta^{2D}$. If the function depends on radial distance r only ($\varphi = \varphi(r)$), then

$$^{St}\Delta^D \varphi(r) = \frac{1}{r^{D-1}} \frac{\partial}{\partial r}\left(r^{D-1} \frac{\partial \varphi(r)}{\partial r}\right) = \frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi(r)}{\partial r}. \tag{45}$$

It is easy to see that the Stillinger's form of Laplacian ${}^{St}\Delta^D$ for radial scalar functions $\varphi(\mathbf{r}) = \varphi(r)$ coincides with the suggested scalar Laplacian ${}^{S}\Delta^D_r$ for this function,

$$^{St}\Delta^D \varphi(r) = {}^{S}\Delta^D \varphi(r). \tag{46}$$

The Stillinger's Laplacian can be applied only for scalar fields and it cannot be used to describe vector fields $\mathbf{u} = u_r(r)\,\mathbf{e}_r$ because Stillinger's Laplacian for $D = 3$ is not equal to the usual vector Laplacian for $\mathbb{R}^3$,

$$^{St}\Delta^3 \mathbf{u}(r) \neq \Delta \mathbf{u}(r) = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2} u_r \right) \mathbf{e}_r. \tag{47}$$

For the vector fields $\mathbf{u} = u_r(r)\,\mathbf{e}_r$, we should use the vector Laplace operators (35) and (35).

Note that the gradient, divergence, curl operator and vector Laplacian are not considered in Stillinger's paper [7]

7. Non-integer dimensional space for complex fractal media with $d \neq D-1$

In general, the dimension D of the ball region B_D of fractal materials and the dimension d of boundary $S_d = \partial B_D$ of this region are not related by the equation $d = D - 1$, i.e.,

$$\dim(\partial B_D) \neq \dim(B_D) - 1, \tag{48}$$

where $\dim(B_D) = D$. We assume that dimension of the boundary $S_d = \partial B_D$ is

$$\dim(S_d) = d. \tag{49}$$

Considering an infinitely thin hollow ball B_D with $R_1 = r$ and $R_2 = r + \Delta r$, we can use

$$\mathrm{Div}^{D,d}_r \mathbf{u} = \lim_{\Delta r \to 0} \frac{\Phi_{\mathbf{u}}(S_d(r; r+\Delta r))}{V(B_D(r; r+\Delta r))}, \tag{50}$$

in order to derive an expression for the divergence for the case $d \neq D - 1$.

The flux of the vector field $\mathbf{u} = u_r(r)\,\mathbf{e}_r$ across a surface S_d is

$$\begin{aligned} \Phi_{\mathbf{u}}(S_d(r; r+\Delta r)) &= \frac{2\,\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \left(u_r(r+\Delta r)\,(r+\Delta r)^d - u_r(r)\, r^d \right) \\ &= \frac{2\,\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \left(r^d \frac{\partial u_r(r)}{\partial r} + d\, r^{d-1} u_r(r) + O(\Delta r) \right) \Delta r. \end{aligned} \tag{51}$$

Using (22) and (51), we get

$$\frac{\Phi_{\mathbf{u}}(S_d(r; r+\Delta r))}{V(B_D(r; r+\Delta r))} = \frac{\pi^{(d+1-D)/2}\,\Gamma(D/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{D-1-d}} \frac{\partial u_r(r)}{\partial r} + \frac{d}{r^{D-d}} u_r(r) \right) + O(\Delta r). \tag{52}$$

Using the limit $\Delta r \to 0$, we get the divergence operator for fractal media with $d \neq D-1$ in the form

$$\mathrm{Div}_r^{D,d}\mathbf{u} = \frac{\pi^{(d+1-D)/2}\,\Gamma(D/2)}{\Gamma((d+1)/2)}\left(\frac{1}{r^{D-1-d}}\frac{\partial u_r(r)}{\partial r} + \frac{d}{r^{D-d}}\,u_r(r)\right). \tag{53}$$

We can define the parameter

$$\alpha_r = D - d, \tag{54}$$

that can be interpreted as a dimension of medium along the radial direction. Using (54), Eq. (53) can be rewritten in the form

$$\mathrm{Div}_r^{D,d}\mathbf{u} = \pi^{(1-\alpha_r)/2}\,\frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)}\left(\frac{1}{r^{\alpha_r-1}}\frac{\partial u_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\,u_r(r)\right). \tag{55}$$

This is divergence operator for non-integer dimensional continuum models of fractal materials with $d \neq D-1$. For $\alpha_r = 1$, i.e. $d = D-1$, equations (53) and (55) give (31).

We can assume that the gradient for the scalar field $\varphi(\mathbf{r}) = \varphi(r)$ may depend on the radial dimension α_r in the form

$$\mathrm{Grad}_r^{D,d}\varphi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\,r^{\alpha_r-1}}\,\frac{\partial \varphi(r)}{\partial r}\,\mathbf{e}_r, \tag{56}$$

because expression (56) can be represented by the equation

$$\mathrm{Grad}_r^{D,d}\varphi = \lim_{\Delta r\to 0}\frac{2(\varphi(r+\Delta r) - \varphi(r))}{V(B_{\alpha_r}(r; r+\Delta r))}, \tag{57}$$

where

$$V(B_{\alpha_r}(r; r+\Delta r)) = \frac{2\,\pi^{\alpha_r/2}}{\Gamma(\alpha_r/2)}\left(r^{\alpha_r-1}\,\Delta r + O((\Delta r)^2)\right). \tag{58}$$

The presence of the factor of 2 in (57) is due to the fact that for $D = 1$, r is integrated from $-R$ to R, and when the limits are taken as 0 and R, one gets a factor of 2.

For $\alpha_r = 1$, Eq. (56) gives (32).

Using the operators (56) and (53) for the fields $\varphi = \varphi(r)$ and $\mathbf{u} = u(r)\,\mathbf{e}_r$, we can get the scalar and vector Laplace operators for the case $d \neq D-1$ by the equation

$${}^S\Delta_r^{D,d}\varphi = \mathrm{Div}_r^{D,d}\mathrm{Grad}_r^{D,d}\varphi, \quad {}^V\Delta_r^{D,d}\mathbf{u} = \mathrm{Grad}_r^{D,d}\mathrm{Div}_r^{D,d}\mathbf{u}. \tag{59}$$

Then the scalar Laplacian for $d \neq D-1$ for the field $\varphi = \varphi(r)$ is

$${}^S\Delta_r^{D,d}\varphi = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2\varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial\varphi}{\partial r}\right), \tag{60}$$

and the vector Laplacian for $d \neq D-1$ for the field $\mathbf{u} = u(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^{D,d}\mathbf{u} = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2 u_r}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial u_r}{\partial r} - \frac{d-\alpha_r}{r^{2\alpha_r}}\,u_r\right)\mathbf{e}_r. \tag{61}$$

The vector differential operators (56), (53), (60) and (61) allow us to describe complex fractal materials with the boundary dimension of the regions $d \neq D-1$.

8. Applications in mechanics and physics

In this section, we consider some applications of vector calculus in non-integer dimensional space to the elasticity theory, the heat processes, and electrodynamics.

8.1. Elasticity of fractal material

For homogenous and isotropic materials, the equation of linear elasticity [79,80] for the displacement vector fields $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$ has the form

$$\lambda\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + 2\mu\,\Delta\mathbf{u} + \mathbf{f} = \rho\,D_t^2\mathbf{u}, \tag{62}$$

where λ and μ are the Lame coefficients, and $\mathbf{f}$ is the external force density vector field.

If the deformation in the material is described by $\mathbf{u}(\mathbf{r}, t) = u(r, t)\,\mathbf{e}_r$, then Eq. (62) has the form

$$(\lambda + 2\mu)\,\Delta\mathbf{u}(r, t) + \mathbf{f}(r, t) = \rho\,D_t^2\mathbf{u}(r, t). \tag{63}$$

A generalization of Eqs. (63) for fractal material in the framework of non-integer dimensional models has the form

$$(\lambda + 2\mu)\,{}^{V}\Delta_r^D \mathbf{u}(r,t) + \mathbf{f}(r,t) = \rho D_t^2 \mathbf{u}(r,t), \tag{64}$$

where ${}^{V}\Delta_r^D$ is defined by (35). Eq. (64) describes dynamics of displacement vector for fractal materials. For static case, Eq. (64) has the form

$${}^{V}\Delta_r^D \mathbf{u}(r) + (\lambda + 2\mu)^{-1} \mathbf{f}(r) = 0, \tag{65}$$

where $\mathbf{u} = u_r \mathbf{e}_r$ and $\mathbf{f} = f(r)\mathbf{e}_r$.

Let us consider some two problems for elasticity of fractal materials.

8.1.1. Elasticity of fractal hollow ball with pressure inside and outside

Let us determine the deformation of a hollow fractal ball with internal radius R_1 and external radius R_2, with the pressure p_1 inside and the pressure p_2 outside.

We can use the spherical polar coordinates with the origin at the center of the ball. The displacement vector $\mathbf{u}$ is everywhere radial, and it is a function of $r = |\mathbf{r}|$ alone. Then the equilibrium equation for fractal ball is

$$(\lambda + 2\mu)\;{}^{V}\Delta_r^D \mathbf{u}(r) = 0, \tag{66}$$

where $\mathbf{u} = u_r \mathbf{e}_r$. Using (35), we represent Eq. (66) in the form

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2}\,u_r(r) = 0. \tag{67}$$

The solution of (67) is

$$u(r) = C_1\, r + C_2\, r^{1-D}. \tag{68}$$

The constants C_1 and C_2 are determined from the boundary conditions for radial stress

$$\sigma_{rr}(R_1) = -p_1, \quad \sigma_{rr}(R_2) = -p_2. \tag{69}$$

Using that the radial components of the stress is

$$\sigma_{rr}(r) = (2\,\mu + \lambda)\,\frac{\partial u_r}{\partial r} + \lambda\,\frac{D-1}{r}\,u_r, \tag{70}$$

we get

$$C_1 = \frac{-(p_2 R_2^D - p_1 R_1^D)}{(2\,\mu + D\,\lambda)\,(R_2^D - R_1^D)}, \tag{71}$$

$$C_2 = \frac{p_2 - p_1}{2\,(1-D)\,\mu\,(R_2^D - R_1^D)}. \tag{72}$$

Then the radial components of the stress is

$$\sigma_{rr}(r) = \frac{-(p_2 R_2^D - p_1 R_1^D)}{R_2^D - R_1^D} + \frac{(p_2 - p_1)\,(R_1 R_2)^D}{R_2^D - R_1^D}\,r^{-D}. \tag{73}$$

The stress distribution in a ball with pressure $p_1 = p$ inside and $p_2 = 0$ outside is gives by

$$\sigma_{rr}(r) = \frac{p\,R_1^D}{R_2^D - R_1^D}\left(1 - \left(\frac{R_2}{r}\right)^D\right). \tag{74}$$

The stress distribution in an infinite elastic medium with spherical cavity with radius R subjected to hydrostatic compression is

$$\sigma_{rr}(r) = -p\left(1 - \left(\frac{R}{r}\right)^D\right) \tag{75}$$

that can be obtained by putting $R_1 = R,\ R_2 \to \infty$, $p_1 = 0$ and $p_2 = p$ in Eq. (73).

8.1.2. Elasticity of cylindrical fractal solid pipe with pressure inside and outside

Let us consider the deformation of a fractal solid cylindrical pipe with internal radius R_1 and external radius R_2 with pressure p_1 inside and pressure p_2 outside. We use the cylindrical coordinates with the z-axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $\mathbf{u} = u_r(r)\,\mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{r}/r$. The equation for the displacement $u_r(r)$ in fractal pine is

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r = 0, \tag{76}$$

where $0 < D \leqslant 3$. If $D = 3$, we get the usual (non-fractal) case.

The general solution of Eq. (76), where $D \neq 1,\ D \neq 2$, has the form

$$u_r(r) = C_1 r + C_2 r^{2-D}. \tag{77}$$

Eqs. (76) with $D = 1$ has the general solution

$$u_r(r) = C_1 r + C_2 r \ln(r). \tag{78}$$

For $D = 2$, Eqs. (76) has the solution

$$u_r(r) = C_1 + C_2 r. \tag{79}$$

Note that dimensions $D = 1$ or $D = 2$ of the fractal pipe material do not correspond to the distribution of matter along the line and surface. These dimensions describe a distribution of matter in 3-dimensional space (in the volume of pipe) such that the mass dimensions are equal to D.

The constants C_1 and C_2 are determined by boundary conditions. Using that pressure is p_1 inside and pressure p_2 outside, we get the boundary condition in the form

$$\sigma_{rr}(R_1) = -p_1, \quad \sigma_{rr}(R_2) = -p_2. \tag{80}$$

Using (77) and

$$\sigma_{rr} = (2\mu + \lambda)\frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r = (2\mu + \lambda(D-1)) C_1 - 2\mu(D-2) C_2 r^{1-D}, \tag{81}$$

the boundary condition (80) gives

$$C_1 = -\frac{p_1 R_2^{1-D} - p_2 R_1^{1-D}}{(2\mu + \lambda(D-1))(R_2^{1-D} - R_1^{1-D})}, \tag{82}$$

$$C_2 = \frac{p_2 - p_1}{2\mu(D-2)(R_2^{1-D} - R_1^{1-D})}. \tag{83}$$

The stress for $2 < D < 3$ or $1 < D < 2$ can represented in the form

$$\sigma_{rr} = \frac{p_1 R_1^{D-1} - p_2 R_2^{D-1}}{(R_2^{D-1} - R_1^{D-1})} - \frac{p_2 - p_1}{(R_2^{D-1} - R_1^{D-1})}\left(\frac{R_1 R_2}{r}\right)^{D-1}. \tag{84}$$

For the boundary conditions $\sigma_{rr}(R_2) = 0$ and $\sigma_{rr}(R_1) = -p$, i.e. $p_2 = 0$ and $p_1 = p$ for (84), we have the solution

$$\sigma_{rr} = \frac{p R_1^{D-1}}{(R_2^{D-1} - R_1^{D-1})}\left(1 - \left(\frac{R_2}{r}\right)^{D-1}\right). \tag{85}$$

This is the deformation of cylindrical pipe with a pressure p inside and no pressure outside. For $D = 3$, Eq. (85) has the well-known form

$$\sigma_{rr} = \frac{p R_1^2}{(R_2^2 - R_1^2)}\left(1 - \left(\frac{R_2}{r}\right)^2\right) \tag{86}$$

that describes the stress of non-fractal material of pipe.

8.2. Heat equation for fractal materials

The heat equation is

$$\frac{\partial \varphi(\mathbf{r}, t)}{\partial t} - a\Delta\varphi(\mathbf{r}, t) = \frac{1}{c_p \rho} q(\mathbf{r}, t), \tag{87}$$

where $\varphi(\mathbf{r}, t)$ is the heat density of a medium, $q(\mathbf{r}, t)$ is the heat source density, and a is the thermal diffusivity $a = k/c_p \rho$, where k is thermal conductivity, ρ is density, c_p is specific heat capacity.

A generalization of Eq. (87) for fractal material, has the form of the heat equation in the non-integer dimensional space

$$\frac{\partial \varphi(r, t)}{\partial t} - a^S \Delta_r^D \varphi(r, t) = \frac{1}{c_p \rho} q(r, t), \tag{88}$$

where we assume that the fields $\varphi(r, t)$ and $q(r, t)$ are not depend on the angles.

Using (34), we get the following equations for the ball

$$\frac{\partial^2 \varphi(r,t)}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi(r,t)}{\partial r} + \frac{1}{c_p \rho} q(r,t) = \frac{1}{a} \frac{\partial \varphi(r,t)}{\partial t}. \tag{89}$$

For the pipe and cylinder, we get

$$\frac{\partial^2 \varphi(r,t)}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi(r,t)}{\partial r} + \frac{1}{c_p \rho} q(r,t) = \frac{1}{a} \frac{\partial \varphi(r,t)}{\partial t}. \tag{90}$$

Steady states is described by the equation

$$\frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi(r)}{\partial r} + \frac{1}{c_p \rho} q(r) = 0. \tag{91}$$

The general solution of Eq. (91) is

$$\varphi(r) = C_1 + C_2 r^{2-D} + \frac{1}{c_p \rho (D-2)} \left(r^{2-D} \int q(r) r^{D-1} dr - \int q(r) r \, dr \right), \tag{92}$$

where the constants C_1 and C_2 are determined by the boundary condition. For the constant heat source density $q(r) = q_0 = \text{const}$, Eq. (92) has the form

$$\varphi(r) = C_1 + C_2 r^{2-D} - \frac{q_0}{2 D c_p \rho} r^2. \tag{93}$$

For $D = 3$, we get the well-known equation for non-fractal material.

8.3. Electric field of fractal charged infinite cylinder

Let us consider a uniformly fractal charged infinite circular cylinder of radius R with a volume charge density $\rho = \text{const}$ and non-integer dimension $2 < D \leqslant 3$. Using the Poisson equation for scalar potential created by an infinite circular cylinder. We assume that the Z-axis is directed along the axis of the cylinder. Due to the axial symmetry of the charge distribution the potential is also axially symmetric. Therefore it is convenient to use a cylindrical coordinate system. The Poisson equation for scalar field $\varphi(r)$ in non-integer dimensional space has the form

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi}{\partial r} = -\frac{\rho}{\varepsilon_0} \quad (0 < r < R), \tag{94}$$

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi}{\partial r} = 0 \quad (r > R). \tag{95}$$

The general solution of equations (94) and (95) are

$$\varphi(r) = C_1 + C_2 r^{3-D} - \frac{\rho r^2}{2 \varepsilon_0 (D-1)} \quad (0 < r < R), \tag{96}$$

$$\varphi(r) = C_3 + C_4 r^{3-D} \quad (r > R), \tag{97}$$

where C_1, C_2, C_3, C_4 are the integration constants, and $2 < D \leqslant 3$. For the case $D = 3$, the general solution of equations (94) and (95) has the well-known form

$$\varphi(r) = C_1 + C_2 \ln(r) - \frac{\rho r^2}{4 \varepsilon_0} \quad (0 < r < R), \tag{98}$$

$$\varphi(r) = C_3 + C_4 \ln(r) \quad (r > R). \tag{99}$$

The electric fields

$$\mathbf{E}(r) = -\mathrm{Grad}_r^D \varphi = -\frac{\partial \varphi(r)}{\partial r} \mathbf{e}_r \tag{100}$$

for potentials (96) and (97) are

$$\mathbf{E}(r) = \left((D-3) C_2 r^{2-D} + \frac{\rho r}{\varepsilon_0 (D-1)} \right) \mathbf{e}_r \quad (0 < r < R), \tag{101}$$

$$\mathbf{E}(r) = (D-3) C_4 r^{2-D} \mathbf{e}_r \quad (r > R). \tag{102}$$

Because the electric field (100) must be finite at all points, and $r^{2-D} \to \infty$ for $r \to 0$ and $2 < D \leqslant 3$, it is necessary put $C_2 = 0$. Conveniently potential normalized by the condition $\varphi(0) = 0$, then we get $C_1 = 0$. Because there are no surface charges, then the electric field (100) at the surface of the cylinder $r = R$ is continuous, i.e. the derivative of the potential should be continuous. The conditions of continuity of the potential and its derivative at $r = R$ give two algebraic equations that allow us to determine the remaining two constants C_3 and C_4 by the equations

$$-\frac{\rho R^2}{2\,\varepsilon_0\,(D-1)} = C_3 + C_4 R^{3-D}, \tag{103}$$

$$\frac{\rho R}{\varepsilon_0\,(D-1)} = (D-3)\,C_4 R^{2-D}. \tag{104}$$

Then we have

$$C_3 = -\frac{\rho R^2}{2\,\varepsilon_0\,(D-3)}, \quad C_4 = \frac{\rho R^{D-1}}{\varepsilon_0\,(D-1)\,(D-3)}. \tag{105}$$

As a result, the potential is

$$\varphi(r) = -\frac{\rho r^2}{2\,\varepsilon_0\,(D-1)} \quad (0 < r \leqslant R), \tag{106}$$

$$\varphi(r) = -\frac{\rho R^2}{2\,\varepsilon_0\,(D-3)} + \frac{\rho R^{D-1}}{\varepsilon_0\,(D-1)\,(D-3)}\, r^{3-D} \quad (r \geqslant R). \tag{107}$$

Using (100), (106) and (107), the electric field has the form

$$\mathbf{E}(r) = \frac{\rho r}{\varepsilon_0\,(D-1)}\,\mathbf{e}_r \quad (0 < r \leqslant R), \tag{108}$$

$$\mathbf{E}(r) = \frac{\rho R^{D-1}\, r^{2-D}}{\varepsilon_0\,(D-1)}\,\mathbf{e}_r \quad (r \geqslant R). \tag{109}$$

For $D = 3$, we get the well-known results of non-fractal case.

Eq. (108) can be represented in the form

$$\mathbf{E}(r) = \frac{\rho r}{2\varepsilon_0 \varepsilon_{eff,in}}\,\mathbf{e}_r \quad (0 < r \leqslant R), \tag{110}$$

where $\varepsilon_{eff,in} = (D-1)/2$ is an effective permittivity of fractal materials. Consider the charge per unit length

$$\tau_D = \rho V_{D-1} = \rho\,\frac{\pi^{(D-1)/2} R^{D-1}}{\Gamma((D+1)/2)}. \tag{111}$$

For $D = 3$, Eq. (111) gives the value $\tau_3 = \rho\,\pi R^2$ for non-fractal charge cylinder. Using (111) Eq. (109) can be represented in the form

$$\mathbf{E}(r) = \frac{1}{2\,\pi\,\varepsilon_0\,\varepsilon_{eff,out}}\,\frac{\tau_D}{r^{D-2}}\,\mathbf{e}_r \quad (r \geqslant R), \tag{112}$$

where the effective permittivity

$$\varepsilon_{eff,out} = \frac{(D-1)}{2\,\pi^{(3-D)/2}\,\Gamma((D-1)/2)}. \tag{113}$$

The electric field in the fractal homogeneous charged cylinder is analogous to the non-fractal case up to the factor $\varepsilon_{eff,in}$. We have a linear dependence on the distance from the cylinder axis for $0 < r \leqslant R$. Electric field outside the fractal charged cylinder differs from non-fractal case. For $r \geqslant R$, we have power-law dependence on the distance from the cylinder axis. In addition the electric field outside the cylinder is reduced by the effective permittivity $\varepsilon_{eff,out}$.

9. Conclusion

In this paper, differential operators of vector calculus for non-integer dimensional space is suggested to describe fractal media and materials in the framework of continuum models. The first and second order differential operators for non-integer dimensional space are proposed for rotationally covariant scalar and vector functions. We consider some applications for the case of spherical and axial symmetries of the fractal material. Elasticity of fractal hollow ball and fractal cylindrical pipe, heat distribution in fractal media, and electrostatic field of fractal charged cylinder are described to illustrate the suggested approach. In general, we can consider not only first and second order differential operators for non-integer dimensional space. The differential and integral operators of fractional orders can also be considered for non-integer dimensional spaces to take into account non-locality of materials. We can note that a dimensional continuation of the Riesz fractional integrals and derivatives [81,82] to generalize differential and integrals of fractional orders for non-integer dimensional space has been considered in [83].

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