



Non-standard extensions of gradient elasticity: Fractional non-locality, memory and fractality



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ABSTRACT

Derivatives and integrals of non-integer order may have a wide application in describing complex properties of materials including long-term memory, non-locality of power-law type and fractality. In this paper we consider extensions of elasticity theory that allow us to describe elasticity of materials with fractional non-locality, memory and fractality. The basis of our consideration is an extension of the usual variational principle for fractional non-locality and fractality. For materials with power-law non-locality described by Riesz derivatives of non-integer order, we suggest a fractional variational equation. Equations for fractal materials are derived by a generalization of the variational principle for fractal media. We demonstrate the suggested approaches to derive corresponding generalizations of the Euler–Bernoulli beam and the Timoshenko beam equations for the considered fractional non-local and fractal models. Various equations for materials with fractional non-locality, fractality and fractional acceleration are considered.

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1. Introduction

Derivatives and integrals of non-integer orders [1–5] have wide applications in mechanics and physics [6–15]. The tools of fractional derivatives and integrals allow us to investigate the behavior of materials and systems that are characterized by power-law non-locality, power-law long-term memory and fractal properties. As concluded from the above listed works, there are different definitions of fractional derivatives such as Riemann–Liouville, Riesz, Caputo, Grünwald–Letnikov, Marchaud, Weyl, Sonin–Letnikov and others. The specific choice of fractional derivatives for a particular application, it thus depends on the taste of the investigator and the nature of the material or system at hand. Many properties of standard differentiation and integration do not extend in the fractional case and fractional counterpart of popular models need to be rederived, each on individual basis.

Usually non-local continuum mechanics are treated with two approaches [16]: The gradient elasticity theory (weak non-locality) and the integral non-local theory (strong non-locality). The fractional calculus can, in fact, be used to formulate a generalization of non-local theory of elasticity in both forms: fractional gradient elasticity (weak power-law non-locality) and fractional integral elasticity (strong power-law non-locality). In this paper, we consider fractional generalizations of

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the gradient elasticity theory only. In particular, we suggest fractional generalizations of a rather popular robust GRADELA model proposed by Aifantis and co-workers [17–22] for the following cases:

- (1) The elasticity of materials with power-law non-locality that can be described by derivatives of non-integer order. Both 1D and 3D models are discussed.
- (2) The elasticity of materials with power-law memory that can be described by fractional time derivatives for the internal inertia or combined strain-acceleration fractional gradient terms.
- (3) The elasticity of materials with fractal structure that can be described by fractional integrals in the framework of fractional continuum models.

The basis of our consideration is an extension of the usual variational principle for materials with fractional non-locality, memory and fractality. For 3D spatial fractional models we also use the apparatus of fractional vector calculus. An extension of the traditional calculus of variations for systems described by fractional derivatives was first proposed by Agrawal in [23] for the Riemann–Liouville derivatives. Then it has been extended for other type of fractional derivatives [24–29], and fractional integrals [30]. For materials with power-law non-locality and memory, we suggest a new fractional variational principle for Lagrangians with Riesz fractional derivatives. A possible generalization of gradient elasticity theory for fractal materials was alluded in [31]. In this paper we describe fractal materials by using the fractional continuum formalism suggested in [32,33] (see also [34–37]). To obtain governing equations for fractional integral continuum models of fractal materials, we employ a generalization of the holonomic variational principle suggested in [36,35]. In this connection, we note that extremum and variational principles for non-gradient but fractal elastic materials within a fractional continuum model framework have been considered in [37,38].

The Euler–Bernoulli beam theory may be viewed as a benchmark example of the classical linear theory of elasticity. It provides tools for calculating the load-carrying and deflection characteristics of beams subjected to lateral loads only. In order to illustrate the implications of the suggested fractional approaches in this paper, we use a variational principle to derive the corresponding generalizations of the static and dynamic Euler–Bernoulli beam model, as well as that of the Timoshenko beam model for the fractional non-local and fractal cases. Solutions to some of these equations for fractional non-local and fractal beams are considered.

Next, we list some non-standard generalizations of constitutive relations for gradient elasticity models. First, we recall the linear elastic constitutive relations for isotopic and homogeneous bodies, i.e.,

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{1}$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain tensor, whereas λ and μ are the Lamé coefficients. In [17–19] it was suggested a generalization of the constitutive relations (1) by a gradient modification that contains the Laplacian Δ in the form

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_s^2 \Delta (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{2}$$

where l_s is an internal length scale parameter [16]. To describe complex materials characterized by non-locality of power-law type, long-term memory, and fractality, we should further generalize Eq. (1) and its gradient counterpart given by Eq. (2). In this paper, we consider the following non-standard generalizations of the gradient stress–strain relation.

- (1) The fractional gradient elasticity models with power-law non-locality

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_s^2 (\alpha) (-^R \Delta)^{\alpha/2} (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{3}$$

where $(-^R \Delta)^{\alpha/2}$ is the fractional generalization of the Laplacian in the Riesz form, and

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_s^2 (\alpha) {}^C \Delta_W^\alpha (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{4}$$

where ${}^C \Delta_W^\alpha$ is the fractional Laplacian in the Caputo form.

- (2) The combined fractional strain gradient-internal inertia model with power-law memory and non-locality

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - \left(l_s^2 (\alpha) (-^R \Delta)^{\alpha/2} + l_d^2 (\beta) ({}^R D_t^\beta)^2 \right) (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{5}$$

where $({}^R D_t^\beta)^2$ is the square of the derivative of non-integer order β with respect to time t , which describes acceleration with power-law memory.

- (3) The gradient elasticity models for fractal materials

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_F^2 (D, d) \Delta^{(D,d)} (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{6}$$

where $\Delta^{(D,d)}$ is the “fractal-Laplacian” that takes into account the power-law density of states of the fractal medium under consideration.

The plan of the paper is as follows: In Section 2, we consider one-dimensional (1D) fractional gradient elasticity models. A variational principle for these models is suggested. Fractional Euler–Bernoulli and Timoshenko beam equations are derived. Solutions for the fractional static and dynamic Euler–Bernoulli beam governing equations are proposed. Corresponding frac-

tional beam models with combined strain-internal inertia gradient terms are also considered. Moreover, solutions of the relevant generalized equation and dispersion law for this model are derived. In Section 3, three-dimensional (3D) fractional gradient elasticity models are formulated and discussed. In particular, 3D problems with spherical symmetry based on the Riesz fractional derivative are considered. In addition, fractional 3D gradient elasticity models based on fractional vector calculus are suggested. The operator split method for solving the relevant fractional gradient elasticity equations is formulated. To illustrate the potential of this method, a simple fractional gradient model is considered as an application and an explicit solution is provided. In Section 4, some basic concepts for extending gradient elasticity models to fractal media are suggested. The equilibrium equations for fractal materials are first derived. A variational principle for obtaining gradient elasticity equations for fractal materials is then proposed. Finally, in Section 5, generalizations of the Euler–Bernoulli and Timoshenko beam equations for fractal materials and the corresponding equations for the combined strain-acceleration gradients fractal beam models are derived.

2. Fractional 1D gradient elasticity

Fractional elasticity models are those for which non-locality of power-law type is described by using derivatives and integrals of non-integer order. We can derive such phenomenological fractional elasticity models by using a variational principle for a Lagrangian with fractional derivatives. A generalization of the traditional calculus of variations for systems described by Riemann–Liouville fractional derivatives has been suggested by Agrawal in [23]. Then, extensions of variational calculus for the Riemann–Liouville derivatives [24], the Caputo derivative [25–27], the Hadamard derivative [28], the Riesz derivative [29], as well as fractional integrals [30], have been derived.

If we use the fractional derivatives of Riemann–Liouville, Caputo, Liouville, Marchaud, then we should take into account the left-sided and the right-sided fractional derivatives in the Lagrangian. The correspondent fractional Euler–Lagrange equations contain the left-sided and the right-sided fractional derivatives also. In addition, the integration by parts, which is used in the derivation of the Euler–Lagrange equations from the variational principle, transforms the left-sided derivatives into the right-sided (see Eq. 2.64 of [1]). As a result, we obtain a mixture of left-sided and the right-sided derivatives in the equations of motion. Unfortunately, these Euler–Lagrange equations can be solved for a very narrow class of Lagrangians only.

In this paper, we suggest a fractional variational principle for systems that are described by Riesz fractional derivatives [1,2]. The suggested principle differs from the one proposed in [29]. We take advantage of the fact that the Riesz derivative does not involve two forms, i.e., left-sided and right-sided derivatives. In addition, integration by parts transforms the Riesz fractional derivative into itself. The corresponding fractional Euler–Lagrange equations can be solved for a wide class of Lagrangians that describe nonlocal materials by the methods described in [2]. Moreover, the Riesz fractional derivatives naturally arise in the elasticity theory based on lattice models [52–56]. As an example, we derive the fractional gradient generalization of the Euler–Bernoulli beam model and provide some general solutions of the corresponding equations for both static and dynamics configurations.

2.1. Fractional 1D gradient elasticity from variational principle

To generalize standard variational principles for fractional nonlocal models, we write all expressions in dimensionless coordinate variables. We can introduce the dimensionless variables $x_k = x'_k/l_0$, $\mathbf{r} = \mathbf{r}'/l_0$, where l_0 is a characteristic scale. This allows us to have usual physical dimensions of measured quantities.

The equation for the fractional gradient elasticity can be derived as the Euler–Lagrange equation of the following action

$$S[w] = \int dt \int dx \mathcal{L}(w, D_t^1 w, {}^R D_x^{\alpha_1} w, {}^R D_x^{\alpha_2} w), \tag{7}$$

where $\mathcal{L}(w, D_t^1 w, {}^R D_x^{\alpha_1} w, {}^R D_x^{\alpha_2} w)$ is the Lagrangian defining the 1D fractional elasticity model, $w = w(x, t)$ denotes the displacement field, and x is the dimensionless coordinate.

The variation of the action functional (7) with respect to $w(x, t)$ and its derivatives is given by

$$\delta S[w] = \int dt \int dx \delta \mathcal{L} = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial w} \delta w + \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) \delta(D_t^1 w) + \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} \right) \delta({}^R D_x^{\alpha_1} w) + \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} \right) \delta({}^R D_x^{\alpha_2} w) \right], \tag{8}$$

where, in the absence of non-holonomic constraints, the variation and fractional derivatives commute, i.e.

$$\delta(D_t^1 w) = D_t^1(\delta w), \quad \delta({}^R D_x^{\alpha_1} w) = {}^R D_x^{\alpha_1}(\delta w), \quad \delta({}^R D_x^{\alpha_2} w) = {}^R D_x^{\alpha_2}(\delta w).$$

In order to utilize the fractional variational principle, we should perform the operation of integration by parts. Unfortunately, integration by parts transforms left-sided derivatives into right-sided ones for the most commonly used types of fractional derivatives. For the Liouville fractional derivatives

$$({}^L D_{\pm}^{\alpha} f)(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_0^{\infty} \frac{f(x \mp z)}{z^{\alpha+1-n}} dz, \tag{9}$$

the integration by parts (see Eq. 5.17 in Section 5.1 of [1]) has the form

$$\int_{-\infty}^{+\infty} f(x) ({}^L D_+^\alpha g)(x) dx = \int_{-\infty}^{+\infty} ({}^L D_-^\alpha f)(x) g(x) dx. \quad (10)$$

For the Marchaud fractional derivatives, which is defined by

$$({}^M D_\pm^\alpha f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x \mp z)}{z^{\alpha+1}} dz, \quad (11)$$

the integration by parts (see Eq. 6.27 in Corollary 2 of Theorem 6.2 of [1]) has the form

$$\int_{-\infty}^{+\infty} f(x) ({}^M D_+^\alpha g)(x) dx = \int_{-\infty}^{+\infty} ({}^M D_-^\alpha f)(x) g(x) dx. \quad (12)$$

This relation is valid for functions $f(x) \in L_s(\mathbb{R})$, $f(x) \in L_t(\mathbb{R})$, such that $({}^M D_+^\alpha g)(x) \in L_p(\mathbb{R})$ and $({}^M D_-^\alpha f)(x) \in L_r(\mathbb{R})$, where $p > 1$, $r > 1$,

$$\frac{1}{p} + \frac{1}{r} = 1 + \alpha, \quad \frac{1}{s} = \frac{1}{p} - \alpha, \quad \frac{1}{t} = \frac{1}{r} - \alpha.$$

We suggest the use of Riesz fractional derivatives. It is known (see Section 20.1 of [1]) that the connection of the Riesz fractional derivative to the Marchaud fractional derivatives has the form

$$({}^R D_x^\alpha f)(x) = \frac{1}{2 \cos(\alpha\pi/2)} \left(({}^M D_+^\alpha f)(x) + ({}^M D_-^\alpha f)(x) \right), \quad (13)$$

where $\alpha > 0$, and $\alpha \neq 1, 2, 3, \dots$. Here ${}^R D_x^\alpha$ is the Riesz fractional derivative defined by the equation

$$({}^R D_x^\alpha f)(x) = -\frac{\alpha}{2\Gamma(1-\alpha) \cos(\alpha\pi/2)} \int_0^\infty \frac{f(x+z) - 2f(x) + f(x-z)}{z^{\alpha+1}} dz, \quad (14)$$

where $x \in \mathbb{R}$. Note that the Riesz derivative for an integer $\alpha = 2n$ gives

$$({}^R D_x^{2n} f)(x) = (-1)^n D_x^{2n} f(x), \quad (15)$$

where $n \in \mathbb{N}$, i.e.

$${}^R D_x^2 = -D_x^2, \quad {}^R D_x^4 = D_x^4, \quad {}^R D_x^6 = -D_x^6. \quad (16)$$

Using relations (13) and (12), we obtain the equation of the integration by parts for the Riesz fractional derivative (13) in the form

$$\int_{-\infty}^{+\infty} f(x) ({}^R D_x^\alpha g)(x) dx = \int_{-\infty}^{+\infty} ({}^R D_x^\alpha f)(x) g(x) dx. \quad (17)$$

As a result, integration by parts in Eq. (17) does not change the type of derivative, and also does not change the sign in front of the integral.

Using the integration by parts given by Eq. (17), we can rewrite the variation in Eq. (8) as

$$\delta S[w] = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial w} \delta w - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + {}^R D_x^{\alpha_1} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} \right) + {}^R D_x^{\alpha_2} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} \right) \right] \delta w. \quad (18)$$

Then, the stationary action principle in the form of the holonomic variational equation

$$\delta S[w] = 0$$

yields the equation

$$\frac{\partial \mathcal{L}}{\partial w} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + {}^R D_x^{\alpha_1} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} \right) + {}^R D_x^{\alpha_2} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} \right) = 0. \quad (19)$$

This is the fractional Euler–Lagrange equation for the model described by the Lagrangian $\mathcal{L} = \mathcal{L}(w, D_t^1 w, {}^R D_x^{\alpha_1} w, {}^R D_x^{\alpha_2} w)$. In the next section, we use this equation to establish a fractional generalization of the Euler–Bernoulli beam model.

2.2. Fractional Euler–Bernoulli beam equation from variational principle

The Lagrangian of Euler–Bernoulli beams with gradient power-law non-locality has the form

$$\begin{aligned} \mathcal{L}(w, D_t^1 w, {}^R D_x^{\alpha_1} w, {}^R D_x^{\alpha_2} w) &= \frac{1}{2} \mu \left(D_t^1 w(x, t) \right)^2 - \frac{1}{2} (EI) \left({}^R D_x^{\alpha_1} w(x, t) \right)^2 - \\ &- \frac{1}{2} (EI) I_s^2(\alpha_2) \left({}^R D_x^{\alpha_2} w(x, t) \right)^2 + q(x, t) w(x, t). \end{aligned} \quad (20)$$

The curve $w(x) = u_y(x)$ describes the deflection of the beam in the y direction at some position x . As we have already noted, x and $I_s^2(\alpha_2)$ are dimensionless values. The first term represents the kinetic energy, where $\mu = \rho A$ is the mass per unit length; the second term describes the potential energy due to internal forces (when considered with a negative sign); and the third term is the potential energy due to the external load $q(x)$. Note that in the Lagrangian of Eq. (20) the second term has a negative sign, since integration by parts in Eq. (17) does not change the sign in front of the integral, in contrast to the standard case.

For the usual case of $\alpha_1 = 2$ and $\alpha_3 = 3$, the Lagrangian given by Eq. (20) is

$$\mathcal{L}(w, D_t^1 w, D_x^2 w, D_x^3 w) = \frac{1}{2} \mu (D_t^1 w(x, t))^2 - \frac{1}{2} (EI) (D_x^2 w(x, t))^2 + \frac{1}{2} (EI) I_s^2 (D_x^3 w(x, t))^2 + q(x, t) w(x, t). \tag{21}$$

For the fractional case, the Lagrangian (20) leads to the expressions

$$\frac{\partial \mathcal{L}}{\partial w} = q(x, t) \quad \frac{\partial \mathcal{L}}{\partial D_t^1 w(x, t)} = \mu D_t^1 w(x, t), \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w(x, t)} = -(EI) {}^R D_x^{\alpha_1} w(x, t), \quad \frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w(x, t)} = -(EI) I_s^2(\alpha_2) {}^R D_x^{\alpha_2} w(x, t). \tag{23}$$

Substitution of Eqs. (22) and (23) into the Euler–Lagrange Eq. (19) gives

$$\mu D_t^2 w + {}^R D_x^{\alpha_1} ((EI) ({}^R D_x^{\alpha_1} w) + {}^R D_x^{\alpha_2} ((EI) I_s^2(\alpha_2) {}^R D_x^{\alpha_2} w) - q(x, t) = 0, \tag{24}$$

which is the governing equation for the dynamics of a fractional non-local Euler–Bernoulli beam.

When the beam is homogeneous, E and I are independent of x , and the fractional Euler–Bernoulli beam equation assumes the simpler form

$$\mu D_t^2 w + (EI) ({}^R D_x^{\alpha_1})^2 w + (EI) I_s^2(\alpha_2) ({}^R D_x^{\alpha_2})^2 w - q(x, t) = 0. \tag{25}$$

For a wide class of functions $w(x)$ the properties of the fractional Riesz derivatives allows us to write Eq. (25) as

$$\mu D_t^2 w + (EI) {}^R D_x^{2\alpha_1} w + (EI) I_s^2(\alpha_2) {}^R D_x^{2\alpha_2} w - q(x, t) = 0. \tag{26}$$

In general, we should consider an effective source term $q_{eff}(x)$ instead of $q(x)$, where $q_{eff}(x)$ contains the function $q(x)$ and deviations from the semigroup property for the Riesz derivatives as described for the fractional gradient model with Caputo derivatives dealt with in [54].

For materials without non-locality and memory, we have $\alpha_1 = 2, \alpha_2 = 3$, and then Eq. (25) obtains the form

$$\mu D_t^2 w + EID_x^4 w - EI I_s^2 D_x^6 w - q(x, t) = 0. \tag{27}$$

This is the gradient elasticity Euler–Bernoulli beam equation derived earlier in [16] for the case of integer-order derivatives and non-fractal media.

2.3. Solution of fractional static Euler–Bernoulli beam equation

For the static case ($D_t^1 w = 0$ and $q(x, t) = q(x)$), Eq. (26) has the form

$${}^R D_x^{2\alpha_1} w + I_s^2(\alpha_2) {}^R D_x^{2\alpha_2} w = (EI)^{-1} q(x). \tag{28}$$

Using Corollary 5.14 of [2], we can state that a particular solution of Eq. (28) is

$$w(x) = (EI)^{-1} \int_{-\infty}^{+\infty} G_{2\alpha_1, 2\alpha_2}(x - x') q(x') dx', \tag{29}$$

where $G_{\alpha_1, \alpha_2}(x)$ is a Green’s type function of the form

$$G_{2\alpha_1, 2\alpha_2}(x) = \int_0^\infty \frac{\cos(\lambda|x|)}{\lambda^{2\alpha_1} + I_s^2(\alpha_2) \lambda^{2\alpha_2}} d\lambda. \tag{30}$$

Here $\alpha_1 > 0, \alpha_2 > 0$ and $I_s^2(\alpha_2) \neq 0$.

For a point load of intensity q_0 , i.e. a load $q(x)$ of the form [66]

$$q(x) = q_0 \delta(x), \tag{31}$$

where $\delta(x)$ denotes the Dirac delta-function, the displacement field $w(x)$ has a simple form $w(x) = (q_0/EI) G_{2\alpha_1, 2\alpha_2}(x)$ given by expression

$$w(x) = \frac{2q_0}{\pi EI} \int_0^\infty \frac{\cos(\lambda|x|)}{\lambda^{2\alpha_1} + I_s^2(\alpha_2) \lambda^{2\alpha_2}} d\lambda, \tag{32}$$

where the definition given by Eq. (30) for $G_{2\alpha_1, 2\alpha_2}(x)$ has been used. For the usual non-fractional case, the solution of the static Euler–Bernoulli beam equation with the external point-load is given by Eq. (32) with $\alpha_1 = 2$ and $\alpha_3 = 3$.

2.4. Solution of fractional dynamic Euler–Bernoulli beam equation

For a plane wave traveling in a fractional non-local material with frequency ω , the governing fractional equation is

$$-\mu\omega^2 w_p(x) + (EI)({}^R D_x^{\alpha_1})^2 w_p(x) + (EI)I_s^2(\alpha_2)({}^R D_x^{\alpha_2})^2 w_p(x) - q_p(x) = 0, \quad (33)$$

where $w(x, t) = e^{-i\omega t} w_p(x)$, and we have also used the notation $q(x, t) = e^{-i\omega t} q_p(x)$. For a wide class of functions $w_p(x)$, Eq. (33) can be expressed as

$${}^R D_x^{2\alpha_1} w_p(x) + I_s^2(\alpha_2) {}^R D_x^{2\alpha_2} w_p(x) - \frac{\mu\omega^2}{EI} w_p(x) = (EI)^{-1} q_p(x). \quad (34)$$

Using Theorem 5.24 of [2], we can obtain a particular solution of Eq. (34) as

$$w_p(x, \omega) = (EI)^{-1} \int_{-\infty}^{+\infty} G_{2\alpha_1, 2\alpha_2}(x - x', \omega) q_p(x') dx', \quad (35)$$

where $G_{2\alpha_1, 2\alpha_2}^{(\omega)}(x)$ is a Green's type function of the form

$$G_{2\alpha_1, 2\alpha_2}(x, \omega) = 2 \int_0^{\infty} \frac{\cos(\lambda|x|)}{\lambda^{2\alpha_1} + I_s^2(\alpha_2)\lambda^{2\alpha_2} - \mu\omega^2/(EI)} d\lambda. \quad (36)$$

Here $\alpha_1 > 0$, $\alpha_2 > 0$, $I_s^2(\alpha_2) \neq 0$ and $\mu\omega^2 \neq 0$, $\mu = \rho A$. For the point-load case (31), the solution given by Eq. (35) is reduced to

$$w_p(x, \omega) = \frac{2q_0}{EI} \int_0^{\infty} \frac{\cos(\lambda|x|)}{\lambda^{2\alpha_1} + I_s^2(\alpha_2)\lambda^{2\alpha_2} - \mu\omega^2/(EI)} d\lambda. \quad (37)$$

For the usual non-fractional case, the solution of dynamic Euler–Bernoulli beam equation with an external point-load is given by Eq. (37) with $\alpha_1 = 2$ and $\alpha_3 = 3$.

2.5. Fractional gradient Timoshenko beam equations

In the Timoshenko beam theory the displacement vector $\mathbf{u}(x, y, z, t)$ of the beam is assumed to be given by

$$u_x(x, y, z, t) = -z\varphi(x, t) \quad u_y(x, y, z, t) = 0, \quad u_z(x, y, z, t) = w(x, t), \quad (38)$$

where (x, y, z) are the coordinates of a point in the beam, (u_x, u_y, u_z) are the corresponding components of the displacement vector, $\varphi = \varphi(x, t)$ is the angle of rotation of the normal to the mid-surface of the beam, and $w = w(x, t)$ is the displacement of the mid-surface in the z -direction.

To obtain a fractional generalization of the relevant gradient beam equation we use a fractional variational principle and a generalization of the Timoshenko beam Lagrangian. The appropriate form of such Lagrangian with fractional gradient non-locality, is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\rho I \left(D_t^1 \varphi(x, t) \right)^2 + \frac{1}{2}\rho A \left(D_t^1 w(x, t) \right)^2 - \frac{1}{2}(kGA) \left({}^R D_x^{\alpha_1} w(x, t) - \varphi(x, t) \right)^2 - \frac{1}{2}(EI) \left({}^R D_x^{\beta_1} \varphi(x, t) \right)^2 \\ & - \frac{1}{2}(kGA) I_s^2 \left({}^R D_x^{\alpha_2} w(x, t) - {}^R D_x^{\beta_1} \varphi \right)^2 - \frac{1}{2}(EI) I_s^2 \left({}^R D_x^{\beta_2} \varphi(x, t) \right)^2, \end{aligned} \quad (39)$$

where (x, y, z) are dimensionless coordinates. Note again that we use dimensionless coordinates, such that the relevant quantities of fractional models have the same physical dimension as corresponding one for non-fractional models.

Then, in view of the expressions

$$\frac{\partial \mathcal{L}}{\partial D_t^1 w} = \rho A D_t^1 w, \quad \frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} = -kGA \left({}^R D_x^{\alpha_1} w - \varphi \right), \quad (40)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = kGA \left({}^R D_x^{\alpha_1} w - \varphi \right) \quad \frac{\partial \mathcal{L}}{\partial D_t^1 \varphi} = \rho I D_t^1 \varphi, \quad (41)$$

$$\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\beta_1} \varphi} = -EI {}^R D_x^{\beta_1} \varphi + I_s^2 kGA {}^R D_x^{\alpha_2} w - I_s^2 kGA {}^R D_x^{\beta_1} \varphi, \quad (42)$$

$$\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} = -I_s^2 kGA \left({}^R D_x^{\alpha_2} w - {}^R D_x^{\beta_1} \varphi \right), \quad \frac{\partial \mathcal{L}}{\partial {}^R D_x^{\beta_2} \varphi} = -I_s^2 EI {}^R D_x^{\beta_2} \varphi, \quad (43)$$

the stationary action principle gives the following Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial w} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + {}^R D_x^{\alpha_1} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} \right) + {}^R D_x^{\alpha_2} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} \right) = 0, \tag{44}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 \varphi} \right) + {}^R D_x^{\beta_1} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\beta_1} \varphi} \right) + {}^R D_x^{\beta_2} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\beta_2} \varphi} \right) = 0. \tag{45}$$

Eqs. (44) and (45) are the Euler–Lagrange equations for the fractional gradient elasticity model described by the Lagrangian (39).

Substitution of Eqs. (40)–(43) into Eqs. (44) and (45) gives the following fractional gradient Timoshenko beam equations for the displacement $w = w(x)$ and the rotation $\varphi = \varphi(x)$,

$$\rho A D_t^2 w = {}^R D_x^{\alpha_1} (-kGA({}^R D_x^{\alpha_1} w - \varphi)) + {}^R D_x^{\alpha_2} (-l_s^2 kGA({}^R D_x^{\alpha_2} w - {}^R D_x^{\beta_1} \varphi)), \tag{46}$$

$$\rho I D_t^2 \varphi = kGA({}^R D_x^{\alpha_1} w - \varphi) + {}^R D_x^{\beta_1} (-EI {}^R D_x^{\beta_1} \varphi + l_s^2 kGA {}^R D_x^{\alpha_2} w - l_s^2 kGA {}^R D_x^{\beta_1} \varphi) + {}^R D_x^{\beta_2} (-l_s^2 EI {}^R D_x^{\beta_2} \varphi). \tag{47}$$

For homogeneous materials, Eqs. (46) and (47) take the form

$$\rho A D_t^2 w = -kGA {}^R D_x^{\alpha_1} ({}^R D_x^{\alpha_1} w - \varphi) - l_s^2 kGA {}^R D_x^{\alpha_2} ({}^R D_x^{\alpha_2} w - {}^R D_x^{\beta_1} \varphi), \tag{48}$$

$$\rho I D_t^2 \varphi = kGA({}^R D_x^{\alpha_1} w - \varphi) - EI {}^R D_x^{\beta_1} {}^R D_x^{\beta_1} \varphi + l_s^2 kGA {}^R D_x^{\beta_1} {}^R D_x^{\alpha_2} w - l_s^2 kGA {}^R D_x^{\beta_1} {}^R D_x^{\beta_1} \varphi - l_s^2 EI {}^R D_x^{\beta_2} {}^R D_x^{\beta_2} \varphi. \tag{49}$$

For a wide class of functions $w(x, t)$ and $\varphi(x, t)$, Eqs. (48) and (49) can be rewritten as

$$\rho A D_t^2 w = -kGA({}^R D_x^{2\alpha_1} w - {}^R D_x^{\alpha_1} \varphi) - l_s^2 kGA({}^R D_x^{2\alpha_2} w - {}^R D_x^{\alpha_2 + \beta_1} \varphi), \tag{50}$$

$$\rho I D_t^2 \varphi = kGA({}^R D_x^{\alpha_1} w - \varphi) - EI {}^R D_x^{2\beta_1} \varphi + l_s^2 kGA {}^R D_x^{\alpha_2 + \beta_1} w - l_s^2 kGA {}^R D_x^{2\beta_1} \varphi - l_s^2 EI {}^R D_x^{2\beta_2} \varphi. \tag{51}$$

If $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$, Eqs. (50) and (51) reduce to the well-known Timoshenko beam equations. If $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 2$, Eqs. (50) and (51) reduce to the form of the gradient generalization of the Timoshenko beam equations. In general, the Riesz fractional derivatives do not commute and

$${}^R D_x^{\alpha} {}^R D_x^{\beta} \neq {}^R D_x^{\alpha + \beta}. \tag{52}$$

In this case, Eqs. (48) and (49) give Eqs. (50) and (51) with an additional term in the form of an effective source terms that contain the deviations from the semigroup property for the Riesz derivatives as it was described in [54].

2.6. Combined strain-acceleration fractional gradients beam model

Let us now consider internal inertia effects, i.e. effects of combined strain-acceleration gradients on fractional nonlocal beams. We start with the governing equation of a gradient elasticity Euler–Bernoulli beam equation with internal inertia or acceleration gradients [16], i.e.,

$$\rho A D_t^2 w + E I D_x^4 w - E I l_s^2 D_x^6 w + \rho I l_d^2 D_t^2 D_x^4 w - q(x, t) = 0, \tag{53}$$

where (ρ, A, E, I) have their usual meaning, (x, t) are dimensionless variables, and (l_s^2, l_d^2) are scale parameters. The fractional generalization of Eq. (53) can be written in the form

$$-\rho A {}^R D_t^{2\beta} w + E I {}^R D_x^{2\alpha_1} w + E I l_s^2 (\alpha_2) {}^R D_x^{2\alpha_2} w - \rho I l_d^2 (\alpha_3) {}^R D_t^{2\beta} {}^R D_x^{2\alpha_3} w - q(x, t) = 0, \tag{54}$$

where ${}^R D_t^{2\beta}$ is the Riesz fractional derivative [2] with respect to time. Using Eqs. (16) and (54) with $\alpha_1 = \alpha_3 = 2$, $\beta = 1$ and $\alpha_2 = 3$ gives Eq. (53).

Eq. (54) can be obtained from the stationary action principle and the correspondent fractional Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial w} + {}^R D_t^{\beta} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_t^{\beta} w} \right) + {}^R D_x^{\alpha_3} {}^R D_t^{\beta} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_3} {}^R D_t^{\beta} w} \right) + {}^R D_x^{\alpha_1} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_1} w} \right) + {}^R D_x^{\alpha_2} \left(\frac{\partial \mathcal{L}}{\partial {}^R D_x^{\alpha_2} w} \right) = 0, \tag{55}$$

where the Lagrangian

$$\mathcal{L} = \frac{1}{2} \rho A ({}^R D_t^{\beta} w(x, t))^2 - \frac{1}{2} E I ({}^R D_x^{\alpha_1} w(x, t))^2 - \frac{1}{2} E I l_s^2 (\alpha_2) ({}^R D_x^{\alpha_2} w(x, t))^2 + \frac{1}{2} \rho I l_d^2 (\alpha_3) ({}^R D_x^{\alpha_3} {}^R D_t^{\beta} w(x, t))^2 + q(x, t) w(x, t), \tag{56}$$

is used and Eq. (16) is also taken into account.

In the above we use the Riesz fractional derivatives (13) with respect to time to derive Eq. (54) from a variational principle, instead of the Caputo derivatives that are commonly used. Moreover, the Riesz fractional derivatives allow us to obtain a general harmonic solution of the combined strain-acceleration fractional gradient beam model, as we will see in the sequel. At the same time, an interpretation of Riesz fractional derivatives with respect to time can be more complicated in comparison with the left-sided Caputo derivative. In any case, the Riesz fractional time derivative describes a special form of power-law material memory (acceleration with memory) and deserves to be explored in its own right.

2.7. Solution for the combined strain-acceleration fractional gradients beam model

Let us consider the Fourier transform \mathcal{F} of the displacement field by utilizing the properties of the Riesz fractional derivative (see Property 2.34 in [2]) with respect to time

$$(\mathcal{F}^R D_t^\alpha w(x, t))(k) = |k|^\alpha (\mathcal{F}w(x, t))(x, \omega), \tag{57}$$

where $w(x, t)$ belongs to the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions on \mathbb{R}^2 with a compact support. Then Eq. (54) takes the form

$$-\rho A |\omega|^{2\beta} \hat{w} + EI^R D_x^{2\alpha_1} \hat{w} + EI I_s^2(\alpha_2)^R D_x^{2\alpha_2} \hat{w} - \rho I I_d^2(\alpha_3) \omega^{2\beta R} D_x^{2\alpha_3} \hat{w} - \hat{q}(x, \omega) = 0, \tag{58}$$

where $\hat{w}(x, \omega) = (\mathcal{F}w(x, t))(x, \omega)$ and $\hat{q}(x, \omega) = (\mathcal{F}q(x, t))(x, \omega)$. By rewriting this equation in the form

$$EI I_s^2(\alpha_2)^R D_x^{2\alpha_2} \hat{w} + EI^R D_x^{2\alpha_1} \hat{w} - \rho I I_d^2(\alpha_3) \omega^{2\beta R} D_x^{2\alpha_3} \hat{w} - \rho A \omega^{2\beta} \hat{w} = \hat{q}(x, \omega), \tag{59}$$

we can solve it by using Theorem 5.24 of [2] with the coefficients

$$a_0 = \rho A \omega^{2\beta}, \quad a_1 = \rho I I_d^2(\alpha_3) \omega^{2\beta}, \quad a_2 = EI, \quad a_3 = EI I_s^2(\alpha_2). \tag{60}$$

Noting that the Fourier transform of the Riesz fractional derivative with respect to coordinates is defined by

$$\mathcal{F}(^R D^{2\alpha} \hat{w}(x, \omega))(k, \omega) = |k|^{2\alpha} (\mathcal{F}\hat{w})(k, \omega), \tag{61}$$

and applying \mathcal{F} to both sides of Eq. (59) by also using Eq. (61), we obtain

$$(\mathcal{F}\hat{w})(k, \omega) = \left(a_3 |k|^{2\alpha_2} + a_2 |k|^{2\alpha_1} - a_1 |k|^{2\alpha_3} - a_0 \right)^{-1} (\mathcal{F}\hat{w})(k, \omega). \tag{62}$$

Next, we define the fractional analogue of Green's function [2] as

$$G_\alpha(x) = \mathcal{F}^{-1} \left[\left(a_3 |k|^{2\alpha_2} + a_2 |k|^{2\alpha_1} - a_1 |k|^{2\alpha_3} - a_0 \right)^{-1} \right](x) = \int_{\mathbb{R}} \left(a_3 |k|^{2\alpha_2} + a_2 |k|^{2\alpha_1} - a_1 |k|^{2\alpha_3} - a_0 \right)^{-1} e^{+ikx} dk, \tag{63}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index. Then, the following relation holds

$$\int_{\mathbb{R}^n} e^{i(\mathbf{k}, \mathbf{r})} f(|\mathbf{k}|) d^n \mathbf{k} = \frac{(2\pi)^{n/2}}{|\mathbf{r}|^{(n-2)/2}} \int_0^\infty f(\lambda) \lambda^{n/2} J_{n/2-1}(\lambda|\mathbf{r}|) d\lambda \tag{64}$$

for any function f such that the integral in the right-hand side of Eq. (64) is convergent (see Lemma 25.1 of [1]). Here J_ν is the Bessel function of the first kind and we can use for $n = 1$ the expression

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z). \tag{65}$$

Using Eq. (64), the Green's function given by Eq. (63) can be rewritten (see Theorem 5.22 of [2]) in the form

$$G_\alpha(\mathbf{r}) = (2\pi)^{1/2} |\mathbf{x}|^{1/2} \int_0^\infty \frac{\lambda^{1/2} J_{-1/2}(\lambda|\mathbf{x}|) d\lambda}{a_3 \lambda^{2\alpha_2} + a_2 \lambda^{2\alpha_1} - a_1 \lambda^{2\alpha_3} - a_0}, \tag{66}$$

where we used $n = 1$. If $\alpha_2 > 1$, and $a_k \neq 0$, then Eq. (59) is solvable [2]. A particular solution of Eq. (59) can be represented as the convolution of the functions $G(x)$ and $q(x)$, i.e.,

$$\hat{w}(x, \omega) = \int_{\mathbb{R}} G_\alpha(x - x') q(x, \omega) dx', \tag{67}$$

where the Green's function $G_\alpha(z)$ is defined by Eq. (66).

For the case $q(x, \omega) = q_0 \delta(x)$, Eq. (67) gives

$$\hat{w}(x, \omega) = 2q_0 \int_0^\infty \frac{\cos(\lambda|x|) d\lambda}{a_3 \lambda^{2\alpha_2} + a_2 \lambda^{2\alpha_1} - a_1 \lambda^{2\alpha_3} - a_0}. \tag{68}$$

Using Eq. (60), we can write Eq. (68) in the form

$$\hat{w}(x, \omega) = \frac{2q_0}{EI} \int_0^\infty \frac{\cos(\lambda|x|) d\lambda}{I_s^2(\alpha_2) \lambda^{2\alpha_2} + \lambda^{2\alpha_1} - (\rho/E) I_d^2(\alpha_3) \omega^{2\beta} \lambda^{2\alpha_3} - (\rho A)/(EI) \omega^{2\beta}}. \tag{69}$$

For a fractional non-local material without memory ($\beta = 1$) and fractional acceleration gradient ($I_d^2(\alpha_3) = 0$), the corresponding solution is

$$\hat{w}(x, \omega) = \frac{2q_0}{EI|x|} \int_0^\infty \frac{\cos(\lambda|x|) d\lambda}{I_s^2(\alpha_2) \lambda^{2\alpha_2} + \lambda^{2\alpha_1} - (\rho A)/(EI) \omega^2}. \tag{70}$$

This is in fact, the solution given by Eq. (37) for the fractional gradient Euler–Bernoulli beam equation of motion for the point load case of Eq. (31).

2.8. Dispersion law and general harmonic solution of the combined strain-acceleration fractional gradient beam model

Let us now obtain a general harmonic solution of the combined strain-acceleration fractional gradient beam model defined by Eq. (54). Using Property 2.34 in [2], the Fourier transform of $({}^R D_x^\alpha w)(x)$ is given by Eq. (57), where $w(x, t)$ belongs to the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions on \mathbb{R}^2 with a compact support.

The Fourier transform \mathcal{F} of the fractional differential Eq. (54) with $q = 0$ gives

$$-\rho A |\omega|^{2\beta} + EI |k|^{2\alpha_1} + EI I_s^2(\alpha_2) |k|^{2\alpha_2} - \rho I I_d^2(\alpha_3) |k|^{2\alpha_3} |\omega|^{2\beta} = 0, \tag{71}$$

implying that

$$\omega^{2\beta} = \frac{E I}{\rho A} \frac{|k|^{2\alpha_1} + I_s^2(\alpha_2) |k|^{2\alpha_2}}{1 + (I/A) I_d^2(\alpha_3) |k|^{2\alpha_3}}. \tag{72}$$

As a result, we obtain

$$\omega = C_e^{1/\beta} R^{1/\beta} |k|^{\alpha_1/\beta} \sqrt[2\beta]{\frac{1 + I_s^2(\alpha_2) |k|^{2(\alpha_2 - \alpha_1)}}{1 + R^2 I_d^2(\alpha_3) |k|^{2\alpha_3}}}, \tag{73}$$

where

$$R = \sqrt{I/A}, \quad C_e = \sqrt{R/\rho}. \tag{74}$$

The parameter R is called the gyration radius.

In the absence of memory, i.e. $\beta = 1$, Eq. (73) yields

$$\omega = C_e R |k|^{\alpha_1} \sqrt{\frac{1 + I_s^2(\alpha_2) |k|^{2(\alpha_2 - \alpha_1)}}{1 + R^2 I_d^2(\alpha_3) |k|^{2\alpha_3}}}. \tag{75}$$

If $\alpha_1 = \alpha_3 = 2$, $\beta = 1$ and $\alpha_2 = 3$, Eq. (73) gives

$$\omega = C_e R k^2 \sqrt{\frac{1 + I_s^2 k^2}{1 + R^2 I_d^2 k^4}}, \tag{76}$$

which is precisely the dispersion relation obtained earlier (Eq. (50) of [16]) for the non-fractional combined strain-acceleration gradient beam model. Using Eq. (73), we can obtain the group velocity $C_g = \partial\omega(k)/\partial k$ for the combined strain-acceleration fractional gradient beam model as

$$\frac{C_g}{C_e} = \frac{1}{2\beta} C_e^{(1-\beta)/\beta} R^{1/\beta} \frac{\left(1 + I_s^2(\alpha_2) |k|^{2(\alpha_2 - \alpha_1)}\right)^{(1-2\beta)/(2\beta)}}{\left(1 + R^2 I_d^2(\alpha_3) |k|^{2\alpha_3}\right)^{(1+2\beta)/(2\beta)}} \cdot \left(2\alpha_1 |k|^{2\alpha_1 - 1} + 2\alpha_2 I_s^2(\alpha_2) |k|^{2\alpha_2 - 1} + 2(\alpha_1 + \alpha_3) R^2 I_d^2(\alpha_3) |k|^{2(\alpha_1 + \alpha_3) - 1} + 2(\alpha_2 + \alpha_3) R^2 I_s^2(\alpha_2) I_d^2(\alpha_3) |k|^{2(\alpha_2 + \alpha_3) - 1}\right). \tag{77}$$

If $\alpha_1 = \alpha_3 = 2$, $\beta = 1$ and $\alpha_2 = 3$, Eq. (77) is reduced to

$$\frac{C_g}{C_e} = R k \frac{2 + 3I_s^2 k^2 + R^2 I_d^2 I_s^2 |k|^6}{(1 + I_s^2 k^2)^{1/2} (1 + R^2 I_d^2 k^4)^{3/2}}. \tag{78}$$

This is the well-known normalized form of the corresponding group velocity (see Eq. (51) of [16]) for non-fractional counterpart of the model.

3. Toward 3D fractional gradient elasticity

To develop a fractional gradient elasticity theory in three-dimensions (3D), the following approaches may be used:

- (1) An approach based on the Riesz fractional derivatives and integrals for \mathbb{R}^3 [1,2,50]. This approach is best suited for 3D problems with spherical symmetry. The Riesz fractional derivative can be considered as a non-integer power of the Laplacian. Such a simple 3D fractional gradient elasticity model based on the Riesz fractional derivatives has already been recently considered by the authors in [54], and it can be naturally derived from lattice models with long-range interactions [52,53,55].
- (2) An approach based on fractional vector calculus. Currently, however fractional vector calculus is formulated for a Cartesian coordinate system only [13,60]. The transformation from Cartesian to cylindrical, spherical or other coordinates is prohibitively complicated for fractional derivatives. It is connected with the fact that the formula of fractional derivative of a composite function (see Eq. 2.209 in Section 2.7.3 of [59]) is very complex, i.e.,

$${}_a D_x^\alpha f(g(x)) = \frac{(x-a)^\alpha}{\Gamma(1-\alpha)} f(g(x)) + \sum_{k=1}^\infty C_k^\alpha \frac{k!(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^k (D_g^m f)(g(x)) \sum_{r=1}^k \prod_{r=1}^k \frac{1}{a_r!} \left(\frac{(D_x^r g)(x)}{r!} \right)^{a_r}, \tag{79}$$

where \sum extends over all combinations of non-negative integer values of a_1, a_2, \dots, a_k such that

$$\sum_{r=1}^k r a_r = k, \quad \sum_{r=1}^k a_r = m. \tag{80}$$

These two approaches which allow us to construct 3D fractional nonlocal models of gradient elasticity are briefly discussed below.

3.1. Fractional gradient elasticity based on Riesz derivative

Three-dimensional fractional gradient elasticity models based on the Riesz fractional derivative are possible due to the fact that this fractional derivative is a generalization of the Laplacian in \mathbb{R}^n and, in fact, it can be considered as a non-integer power of the Laplacian. The corresponding 3D fractional gradient elasticity model is described by the following equation (for details see [54])

$$c_\alpha ((-{}^R\Delta)^{\alpha/2} u)(\mathbf{r}) + c_\beta ((-{}^R\Delta)^{\beta/2} u)(\mathbf{r}) = f(\mathbf{r}) \quad (\alpha > \beta), \tag{81}$$

where $\mathbf{r} \in \mathbb{R}^3$ and $r = |\mathbf{r}|$ are dimensionless variables, and $(-{}^R\Delta)^{\alpha/2}$ is the Riesz fractional Laplacian of order α [2]. The coefficients (c_α, c_β) are phenomenological constants and the rest of the symbols have their usual meaning, with u denoting the radial component of the displacement.

For $\alpha > 0$ and suitable functions $u(\mathbf{r}), \mathbf{r} \in \mathbb{R}^3$, the Riesz fractional derivative can be defined [2] in terms of the inverse Fourier transform \mathcal{F}^{-1} by

$$((-{}^R\Delta)^{\alpha/2} u)(\mathbf{r}) = \mathcal{F}^{-1} (|\mathbf{k}|^\alpha (\mathcal{F}u)(\mathbf{k})), \tag{82}$$

where \mathbf{k} denotes the wave vector, $\alpha > 0$ and $x \in \mathbb{R}^n$. The fractional Laplacian in the Riesz form is usually defined in terms of the hypersingular integral

$$((-{}^R\Delta)^{\alpha/2} f)(x) = \frac{1}{d_n(m, \alpha)} \int_{\mathbb{R}^n} \frac{1}{|z|^{\alpha+n}} (\Delta_z^m f)(x) dz, \tag{83}$$

where $m > \alpha > 0$, and $(\Delta_z^m f)(x)$ is a finite difference of order m of a function $f(x)$ with a vector step $z \in \mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$:

$$(\Delta_z^m f)(x) = \sum_{k=0}^m (-1)^k \frac{m!}{k!(m-k)!} f(x - kz),$$

where the constant $d_n(m, \alpha)$ is defined by

$$d_n(m, \alpha) = \frac{\pi^{1+n/2} A_m(\alpha)}{2^\alpha \Gamma(1 + \alpha/2) \Gamma(n/2 + \alpha/2) \sin(\pi\alpha/2)},$$

with

$$A_m(\alpha) = \sum_{j=0}^m (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha.$$

The definition given by Eq. (83) for the fractional Laplacian of order α does not depend on the choice of $m > \alpha$. Its Fourier transform \mathcal{F} satisfies the relationship $(\mathcal{F}(-\Delta)^{\alpha/2}f)(k) = |k|^\alpha(\mathcal{F}f)(k)$, which is valid for the Lizorkin space [1] and the space $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions on \mathbb{R}^n with compact support.

If $\alpha = 4$ and $\beta = 2$, we have the well-known equation of gradient elasticity [16] for the non-fractional case, i.e.,

$$c_2 \Delta u(\mathbf{r}) - c_4 \Delta^2 u(\mathbf{r}) + f(\mathbf{r}) = 0, \tag{84}$$

where

$$c_2 = E, \quad c_4 = \pm l^2 E. \tag{85}$$

Eq. (81) is a fractional partial differential equation with a particular solution (Section 5.5.1. of [2]) of the form

$$u(\mathbf{r}) = \int_{\mathbb{R}^3} G_{\alpha,\beta}^3(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d^3 \mathbf{r}', \tag{86}$$

where the Green's type function is given by the expression

$$G_{\alpha,\beta}^3(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{c_\alpha |\mathbf{k}|^\alpha + c_\beta |\mathbf{k}|^\beta} e^{+i(\mathbf{k},\mathbf{r})} d^3 \mathbf{k}. \tag{87}$$

Using Lemma 25.1 of [1], the kernel function in Eq. (87) can be represented by the equation

$$G_{\alpha,\beta}^3(\mathbf{r}) = \frac{1}{(2\pi)^{3/2} \sqrt{|\mathbf{r}|}} \int_0^\infty \frac{\lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)}{c_\alpha \lambda^\alpha + c_\beta \lambda^\beta} d\lambda, \tag{88}$$

where $J_{1/2}(z) = \sqrt{2/(\pi z)} \sin(z)$ denotes Bessel function of the first kind.

If we consider the deformation of an infinite elastic continuum due to an external field $f(\mathbf{r})$ applied to a very small region, then for distances $|\mathbf{r}|$ which are large in comparison with the size of the region (neighborhood) of load application, we can suppose that $f(\mathbf{r})$ is applied at a point [66]:

$$f(\mathbf{r}) = f_0 \delta(\mathbf{r}). \tag{89}$$

Then, the displacement field $u(\mathbf{r})$ has a simple form $u(\mathbf{r}) = f_0 G_{\alpha,\beta}^3(\mathbf{r})$ given by

$$u(\mathbf{r}) = \frac{1}{(2\pi)^{3/2} \sqrt{|\mathbf{r}|}} \int_0^\infty \frac{\lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)}{c_\alpha \lambda^\alpha + c_\beta \lambda^\beta} d\lambda. \tag{90}$$

3.2. Fractional vector calculus and 3D models

3.2.1. Fractional vector calculus

Fractional vector calculus is a very important tool for describing processes in complex media and materials with non-local properties. It allows us to formulate a dynamical theory of materials with non-locality of power-law type in three dimensions. At present, however, several formulations of fractional vector calculus are either incorrect or inconsistent, leading to errors. It seems that it is possible to define a generalization of grad, div and curl operators by using a fractional derivative $D_{x_k}^\alpha$ instead of the usual derivative $D_{x_k}^1$, where $D_{x_k}^\alpha$ are fractional (Liouville, Riemann–Liouville, Caputo, etc.) derivatives of order α with respect to x_k , $k = 1, 2, 3$. In such an approach, there is considerable arbitrariness in the definition of vector operators. The main problem in fractional vector calculus, however, appears when we try to generalize not only differential vector operators, but also the related integral theorems [60]. In general, a robust framework of fractional vector calculus must include generalizations of the differential operators (gradient, divergence, curl), the integral operations (flux, circulation), and the theorems of Gauss, Stokes and Green.

The main problem in the formulation of fractional integral vector operations is connected with the complex form of the fractional analogue of the Newton–Leibniz formula ${}_a I_b^1 {}_a D_x^1 f(x) = f(b) - f(a)$. In fact, the non-commutativity of D_x^α and ${}_a I_b^\alpha$ does not allow us to derive a convenient Riemann–Liouville fractional counterpart of the Newton–Leibniz formula. For fractional Riemann–Liouville integrals and derivatives, we have the relation

$${}_a I_b^\alpha {}_a D_x^\alpha f(x) = f(b) - \sum_{j=1}^n \frac{(b-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} (D_x^{n-j} {}_a I_x^{n-\alpha} f)(a) \tag{91}$$

holding almost everywhere in $[a, b]$, where $D_x^{n-j} = d^{n-j}/dx^{n-j}$ are integer derivatives, and $n - 1 < \alpha < n$. Here $f(x)$ is a Lebesgue measurable function on $[a, b]$ for which ${}_a I_b^1 f(x) < \infty$, and ${}_a I_b^{n-\alpha} f(x)$ has absolutely continuous derivatives up to order $(n - 1)$ on $[a, b]$. This relation was proved in [1] (see Theorem 2.4 of Section 2.6). For $0 < \alpha < 1$, Eq. (91) gives

$${}_a I_b^\alpha {}_a D_x^\alpha f(x) = f(b) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} {}_a I_b^{1-\alpha} f(x). \tag{92}$$

Obviously, that Eqs. (92) and (91) do not have the usual form of the Newton-Leibniz formula.

A consistent formulation of fractional vector calculus has been realized in [60] by using fractional derivatives and fractional integrals of different types. For this purpose, the Riemann–Liouville integration and the Caputo differentiation are used. The main property is that the Caputo fractional derivative provides an operation that is inverse to the Riemann–Liouville fractional integration from the left. As a result, we can formulate a fractional analogue of the Newton–Leibniz formula in the usual form if the integral is of Riemann–Liouville type and the derivative is of the Caputo type. i.e.,

$${}_a I_b^\alpha {}_a^C D_x^\alpha f(x) = f(b) - f(a), \quad (0 < \alpha < 1), \quad (93)$$

where ${}_a^C D_x^\alpha$ is the Caputo fractional derivative defined by the equation

$${}_a^C D_x^\alpha F(x) = {}_a I_x^{n-\alpha} D_x^n F(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{dx' D_x^n F(x')}{(x-x')^{1+\alpha-n}},$$

where $n-1 < \alpha < n$, and ${}_a I_x^\alpha$ is the Riemann–Liouville fractional integral

$${}_a I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(x')}{(x-x')^{1-\alpha}} dx'.$$

Here $f(x)$ is a real-valued function defined on a closed interval $[a, b]$ such that $f(x) \in AC^1[a, b]$ or $f(x) \in C^1[a, b]$. For details, the reader may consult [60], where the fractional differential operators are defined such that fractional generalizations of integral theorems (Green's, Stokes', Gauss') can be realized. Using this fractional vector calculus [60], fractional differential equations for the conservation of mass, momentum and energy can be obtained for a continuum with power-law non-locality. This allows us to formulate 3D fractional models of continuum mechanics for fluids and solids with non-local properties. In the next subsection, we show how the fractional vector calculus can be used to formulate a fractional generalization of gradient elasticity for the 3D case.

3.2.2. Fractional differential vector operators

To properly define fractional vector operations, we will first introduce the operators that correspond to fractional differentiation and fractional integration. The left-sided Riemann–Liouville fractional integral operator is defined as

$${}_a I_x^\alpha [x'] := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{dx'}{(x-x')^{1-\alpha}}, \quad (\alpha > 0). \quad (94)$$

To designate that the operator given by Eq. (94) acts on a real-valued function $f(x) \in L_1[a, b]$, we employ the notation ${}_a I_x^\alpha [x'] f(x')$. We define the left-sided Caputo fractional differential operator on $[a, b]$ in the form

$${}_a^C D_x^\alpha [x'] := \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{dx'}{(x-x')^{1+\alpha-n}} \frac{\partial^n}{\partial x'^n}, \quad (n-1 < \alpha < n). \quad (95)$$

The Caputo operator defined by Eq. (95) acts on real-valued functions $f(x) \in AC^n[a, b]$ as ${}_a^C D_x^\alpha [x'] f(x')$. We note that the Caputo operator can be represented as

$${}_a^C D_x^\alpha [x'] = {}_a I_x^{n-\alpha} [x'] D^n [x'], \quad (n-1 < \alpha < n).$$

Eq. (93) can be rewritten in the form

$${}_a I_b^\alpha [x] {}_a^C D_x^\alpha [x'] f(x') = f(b) - f(a), \quad (0 < \alpha < 1). \quad (96)$$

In the notations ${}_a I_b^\alpha [x]$ and ${}_a^C D_x^\alpha [x']$, we indicate the variable of integration by the brackets $[\]$, and the lower indices show the limits of integration. Note that in Eq. (96) the variable of integration is x since the result of the integration with respect to x' in the operator ${}_a^C D_x^\alpha [x']$ depends on x only. These notations are more convenient than the ones usually used (see Eq. (93)), since it allows us to take into account the variables of integration and the domain of the operators.

We define a fractional generalization of nabla operator by

$$\nabla_W^\alpha = {}^C D_W^\alpha = \mathbf{e}_1 {}^C D_W^\alpha [x] + \mathbf{e}_2 {}^C D_W^\alpha [y] + \mathbf{e}_3 {}^C D_W^\alpha [z], \quad (n-1 < \alpha < n), \quad (97)$$

where ${}_a^C D_W^\alpha [x_m]$ denotes the Caputo fractional derivative with respect to coordinates x_m . For the parallelepiped $W := \{a \leq x \leq b, c \leq y \leq d, g \leq z \leq h\}$, we have

$${}_a^C D_W^\alpha [x] = {}_a^C D_b^\alpha [x], \quad {}^C D_W^\alpha [y] = {}_c^C D_d^\alpha [y], \quad {}^C D_W^\alpha [z] = {}_g^C D_h^\alpha [z].$$

Let us now give the definitions of fractional gradient, divergence and curl operators in Cartesian coordinates [13,60]. We assume that $f(x)$ and $\mathbf{F}(x)$ are real-valued functions with continuous derivatives up to order $(n-1)$ on $W \subset \mathbb{R}^3$, such that their $(n-1)$ derivatives are absolutely continuous, i.e., $f, \mathbf{F} \in AC^n[W]$.

(1) The fractional gradient is defined by

$$\text{Grad}_W^\alpha f = {}^C \mathbf{D}_W^\alpha f = \mathbf{e}_1 {}^C D_W^\alpha [x_i] f(x, y, z) = \mathbf{e}_1 {}^C D_W^\alpha [x] f(x, y, z) + \mathbf{e}_2 {}^C D_W^\alpha [y] f(x, y, z) + \mathbf{e}_3 {}^C D_W^\alpha [z] f(x, y, z), \quad (98)$$

here $f = f(x, y, z)$ is a $(n - 1)$ times continuously differentiable scalar field such that the derivative $D_{x_i}^{n-1} f$ is absolutely continuous.

(2) The fractional divergence is defined by the equation

$$\text{Div}_W^\alpha \mathbf{F} = ({}^C \mathbf{D}_W^\alpha, \mathbf{F}) = {}^C D_W^\alpha [x_i] F_i(x, y, z) = {}^C D_W^\alpha [x] F_x(x, y, z) + {}^C D_W^\alpha [y] F_y(x, y, z) + {}^C D_W^\alpha [z] F_z(x, y, z), \quad (99)$$

where $\mathbf{F}(x, y, z)$ is a $(n - 1)$ times continuously differentiable vector field such that the derivatives $D_{x_i}^{n-1} F_i$ are absolutely continuous.

(3) The fractional curl operator is defined by

$$\begin{aligned} \text{Curl}_W^\alpha \mathbf{F} &= [{}^C \mathbf{D}_W^\alpha, \mathbf{F}] = \mathbf{e}_i \varepsilon_{lmk} {}^C D_W^\alpha [x_m] F_k \\ &= \mathbf{e}_1 ({}^C D_W^\alpha [y] F_z - {}^C D_W^\alpha [z] F_y) + \mathbf{e}_2 ({}^C D_W^\alpha [z] F_x - {}^C D_W^\alpha [x] F_z) + \mathbf{e}_3 ({}^C D_W^\alpha [x] F_y - {}^C D_W^\alpha [y] F_x), \end{aligned} \quad (100)$$

where $F_k = F_k(x, y, z) \in AC^n[W]$, $(k = 1, 2, 3)$.

(4) Using the notation introduced in Eq. (97), the operator $({}^C \mathbf{D}_W^\alpha)^2$ can be considered as the fractional Laplacian of the Caputo type:

$${}^C \Delta_W^\alpha = ({}^C \mathbf{D}_W^\alpha, {}^C \mathbf{D}_W^\alpha) = ({}^C \mathbf{D}_W^\alpha)^2 = \sum_{l=1}^3 ({}^C D_W^\alpha [x_l])^2. \quad (101)$$

Note that in the general case we have the inequality

$$({}^C D_W^\alpha [x_l])^2 \neq {}^C D_W^{2\alpha} [x_l]. \quad (102)$$

Let us now give the basic relations for the fractional differential vector operators (for details of proofs see [13,60]).

(i) For the scalar field $f = f(x, y, z)$, we have

$$\text{Div}_W^\alpha \text{Grad}_W^\alpha f = {}^C D_W^\alpha [x_i] {}^C D_W^\alpha [x_i] f = \sum_{l=1}^3 ({}^C D_W^\alpha [x_l])^2 f. \quad (103)$$

Using then the notations introduced in Eqs. (97) and (101), we conclude

$$\text{Div}_W^\alpha \text{Grad}_W^\alpha = ({}^C \mathbf{D}_W^\alpha, {}^C \mathbf{D}_W^\alpha) = {}^C \Delta_W^\alpha. \quad (104)$$

(ii) The second relation for the scalar field $f = f(x, y, z)$ is

$$\text{Curl}_W^\alpha \text{Grad}_W^\alpha f = 0. \quad (105)$$

(iii) For the vector field $\mathbf{F} = \mathbf{e}_m F_m$, it is easy to prove the relation

$$\text{Div}_W^\alpha \text{Curl}_W^\alpha \mathbf{F}(x, y, z) = 0. \quad (106)$$

(iv) The following identity also holds for the double curl operator

$$\text{Curl}_W^\alpha \text{Curl}_W^\alpha \mathbf{F} = \text{Grad}_W^\alpha \text{Div}_W^\alpha \mathbf{F} - ({}^C \mathbf{D}_W^\alpha)^2 \mathbf{F}. \quad (107)$$

(v) The Leibniz rule for fractional differential vector operators [58] does not hold, i.e.,

$$\text{Grad}_W^\alpha (fg) \neq (\text{Grad}_W^\alpha f)g + (\text{Grad}_W^\alpha g)f, \quad (108)$$

$$\text{Div}_W^\alpha (f\mathbf{F}) \neq (\text{Grad}_W^\alpha f, \mathbf{F}) + f \text{Div}_W^\alpha \mathbf{F}. \quad (109)$$

We define the fractional differential vector operators such that the fractional vector integral operators (circulation, flux, and volume integral) exist as inverse operations. This allows us to establish the fractional analogues of Green's, Stokes' and Gauss' integral theorems [13,60]. It is also noted that the fractional differential operators are nonlocal by definition. The fractional gradient, divergence and curl operators depend on the region W . This property allows for the use of fractional vector calculus to describe complex materials with power-law non-locality in three dimensional space. Note that these continuum fractional vector operators can be connected with the fractional-order operators on lattices with long-range interactions [61].

3.3. Fractional 3D gradient elasticity model

The simplest form of the stress-strain relation of gradient elasticity theory can be written [16] as

$$\sigma_{ij} = C_{ijkl} \left(\varepsilon_{kl} \pm l_s^2 \Delta \varepsilon_{kl} \right), \quad (110)$$

where C_{ijkl} is the matrix of elastic modulus, l_s is a length scale parameter, σ_{ij} is the stress, and ε_{kl} is the strain tensor. For homogenous and isotropic materials we have

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}, \quad (111)$$

where λ and μ are the usual Lamé constants, and δ_{ij} is the Kronecker delta.

The equation of motion based on Eq. (110) has the form

$$C_{ijkl} \left(D_{x_j}^1 D_{x_l}^1 \pm l_s^2 D_{x_j}^1 (D_{x_m}^1 D_{x_m}^1) D_{x_l}^1 \right) u_k + f_i = \rho D_t^2 u_i, \quad (112)$$

where f_i are the components of the external force field, and u_k are the components of the displacement vector field. For homogenous and isotropic materials, Eq. (112) can be written as

$$\lambda \left(D_{x_i}^1 D_{x_k}^1 \pm l_s^2 D_{x_i}^1 (D_{x_m}^1 D_{x_m}^1) \right) u_k + 2\mu \left((D_{x_l}^1 D_{x_l}^1) \pm l_s^2 (D_{x_l}^1 D_{x_l}^1) (D_{x_m}^1 D_{x_m}^1) \right) u_l + f_i = \rho D_t^2 u_i. \quad (113)$$

Using now operations of the vector calculus operators, this equation can be rewritten in the following vector form

$$\lambda \left(1 \pm l_s^2 \Delta \right) \text{grad div } \mathbf{u} + 2\mu \left(\Delta \pm l_s^2 \Delta^2 \right) \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (114)$$

A formal fractional generalization of Eq. (112) can be obtained in the form

$$C_{ijkl} \left({}^C D_{x_j}^{\alpha_j} {}^C D_{x_l}^{\alpha_l} \pm l_s^2 (\alpha) {}^C D_{x_j}^{\alpha_j} ({}^C D_{x_m}^{\alpha_m} {}^C D_{x_m}^{\alpha_m}) {}^C D_{x_l}^{\alpha_l} \right) u_k + f_i = \rho D_t^2 u_i, \quad (115)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index. For the isotropic case ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha$), we have the fractional equation

$$C_{ijkl} \left({}^C D_{x_j}^{\alpha} {}^C D_{x_l}^{\alpha} \pm l_s^2 (\alpha) {}^C D_{x_j}^{\alpha} ({}^C D_{x_m}^{\alpha} {}^C D_{x_m}^{\alpha}) {}^C D_{x_l}^{\alpha} \right) u_k + f_i = \rho D_t^2 u_i. \quad (116)$$

Using the properties of the fractional differential vector operators, Eq. (116) for homogenous and isotropic materials can be rewritten in the following vector form

$$\lambda \left(1 \pm l_s^2 (\alpha) {}^C \Delta_W^{\alpha} \right) \text{Grad}_W^{\alpha} \text{Div}^{\alpha} \mathbf{u} + 2\mu \left({}^C \Delta_W^{\alpha} \pm l_s^2 (\alpha) ({}^C \Delta_W^{\alpha})^2 \right) \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (117)$$

Note that, in general, the following inequality holds

$$\left({}^C \Delta_W^{\alpha} \right)^2 \neq {}^C \Delta_W^{2\alpha}, \quad (118)$$

since $\left({}^C D_x^{\alpha} \right)^2 \neq {}^C D_x^{2\alpha}$.

In general, the fractional equations of motion may contain expressions of the form $A_{\alpha}(x_j) {}^C D_{x_j}^{\alpha}$ with a given function $A_{\alpha}(x)$ instead of the fractional derivative ${}^C D_{x_j}^{\alpha}$. The explicit form of the function $A_{\alpha}(x)$ is deduced by the conservation law for non-local media by using the fractional vector calculus [13,60]. In this case, the resulting 3D gradient elasticity models are more complicated and the corresponding equations of motion are much more difficult to solve. To solve the governing equations of 3D fractional models we should also take into account an explicit form of the violation of the semigroup property for the Caputo derivative [54] that gives the relationship between the product ${}^C D_{a+}^{\alpha} {}^C D_{a+}^{\beta}$ and the derivative ${}^C D_{a+}^{\alpha+\beta}$.

Using Eq. (107) in the form

$$\text{Grad}_W^{\alpha} \text{Div}_W^{\alpha} \mathbf{u} = \text{Curl}_W^{\alpha} \text{Curl}_W^{\alpha} \mathbf{u} + {}^C \Delta_W \mathbf{u}, \quad (119)$$

we can rewrite Eq. (117) as

$$\lambda \left(1 \pm l_s^2 (\alpha) {}^C \Delta_W^{\alpha} \right) \text{Curl}_W^{\alpha} \text{Curl}_W^{\alpha} \mathbf{u} + (\lambda + 2\mu) \left({}^C \Delta_W^{\alpha} \pm l_s^2 (\alpha) ({}^C \Delta_W^{\alpha})^2 \right) \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (120)$$

If we further assume that the displacement vector \mathbf{u} is radial and function of $r = |\mathbf{r}|$ alone ($u_k = u_k(|\mathbf{r}|)$), we have

$$\text{Curl}_W^{\alpha} \mathbf{u} = \mathbf{0},$$

and, as a result, Eq. (120) has the form

$$(\lambda + 2\mu) \left(1 \pm l_s^2 (\alpha) {}^C \Delta_W^{\alpha} \right) {}^C \Delta_W^{\alpha} \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (121)$$

This is the governing fractional gradient elasticity equation for homogenous and isotropic materials with spherical symmetry.

3.4. The square of fractional derivative is not equal to a dual-order derivative

In order to solve the governing equations of fractional gradient elasticity, we should give first the explicit form of the relationship between the square of the Caputo derivative $({}^C D_{a+}^\alpha)^2$ and the Caputo derivative ${}^C D_{a+}^{2\alpha}$. To obtain this relation we use Eq. 2.4.6 of [2], in the form

$$({}^C D_{a+}^\alpha f)(x) = ({}^{RL} D_{a+}^\alpha f)(x) - \sum_{k=0}^{n-1} \frac{(D^k f)(a)}{\Gamma(k - \alpha + 1)} (x - a)^{k-\alpha}, \tag{122}$$

and Eq. 2.1.16 of [2], in the form

$$I_{a+}^\alpha (x - a)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta)} (x - a)^{\beta+\alpha}, \tag{123}$$

where $\alpha > 0$ and $\beta > -1$. The condition $\beta > -1$ gives another restriction for α in the form $\alpha < 1$. The relationship between the square of the Caputo derivative of order α and the Caputo derivative of order 2α takes then the form

$$({}^C D_{a+}^\alpha)^2 f(x) = {}^C D_{a+}^{2\alpha} f(x) + \frac{f'(a)}{\Gamma(1 - 2\alpha)} (x - a)^{1-2\alpha}, \quad (0 < \alpha \leq 1), \tag{124}$$

where $\alpha \neq 1/2$. Using Eq. (124), we can represent the fractional Laplacian of Caputo type as

$${}^C \Delta_W^\alpha = \sum_{k=1}^3 {}^C D_{x_k}^{2\alpha} + \sum_{k=1}^3 \frac{(D_{x_k}^1 f)(a)}{\Gamma(1 - 2\alpha)} (x_k - a_k)^{1-2\alpha}. \tag{125}$$

Note that the relation given by Eq. (124) cannot be used for $\alpha > 1$. As a result, additional difficulties for solving fractional gradient equations arise. To solve these problems, we can use a generalization of the Ru–Aifantis operator split method [19,22].

3.5. Operator split method for fractional gradient elasticity

In 1993, Ru and Aifantis [19] suggested an operator split method to solve static problems of gradient elasticity. Let us consider a generalization of this method to solve the fractional gradient elasticity problems. For the static case, Eq. (117) can be written in the form

$$\left(1 \pm I_s^2(\alpha) {}^C \Delta_W^\alpha\right) [\lambda \text{Grad}_W^\alpha \text{Div}^\alpha + 2\mu {}^C \Delta_W^\alpha] \mathbf{u} + \mathbf{f} = 0. \tag{126}$$

By introducing $I_s^2(\alpha) = 0$ in Eq. (126), we obtain the fractional differential equation

$$L^{(\alpha)} \mathbf{u} + \mathbf{f} = 0, \tag{127}$$

where we use the fractional operator

$$L^{(\alpha)} = \lambda \text{Grad}_W^\alpha \text{Div}^\alpha + 2\mu {}^C \Delta_W^\alpha. \tag{128}$$

For the gradient-dependent case $I_s^2(\alpha) \neq 0$, Eq. (126) has the form

$$\left(1 \pm I_s^2(\alpha) {}^C \Delta_W^\alpha\right) L^{(\alpha)} \mathbf{u} + \mathbf{f} = 0. \tag{129}$$

In general, it is necessary to solve the fractional partial differential equation of order 4α , which has a very complex form caused by the inequality $({}^C \Delta_W^\alpha)^2 \neq {}^C \Delta_W^{2\alpha}$ for the fractional Laplacian of Caputo type. The following observation can reduce the complexity of this task and greatly facilitate the obtaining of solutions in certain cases. For the radial displacement case ($\text{Curl}_W^\alpha \mathbf{u} = 0$), the operators $L^{(\alpha)}$ and ${}^C \Delta_W^\alpha$ commute, i.e.,

$$L^{(\alpha)} {}^C \Delta_W^\alpha - {}^C \Delta_W^\alpha L^{(\alpha)} = 0.$$

Therefore, we can see from Eq. (129) that the vector field $\left(1 \pm I_s^2(\alpha) {}^C \Delta_W^\alpha\right) \mathbf{u}$ satisfies the non-gradient expression of Eq. (127) for the field \mathbf{u} . Thus, if $\left(1 \pm I_s^2(\alpha) {}^C \Delta_W^\alpha\right) \mathbf{u}$ can be identified with the non-gradient displacement field \mathbf{u}^c of fractional non-gradient elasticity theory given by Eq. (127), which can be solved, then the original fractional gradient elasticity theory given by Eq. (126) is reduced to the following fractional equation

$$\left(1 \pm I_s^2(\alpha) {}^C \Delta_W^\alpha\right) \mathbf{u}^g = \mathbf{u}^c, \tag{130}$$

where \mathbf{u}^c is a classical (“non-gradient”) solution of the fractional equation

$$L^{(\alpha)} \mathbf{u}^c + \mathbf{f} = 0.$$

Obviously, the solution of Eq. (130) can be more conveniently obtained. This establishes a connection between the “gradient” (g) and the non-gradient “classical” (c) fractional elasticity solutions. For the non-radial case ($\text{Curl}_W^\alpha \mathbf{u} \neq 0$), the fractional gradient elasticity theory given by Eq. (116) takes the form

$$L_{ik}^{(\alpha)} \left(1 \pm I_s^{2\alpha}(\alpha) {}^C \Delta_W^\alpha \right) u_k + f_i = 0, \quad (131)$$

where

$$L_{ik}^{(\alpha)} = C_{ijkl} {}^C D_W^\alpha [x_j] {}^C D_W^\alpha [x_l]. \quad (132)$$

Using the operator split approach, Eq. (131) can be solved as an uncoupled sequence of two sets of fractional equations, that is

$$L_{ik}^{(\alpha)} u_k^c + f_i = 0 \quad (133)$$

followed by

$$\left(1 \pm I_s^{2\alpha}(\alpha) {}^C \Delta_W^\alpha \right) u_k^g = u_k^c, \quad (134)$$

where two separate displacement fields are distinguished. Firstly, u_k^c obeys the non-gradient fractional elasticity as given by Eq. (133). Secondly, u_k^g are the same as u_k in Eq. (131), but they are now appended with a superscript g to emphasize that they incorporate fractional gradient effects.

3.6. Solutions by fractional operator split method

Unfortunately, the applicability of fractional vector calculus to solve 3D fractional differential equations, such as Eq. (117), is very limited due to the weak development of this area of mathematics. Therefore, we demonstrate an application of the suggested generalization of the operator split method to obtain solutions of fractional gradient elasticity equation for a 1D case only. The 1D counterpart of Eq. (121) reads

$$(\lambda + 2\mu) \left(1 \pm I_s^{2\alpha}(\alpha) {}^C \Delta_x^\alpha \right) {}^C \Delta_x^\alpha u(x) + f(x) = 0. \quad (135)$$

Using the operator split method in Eq. (135), we derive two uncoupled fractional equations

$$L^{(\alpha)} u^c(x) + f(x) = 0, \quad (136)$$

and

$$\left(1 \pm I_s^{2\alpha}(\alpha) {}^C \Delta_x^\alpha \right) u^g(x) = u^c(x), \quad (137)$$

where the notation $L^{(\alpha)} = (\lambda + 2\mu) {}^C \Delta_x^\alpha$ was used.

Let us first consider the equation for the non-gradient case. Using (124), Eq. (136) can be represented as

$$(\lambda + 2\mu) {}^C D_{a+}^{2\alpha} u(x) + \frac{(\lambda + 2\mu) u'(a)}{\Gamma(1 - 2\alpha)} (x - a)^{1-2\alpha} + f(x) = 0. \quad (138)$$

We can rewrite this equation in the form

$$(\lambda + 2\mu) {}^C D_{a+}^{2\alpha} u(x) + f_{\text{eff}}(x) = 0, \quad (139)$$

where we have used the effective body force given by the expression

$$f_{\text{eff}}(x) = \frac{(\lambda + 2\mu) u'(a)}{\Gamma(1 - 2\alpha)} (x - a)^{1-2\alpha} + f(x). \quad (140)$$

If $f_{\text{eff}}(x) \in C_\gamma[a; b]$ with $0 \leq \gamma < 1$ and $\gamma \leq 2\alpha$, then (see Section 4.1.3 and Theorem 4.3 of [2]) Eq. (139) has a unique solution $u^c(x)$ belonging to the space $C_\gamma^{2\alpha, n}[a; b]$, where $n - 1 < 2\alpha < n$, defined by the expression

$$u(x) = u^c(x) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (x - a)^k - \frac{1}{(\lambda + 2\mu) \Gamma(2\alpha)} \int_a^x \frac{f_{\text{eff}}(z)}{(x - z)^{1-2\alpha}}, \quad (141)$$

where $n - 1 < 2\alpha < n$.

Next, we consider the corresponding equation for the gradient case. Eq. (137) can be rewritten as

$${}^C D_{a+}^{2\alpha} u^g(x) \pm I_s^{-2\alpha}(\alpha) u^g(x) = \pm u^c(x). \quad (142)$$

Using (124), Eq. (142) can be represented as

$${}^C D_{a+}^{2\alpha} u^g(x) + \frac{u'(a)}{\Gamma(1 - 2\alpha)} (x - a)^{1-2\alpha} \pm I_s^{-2\alpha}(\alpha) u^g(x) = \pm u^c(x), \quad (143)$$

where $u^c(x)$ is defined by Eq. (141). We rewrite this equation in the form

$${}^C\Delta_x^{3\alpha} u^g(x) \pm I_s^{-2}(\alpha) u^g(x) = u_{eff}^c(x), \tag{144}$$

$$u_{eff}^c(x) = \pm u^c(x) - \frac{u'(a)}{\Gamma(1-2\alpha)}(x-a)^{1-2\alpha}. \tag{145}$$

If $u_{eff}^c(x) \in C_\gamma[a; b]$ with $0 \leq \gamma < 1$ and $\gamma \leq 2\alpha$, then (see Theorem 4.3 of [2]) Eq. (144) has a unique solution $u^g(x)$ belonging to the space $C_\gamma^{2\alpha, n}[a; b]$, where $n-1 < 2\alpha < n$, defined by the expression

$$u^g(x) = \sum_{k=0}^{n-1} u^g(a) (x-a)^k E_{2\alpha, k+1} [\mp I_s^{-2}(\alpha) (x-a)^{2\alpha}] + \int_a^x (x-z)^{2\alpha-1} E_{2\alpha, 2\alpha} [\pm I_s^{-2}(\alpha) (x-z)^{2\alpha}] u_{eff}^c(z) dz. \tag{146}$$

The quantity $E_{\alpha, \beta}(z)$ is the Mittag–Leffler function [2] defined by the relation

$$E_{\alpha, \beta}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta \in \mathbb{R}). \tag{147}$$

Note also that $E_{1,1}[z] = e^z$.

4. Toward gradient elasticity of fractal materials

Fractals are measurable metric sets with non-integer Hausdorff dimension [44,45] that should be observed on all scales.- Real fractal materials can be characterized by an asymptotic relation between the mass $M(W)$ and the volume $V(W)$ of regions W of the fractal medium. For example, for a homogeneous fractal medium, a ball of radius $R \gg R_0$ contains the mass $M_D(W) = M_0(R/R_0)^D$, where the number D is called the mass dimension, and R_0 is a characteristic size related to the arrangement of the medium particles. The mass dimension D does not depend on the shape of the region W , or on the packing of particles (close packing, random packing or porous packing with uniform distribution of holes).

As a result, we can define a fractal material as a medium with non-integer mass (or number of particles) dimension. Although, the non-integer dimension does not reflect completely the geometric and dynamic properties of a fractal medium, it nevertheless permits a number of important conclusions about its behavior.

4.1. Fractional continuum model for fractal materials

In general, a fractal material cannot be considered as a usual continuum, since there are places and areas that are not filled with particles. Nevertheless it can be described by special continuum models [32,33,13] based on the use of the integrals with non-integer order. The order of these integrals should be defined by the fractal mass dimension. The kernel of the fractional integral operator describes a density of permitted states (permitted places) in space. The fractional-order integrals can be considered as integrals over a non-integer dimensional space up to a numerical factor by using the well-known formulas of dimensional regularization [46].

Fractional integral continuum models of fractal media may have a wide range of applications [13] due to the relatively small numbers of parameters that define a fractal material of great complexity and rich structure. One of the advantages of such models is the ability to describe dynamics of fractal materials and media (for details see [13]).

To describe fractal materials by a fractional integral continuum model, we use two different notions: the density of states $c_n(D, \mathbf{r})$ and the distribution function $\rho(\mathbf{r}, t)$.

- (1) The function $c_n(D, \mathbf{r})$ is a density of states in the n -dimensional Euclidean space \mathbb{R}^n . The density of states describes how closely packed permitted states of particles in the space \mathbb{R}^n . The expression $c_n(D, \mathbf{r}) dV_n$ represents the number of states (permitted places) between V_n and $V_n + dV_n$. We note that the symmetry of the density of states $c_n(D, \mathbf{r})$ must be the defined by the symmetry properties of the fractal medium.
- (2) The function $\rho(\mathbf{r}, t)$ is a distribution function in the n -dimensional Euclidean space \mathbb{R}^n . It describes the distribution of physical values (for example, mass, electric charge, number of particles, probability) on a set of possible (permitted) states in the space \mathbb{R}^n .

For example, the mass of a region dV_n in fractal media is defined by the equation

$$dM(\mathbf{r}, t) = \rho(\mathbf{r}, t) c_n(D, \mathbf{r}) dV_n.$$

In general, we cannot consider the value $\rho(\mathbf{r}, t) c_n(D, \mathbf{r})$ as a new distribution function or a particle number density, since the notions of density of states and of distribution function are different. We cannot reduce all properties of the system to a description of the distribution function. This fact is well-known in statistical and condensed matter physics, where the density of states is usually considered as a density of energy states or as a density of wave vector states [57] that describe how closely packed the allowed states in energy or wave-vector spaces. For fractal distributions of particles in a coordinate space

\mathbb{R}^n , we must use a density of states in this space. The density of states $c_n(D, \mathbf{r})$ in \mathbb{R}^n is chosen such that $d\mu_D(\mathbf{r}, n) = c_n(D, \mathbf{r})dV_n$ describes the number of states in dV_n . We use the notations

$$dV_D = c_3(D, \mathbf{r})dV_3, \quad dS_d = c_2(d, \mathbf{r})dS_2, \quad dl_\beta = c_1(\beta, \mathbf{r})dl_1$$

to describe densities of states in n -dimensional Euclidean spaces with $n = 1, 2, 3$.

4.2. Mass of fractal materials

The cornerstone of fractal media is the non-integer mass dimension. One of the best static experimental methods to determine the mass dimension D of fractal materials is the box-counting method (see, for example [49] and references therein). It involves the selection of a box of size R and counting the mass inside to estimate D from corresponding power law relation $M \sim R^D$.

Let us now consider a region W of a fractal material in the Euclidean space \mathbb{R}^3 , with its boundary denoted by ∂W . Suppose that the medium in the region W has a mass dimension D , and the medium on the boundary ∂W has a dimension d . In general, the dimension d is not equal to $(D - 1)$ and it is not equal to 2. The mass of the region W in the fractal medium is denoted by $M_D(W)$. The fractality means that the mass in any region $W \subset \mathbb{R}^3$ increases slower than the 3D volume of this region, i.e., according to the power law $M_D(W) \sim R^D$, where R is the radius of the ball used to measure D .

A fractal material is called homogeneous if the power law $M_D(W) \sim R^D$ does not depend on the translation of the region W . In other words, for any two regions W_1 and W_2 of the homogeneous fractal material with equal volumes $V_D(W_1) = V_D(W_2)$, the corresponding masses are equal $M_D(W_1) = M_D(W_2)$. A wide class of fractal media satisfies the homogeneous property. Many porous materials, polymers, colloid aggregates, and aerogels can be considered as homogeneous fractal materials. However, the fact that a material is porous or random does not necessarily imply that this material is fractal. To describe fractal materials by a fractal integral continuum model, the fractality and homogeneity properties are implemented as follows:

- Homogeneity: The local density of a homogeneous fractal material can be described by the constant density $\rho(\mathbf{r}) = \rho_0 = \text{const}$. This property means that if $\rho(\mathbf{r}) = \text{const}$ and $V(W_1) = V(W_2)$, then $M_D(W_1) = M_D(W_2)$.
- Fractality: The mass of the ball region W of a fractal homogeneous material obeys a power law relation $M \sim R^D$, where $0 < D < 3$, and R is the radius of the ball. If $V_n(W_1) = \lambda^n V_n(W_2)$ and $\rho(\mathbf{r}, t) = \text{const}$, then fractality implies that $M_D(W_1) = \lambda^D M_D(W_2)$.

These two conditions cannot be satisfied if the mass of the medium is described by an integral of integer order. In this case the mass is expressed by the fractional-order integral equation

$$M_D(W, t) = \int_W \rho(\mathbf{r}, t)dV_D, \quad dV_D = c_3(D, \mathbf{r})dV_3, \tag{148}$$

where \mathbf{r} is a dimensionless vector variable. As already noted, $\rho(\mathbf{r}, t)$ is a distribution function, and $c_3(D, \mathbf{r})$ is a density of states in the Euclidean space \mathbb{R}^3 . The order of the integral in Eq. (148) is defined by the fractal mass dimension of the material. The kernel of the fractional integral operator describes a density of permitted states $c_3(D, \mathbf{r})$ in space, and its symmetry is defined by the symmetry of the material structure. The particular form (Riesz, Riemann–Liouville, etc.) of the function $c_3(D, \mathbf{r})$ is defined by the properties of the fractal material at hand. Note that the final field equations that relate the various physical variables of the system have a form that is independent of the numerical factor in the function $c_3(D, \mathbf{r})$. However the dependence on \mathbf{r} is important in these equations. In addition, we note that for $D = 2$, we have the fractal mass distribution in 3D Euclidean space \mathbb{R}^3 . In general, this case is not equivalent to the distribution on a 2D surface.

4.3. Moment of inertia for fractal materials

A method for calculating the moment of inertia of fractal materials has been suggested in [34]. The moment of inertia has two forms, a scalar form $I(t)$, which is used when the axis of rotation is known, and a more general tensor form that does not require knowing the axis of rotation. The scalar moment of inertia (often called simply the “moment of inertia”) of a rigid body with density $\rho'(\mathbf{r}', t)$ with respect to a given axis is defined by the volume integral

$$I(t) = \int_W \rho'(\mathbf{r}', t) \mathbf{r}'_{\perp}{}^2 dV'_3, \tag{149}$$

where $\mathbf{r}'_{\perp}{}^2$ is the square of the perpendicular distance from the axis of rotation, and $dV'_3 = dx'_1 dx'_2 dx'_3$. If $\mathbf{r}' = x'_k \mathbf{e}_k$ denotes the position vector from the origin to a point ($x'_k, k = 1, 2, 3$, are components of \mathbf{r}'), then the tensor form of the moment of inertia is

$$I'_{kl}(t) = \int_W \rho'(\mathbf{r}', t) \left((\mathbf{r}')^2 \delta_{kl} - x'_k x'_l \right) dV'_3, \tag{150}$$

where δ_{kl} is the Kronecker delta. We note that the SI units of I'_{kl} is $\text{kg} \cdot \text{m}^2$, i.e., $[I'_{kl}] = \text{kg} \cdot \text{m}^2$.

To generalize Eqs. (149) and (150) for fractional media, we express these equations through dimensionless coordinates. We thus introduce the dimensionless variables $x_k = x'_k/l_0$, $\mathbf{r} = \mathbf{r}'/l_0$, where l_0 is a characteristic length scale, and write the density as $\rho(\mathbf{r}, t) = l_0^3 \rho'(\mathbf{r}l_0, t)$ so its SI units is m , i.e., $[\rho] = kg$. We then define the following moments of inertia $I_{kl}(t) = l_0^{-2} I'_{kl}(t)$, $I(t) = l_0^{-2} I'(t)$ to finally obtain the relations

$$I(t) = \int_W \rho(\mathbf{r}, t) \mathbf{r}_\perp^2 dV_3, \quad I_{kl}(t) = \int_W \rho(\mathbf{r}, t) (\mathbf{r}^2 \delta_{kl} - x_k x_l) dV_3, \tag{151}$$

where $dV_3 = dx_1 dx_2 dx_3$ for Cartesian coordinates, and the variables x_k , $k = 1, 2, 3$ are now dimensionless. We note that the SI units of I_{kl} is kg , i.e., $[I_{kl}] = kg$. This representation allows us to generalize Eq. (151) to fractal materials in the form

$$I^{(D)}(t) = \int_W \rho(\mathbf{r}, t) \mathbf{r}_\perp^2 dV_D, \quad I_{kl}^{(D)}(t) = \int_W \rho(\mathbf{r}, t) (\mathbf{r}^2 \delta_{kl} - x_k x_l) dV_D, \tag{152}$$

where $dV_D = c_3(D, \mathbf{r}) dV_3$ with D denoting, as usual, the mass dimension of the fractal material.

4.4. Equilibrium equations for fractal materials

Let us now derive the equilibrium equations for a fractal material with mass dimension D . Consider a finite region W in the fractal material, supporting a volume force and a surface force. Let the density of force $\mathbf{f}(\mathbf{r}, t)$ be a function of the dimensionless vector \mathbf{r} , and time t . The volume or mass force $\mathbf{F}_M(W)$, i.e. the force acting on a region W of a fractal medium with dimension D , is defined by

$$\mathbf{F}_M(W) = \int_W \mathbf{f}(\mathbf{r}, t) dV_D. \tag{153}$$

The surface force $\mathbf{F}_S(W)$, i.e. the force acting on the surface ∂W with dimension d , is defined by

$$\mathbf{F}_S(W) = \int_{\partial W} \sigma^n(\mathbf{r}, t) dA_d, \tag{154}$$

where $\sigma = \sigma(\mathbf{r}, t)$ is the traction vector on a surface with unit normal \mathbf{n} . As already mentioned, in general the dimension d is not equal to $(D - 1)$ and it is not equal to 2. The resultant force that acts on the region W is then

$$\mathbf{F}_\Sigma(W) = \mathbf{F}_M(W) + \mathbf{F}_S(W), \tag{155}$$

and by substituting Eqs. (153) and (154) into Eq. (155), we obtain

$$\mathbf{F}_\Sigma(W) = \int_W \mathbf{f}(\mathbf{r}, t) dV_D + \int_{\partial W} \sigma^n(\mathbf{r}, t) dA_d. \tag{156}$$

This fractional integral equation represents the resultant force acting on any region W of the fractal material. For $D = 3$ and $d = 2$, Eq. (156) gives the usual equation for the resultant force in a non-fractal continuum. The force equilibrium condition for the region W requires $\mathbf{F}_\Sigma(W) = 0$. Therefore, we have the fractional integral equation of equilibrium

$$\int_W \mathbf{f}(\mathbf{r}, t) dV_D + \int_{\partial W} \sigma^n(\mathbf{r}, t) dA_d = 0. \tag{157}$$

In component form, this equation reads

$$\int_W f_k(\mathbf{r}, t) dV_D + \int_{\partial W} \sigma_k^n(\mathbf{r}, t) dA_d = 0, \tag{158}$$

where we use $\mathbf{f} = f_k \mathbf{e}_k$ and $\sigma^n = \sigma_k^n \mathbf{e}_k$. Using the normal vector $\mathbf{n} = n_j \mathbf{e}_j$, we can represent σ_k^n in the form $\sigma_k^n = \sigma_{ij} n_j$, where σ_{ij} is the stress tensor.

The differential form of equilibrium equations follows directly from Eq. (158). Using the generalization of the Gauss theorem for fractal media [33], the surface integral can be represented as

$$\int_{\partial W} \sigma^n dA_d = \int_{\partial W} c_2(d, \mathbf{r}) \sigma^n dA_2 = \int_W \frac{\partial(c_2(d, \mathbf{r}) \sigma_l)}{\partial x_l} c_3^{-1}(D, \mathbf{r}) dV_D = \int_W \nabla_l^{(D,d)} \sigma_l dV_D, \tag{159}$$

where a generalization of the nabla operator for fractal materials [60] was also used in the form

$$\nabla_k^{(D,d)} B = c_3^{-1}(D, \mathbf{r}) \frac{\partial(c_2(d, \mathbf{r}) B)}{\partial x_k}, \tag{160}$$

where $B = B(\mathbf{r})$ is a function of the coordinates. This operator will be called “fractal-nabla” operator. We note that the operator given by Eq. (160) is not a fractional derivative [2] or an operator on a fractal set [69]. For example, if we use the density of states $c_3(D, \mathbf{r})$ and $c_2(d, \mathbf{r})$ in the form

$$c_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}|^{D-3}, \quad (161)$$

$$c_2(d, \mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}|^{d-2}, \quad (162)$$

then the “fractal-nabla” operator is given by

$$\nabla_k^{(D,d)} B = \frac{2^{D-d-1}\Gamma(D/2)}{\Gamma(3/2)\Gamma(d/2)} |\mathbf{r}|^{3-D} \frac{\partial}{\partial x_k} (|\mathbf{r}|^{d-2} B). \quad (163)$$

For non-fractal materials ($D = 3$ and $d = 2$), we have

$$\nabla_k^{(3,2)} B = \frac{\partial B}{\partial x_k}.$$

We note that the rule of term-by-term differentiation for the operator $\nabla_k^{(D,d)}$ is not satisfied, i.e.

$$\nabla_k^{(D,d)}(BC) \neq B\nabla_k^{(D,d)}(C) + C\nabla_k^{(D,d)}(B).$$

The operator $\nabla_k^{(D,d)}$ satisfies the following rule

$$\nabla_k^{(D,d)}(BC) = B\nabla_k^{(D,d)}(C) + c(D, d, \mathbf{r}) C \nabla_k^1 B, \quad (164)$$

where

$$c(D, d, \mathbf{r}) = c_3^{-1}(D, \mathbf{r}) c_2(d, \mathbf{r}).$$

For example, the density of states given by Eqs. (161) and (162), can be expressed as

$$c(D, d, \mathbf{r}) = \frac{2^{D-d-1}\Gamma(D/2)}{\Gamma(3/2)\Gamma(d/2)} |\mathbf{r}|^{d+1-D}.$$

Note that, in general, $\nabla_k^{(D,d)}(1) \neq 0$ since

$$\nabla_k^{(D,d)}(1) = c(D, d, \mathbf{r}) (d-2) \frac{x_k}{r^2}.$$

Using now Eqs. (159) and (157) takes the form

$$\int_W (\mathbf{f} + \nabla_l^{(D,d)} \sigma_l) dV_D = 0, \quad (165)$$

or in components form (with $\mathbf{f} = f_k \mathbf{e}_k$, and $\sigma_l^n = \sigma_{kl} \mathbf{e}_k$), we have

$$\int_W (f_k + \nabla_l^{(D,d)} \sigma_{kl}) dV_D = 0, \quad (k = 1, 2, 3). \quad (166)$$

This equation is satisfied for all regions W . As a result, we have

$$\nabla_l^{(D,d)} \sigma_{kl} + f_k = 0, \quad (k = 1, 2, 3). \quad (167)$$

Using the usual notation, we have

$$c_3^{-1}(D, \mathbf{r}) D_{x_l}^1 (c_2(d, \mathbf{r}) \sigma_{kl}) + f_k = 0, \quad (k = 1, 2, 3). \quad (168)$$

These are the differential equations of equilibrium for fractal materials.

Let us derive next, the equilibrium equation for the moment of forces. The moment $\mathbf{M}_M(W)$ of the mass force (153), can be written as

$$\mathbf{M}_M(W) = \int_W [\mathbf{r}, \mathbf{f}] dV_D. \quad (169)$$

The moment $\mathbf{M}_S(W)$ of the surface force (154) is given by

$$\mathbf{M}_S(W) = \int_{\partial W} [\mathbf{r}, \sigma^n] dA_d. \quad (170)$$

In Eqs. (169) and (170), the brackets $[\cdot, \cdot]$ denotes vector product of vector fields. The resultant moment $\mathbf{M}_\Sigma(W)$ is the sum

$$\mathbf{M}_\Sigma(W) = \mathbf{M}_M(W) + \mathbf{M}_S(W). \quad (171)$$

Substituting Eqs. (169) and (170) into Eq. (171), we obtain

$$\mathbf{M}_\Sigma(W) = \int_W [\mathbf{r}, \mathbf{f}] dV_D + \int_{\partial W} [\mathbf{r}, \sigma^n] dA_d. \tag{172}$$

The equilibrium condition for the region W surrounded by its surface ∂W of a fractal material leads to $\mathbf{M}_\Sigma(W) = 0$, yielding the fractional integral equation

$$\int_W [\mathbf{r}, \mathbf{f}] dV_D + \int_{\partial W} [\mathbf{r}, \sigma^n] dA_d = 0. \tag{173}$$

In component form, this equation reads

$$\int_W \epsilon_{ijk} x_j f_k dV_D + \int_{\partial W} \epsilon_{ijk} x_j \sigma_{kl} n_l dA_d, \tag{174}$$

where ϵ_{ijk} is the Levi–Civita symbol. Using then the generalization of Gauss theorem for fractal materials given by Eq. (159), we obtain

$$\begin{aligned} \int_{\partial W} \epsilon_{ijk} x_j \sigma_{kl} n_l dA_d &= \int_{\partial W} \epsilon_{ijk} x_j \sigma_{kl} n_l c_2(d, \mathbf{r}) dA_2 = \int_{\partial W} \epsilon_{ijk} D_l^1 (x_j \sigma_{kl} c_2(d, \mathbf{r})) dV_3 \\ &= \int_{\partial W} \epsilon_{ijk} c_3^{-1}(D, \mathbf{r}) D_l^1 (x_j c_2(d, \mathbf{r}) \sigma_{kl}) dV_D = \int_{\partial W} c(D, d, \mathbf{r}) \epsilon_{ilk} \sigma_{kl} dV_D + \int_{\partial W} \epsilon_{ijk} x_j \nabla_l^{(D,d)} \sigma_{kl} dV_D \\ &= \int_{\partial W} c(D, d, \mathbf{r}) \epsilon_{ilk} \sigma_{kl} dV_D - \int_{\partial W} \epsilon_{ijk} x_j f_k dV_D, \end{aligned} \tag{175}$$

where Eq. (167) is also used. Substitution of Eq. (175) into Eq. (174) gives

$$\int_{\partial W} c(D, d, \mathbf{r}) \epsilon_{ilk} \sigma_{kl} dV_D = 0. \tag{176}$$

This equation is satisfied for all regions W . Therefore we have the condition

$$\epsilon_{ijk} \sigma_{kj} = 0, \tag{177}$$

or, equivalent,

$$\sigma_{ij} = \sigma_{ji}. \tag{178}$$

This equilibrium equation for the moment of the force in fractal materials is the same as for the non-fractal case, and suggests that the stress tensor is symmetric.

4.5. Conservation laws for fractal materials

In the framework of fractional integral continuum model, the fractional conservation laws for fractal media have been derived in [33] (see also [39,13]). For future reference, the differential equations of the conservation laws are also summarized below:

(1) The conservation law for mass

$$\left(\frac{d}{dt}\right)_{(D,d)} \rho = -\rho \nabla_k^{(D,d)} u_k. \tag{179}$$

(2) The conservation law for momentum

$$\rho \left(\frac{d}{dt}\right)_{(D,d)} u_k = f_k + \nabla_l^{(D,d)} \sigma_{kl}. \tag{180}$$

(3) The conservation law for energy

$$\rho \left(\frac{d}{dt}\right)_{(D,d)} e = c(D, d, \mathbf{r}) \sigma_{kl} D_l^1 u_k + \nabla_k^{(D,d)} q_k. \tag{181}$$

It is noted that these equations are differential equations with derivatives of integer order (see Eq. (163)). It is also pointed out that the generalized total time derivative is defined by

$$\left(\frac{d}{dt}\right)_{(D,d)} = \frac{\partial}{\partial t} + c(D, d, \mathbf{r}) u_l D_l^1, \tag{182}$$

where $r = |\mathbf{r}|$, x_k , $k = 1, 2, 3$, are dimensionless variables, the operator D_l^1 is defined as usual by $D_l^1 = \partial/\partial x_k$, and

$$c(D, d, \mathbf{r}) = c_3^{-1}(D, \mathbf{r}) c_2(d, \mathbf{r}).$$

The above listed differential equations of balance for the density of mass, the density of momentum, and the density of internal energy make up a set of five equations, which are not closed. In addition to the fields $\rho(\mathbf{r}, t)$, $\mathbf{u}(\mathbf{r}, t)$, $e(\mathbf{r}, t)$, Eqs. (181) and (182) include the tensor of stress $\sigma_{kl}(\mathbf{r}, t) = \sigma_{lk}(\mathbf{r}, t)$ and the vector of thermal flux $q_k(\mathbf{r}, t)$. It is also remarked that the conservation laws for fractal media, which are suggested in [39] are different from the conservation laws given by (179) and (181) derived in [33,13]. In [39] all equations contain the derivatives $c_1^{-1}(\alpha_{x_i}, x_i) D_{x_i}^1$ only, where the density of states $c_1^{-1}(\alpha_{x_i}, x_i)$ can be considered as $c_3^{-1}(D, \mathbf{r}) c_2(D - \alpha_{x_i}, \mathbf{r} - x_i \mathbf{e}_i)$. 179,181,182 contain two types of derivatives: D_i^1 and $\nabla_k^{(D,d)}$.

4.6. Constitutive relations for fractal materials

For the theory of non-fractal gradient elasticity of isotropic materials the constitutive relations [17–19] has the form

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l^2 \Delta (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{183}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors and l denotes an internal length. As usual, λ and μ are the Lamé coefficients; and Δ is the Laplace operator defined by the scalar product of the nabla operators

$$\Delta = (\nabla, \nabla) = \sum_k^2 (\nabla_k)^2. \tag{184}$$

It is easy to see that the balance equations for fractal media considered herein contain in addition to the usual derivatives D_k^1 the “fractal-nabla” operator $\nabla_k^{(D,d)}$ of Eq. (160),

$$\nabla_k^{(D,d)}(\cdot) = c_3^{-1}(D, \mathbf{r}) \nabla_k(c_2(d, \mathbf{r}) \cdot) \tag{185}$$

that takes into account the density of states of fractal media with non-integer mass dimensions. Therefore, we can assume that corresponding generalizations of constitutive relations can be obtained by the replacement of the usual nabla operator by the “fractal-nabla” operator. For example, a fractal generalization of the gradient elasticity model given by Eq. (183) can be represented by the constitutive relations in the form

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_F^2 \Delta^{(D,d)} (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{186}$$

where we use the “fractal-Laplacian” that is defined by

$$\Delta^{(D,d)} = (\nabla^{(D,d)}, \nabla^{(D,d)}) = \sum_k^2 (\nabla_k^{(D,d)})^2. \tag{187}$$

For non-fractal materials, we have $D = 3$, $d = 2$ and $\Delta^{(3,2)} = \Delta$. More generally, we can assume that the constitutive relations for fractal materials are of the form

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_S^2 \Delta (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_F^2 \Delta^{(D,d)} (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}), \tag{188}$$

where two types of Laplacians are taken into account.

In general, fractal materials cannot be defined as media distributed over a fractal set. Naturally, in real materials the fractal structure cannot be observed on all scales. Materials demonstrate fractality only in a range of scales $R_{min} < R < R_{max}$. If the sample material has a size R_S greater than R_{max} , or the region of scales $[R_{min}, R_{max}]$ is narrow, then the material is “semi-fractal” material. The parameter l_S^2 in constitutive relation given by Eq. (188) is a measure of spatial non-fractality of the material, whereas the parameter l_F^2 is a measure of spatial fractality of material for the fractal gradient elasticity theory considered herein. Which of the two models of Eq. (186) or Eq. (188) is more appropriate to describe a particular fractal material, must be determined experimentally.

4.7. Strain–displacement relation for fractal materials

In [39–41] it is postulated that the strain ε_{ij} for small deformations of fractal materials is given in terms of the displacement u_k by the equation

$$\varepsilon_{ij} = \frac{1}{2} (c_1^{-1}(\alpha_{x_i}, x_i) D_{x_i}^1 u_j + c_1^{-1}(\alpha_{x_j}, x_j) D_{x_j}^1 u_i). \tag{189}$$

The one-dimensional analogue of Eq. (189) has been considered in [40,41] in the form

$$\varepsilon(x) = c_1^{-1}(\alpha, x) D_x^1 u(x), \tag{190}$$

where $c_1(\alpha, x)$ is the density of states. As a basis for using this definition, reference is made the differential form of a linear element $dl_x = c_1^{-1}(\alpha_x, x) dx$, which takes into account the 1D density of states. Another argument [40,41] to support this choice is a possibility to obtain the same 1D elastic wave equation from a variational principle, as the wave equation obtained from the balance equations. However, it is not quite clear the necessity to consider the density of states in the definition of the strain.

It thus seems that the definitions given by Eqs. (189) or (190) are not sufficiently rigorously justified. The inclusion of the density of states $c_1(\alpha, x)$ into the strain–displacement relation looks like an artificial reception. The relation between the strain tensor ε_{ij} and the displacement vector u_k should be derived directly from the relevant distance changes (for example, see Section 1.1 of [66]), and this relation should not be postulated in definition. For example, in the 1D case, the strain–displacement relation for fractal materials should be derived from the equation

$$(dl'_x)^2 = (dl_x)^2 (1 + 2\varepsilon(x)) \tag{191}$$

that describes the deformation of a linear element $dl_x = c_1(\alpha, x)dx$ of 1D fractal medium. From Eq. (191) it is apparent that the strain $\varepsilon(x)$ does not contain the density of states $c_1(\alpha, x)$. The relation between strain and displacement should define the deformation of a volume element $dV_D = c_3(D, \mathbf{r})dV_3$ of a fractal material through the condition

$$dV'_D = dV_D [1 + \varepsilon_{11}(x) + \varepsilon_{22}(x) + \varepsilon_{33}(x)], \tag{192}$$

which is the fractal analogue of Eq. 1.6 of [66], we see that $\varepsilon_{ii}(x)$ does not contain the density of states also.

4.8. Variational principle for fractal materials

Another way to derive the governing equations for fractional integral continuum models for fractal materials is the use of variational principles. A holonomic variational principle for fractal materials has been suggested in [35,36] in the framework of a fractional integral continuum model. Variational principles for fractal elasticity are also considered in [37,38]. The equation for fractal elasticity can be derived as the Euler–Lagrange equations from a holonomic functional.

Let us consider a fractional integral continuum model for fractal materials in \mathbb{R}^3 that is described by the action

$$S_F[u] = \int dt \int_{\mathbb{R}^3} dV_D \mathcal{L}(u_i, u_{i,t}, u_{i,k}, u_{i,kl}, u_{i,klm}) \tag{193}$$

with Lagrangian $\mathcal{L}(u_i, u_{i,t}, u_{i,k}, u_{i,kl}, u_{i,klm})$, where $u_i = u_i(\mathbf{r}, t)$ is the displacement vector. To take into account the fractality of the material in coordinate space \mathbb{R}^3 , we use

$$dV_D = c_3(D, \mathbf{r})dV_3,$$

where the function $c_3(D, \mathbf{r})$ describes the density of states in \mathbb{R}^3 . Note that x, y, z and \mathbf{r} are dimensionless variables.

The variation of the action functional given by Eq. (193) is

$$\begin{aligned} \delta S_F[u] &= \int dt \int_{\mathbb{R}^3} dV_D \delta \mathcal{L} \\ &= \int dt \int_{\mathbb{R}^3} dV_D \left[\frac{\partial \mathcal{L}}{\partial u_i} \delta u_i + \left(\frac{\partial \mathcal{L}}{\partial u_{i,t}} \right) \delta u_{i,t} + \left(\frac{\partial \mathcal{L}}{\partial u_{i,k}} \right) \delta u_{i,k} + \left(\frac{\partial \mathcal{L}}{\partial u_{i,kl}} \right) \delta u_{i,kl} + \left(\frac{\partial \mathcal{L}}{\partial u_{i,klm}} \right) \delta u_{i,klm} \right]. \end{aligned} \tag{194}$$

If the fractal material is not subjected to non-holonomic constraints, then the variation and fractional derivatives commute,

$$\delta u_{i,t} = D_t^1(\delta w), \quad \delta u_{i,k} = \nabla_k(\delta w), \quad \delta u_{i,kl} = \nabla_k \nabla_l(\delta w), \quad \delta u_{i,klm} = \nabla_k \nabla_l \nabla_m(\delta w).$$

Using integration by parts, we can express Eq. (194) in the form

$$\begin{aligned} \delta S_F[u] &= \int dt \int_{\mathbb{R}^3} dV_3 \left[c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_i} \delta u_i - D_t^1 \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,t}} \right) + \nabla_k \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,k}} \right) \right. \\ &\quad \left. + \nabla_k \nabla_l \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,kl}} \right) - \nabla_k \nabla_l \nabla_m \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,klm}} \right) \right] \delta u_i. \end{aligned} \tag{195}$$

Then, the stationary action principle, in the form of the holonomic variational equation $\delta S_F[u] = 0$, gives the Euler–Lagrange equations for the fractional integral continuum model of the fractal material considered in the form

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_i} \delta u_i - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial u_{i,t}} \right) - c_3^{-1}(D, \mathbf{r}) \nabla_k \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,k}} \right) + c_3^{-1}(D, \mathbf{r}) \nabla_k \nabla_l \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,kl}} \right) \\ - c_3^{-1}(D, \mathbf{r}) \nabla_k \nabla_l \nabla_m \left(c_3(D, \mathbf{r}) \frac{\partial \mathcal{L}}{\partial u_{i,klm}} \right) = 0. \end{aligned} \tag{196}$$

It follows that a mathematical model for a fractal material is entirely determined by the choice of the Lagrangian. We demonstrate an application of this approach by considering the example of the Euler–Bernoulli fractal beam in the next section.

5. Gradient elasticity model for fractal beam

In this section we derive a gradient elasticity model for fractal materials in the form of the Euler–Bernoulli beam equation of motion by using the holonomic variational principle for fractal media [35,36]. We will consider the gradient fractal beam

by using the fractional integral continuum approach suggested in [33,32,13]. In this connection, it is noted that a non-gradient fractal beam has been considered in [40,41] in the framework of a fractional integral continuum model.

5.1. Variational equation for 1-dimensional model of fractal materials

Let us consider a 1D fractional continuum model for fractal materials described by the action

$$S_F[w] = \int dt \int dl_{\alpha_x} \mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w) \quad (197)$$

with Lagrangian $\mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w)$, where $dl_{\alpha_x} = c_1(\alpha_x, x) dx$ and x is dimensionless. The function $c_1(\alpha_x, x)$ denotes the density of states along the x -axis. For the Euler–Bernoulli fractal beam model, the field $w(x) = u_y(x)$ is the curve that describes the deflection of the beam in the y direction at some position x .

The variation of the action functional given by Eq. (197) is

$$\begin{aligned} \delta S_F[w] &= \int dt \int dx c_1(\alpha_x, x) \delta \mathcal{L} \\ &= \int dt \int dx c_1(\alpha_x, x) \left[\frac{\partial \mathcal{L}}{\partial w} \delta w + \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) \delta(D_t^1 w) + \left(\frac{\partial \mathcal{L}}{\partial D_x^2 w} \right) \delta(D_x^2 w) + \left(\frac{\partial \mathcal{L}}{\partial D_x^3 w} \right) \delta(D_x^3 w) \right]. \end{aligned} \quad (198)$$

If non-holonomic constraints are not involved, the variation and fractional derivatives commute, i.e.

$$\delta(D_t^1 w) = D_t^1(\delta w), \quad \delta(D_x^2 w) = D_x^2(\delta w), \quad \delta(D_x^3 w) = D_x^3(\delta w).$$

Using integration by parts, we express Eq. (198) in the form

$$\delta S_F[w] = \int dt \int dx \left[c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial w} \delta w - D_t^1 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + D_x^2 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^2 w} \right) - D_x^3 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^3 w} \right) \right] \delta w. \quad (199)$$

The stationary action principle implies the holonomic variational equation $\delta S_F[w] = 0$. This equation gives the Euler–Lagrange equation in the form

$$\frac{\partial \mathcal{L}}{\partial w} - c_1(\alpha_x, x) D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + D_x^2 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^2 w} \right) - D_x^3 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^3 w} \right) = 0. \quad (200)$$

This equation describes the fractional continuum model of a fractal material distributed in \mathbb{R}^1 with dimension α_z .

5.2. Euler–Lagrange equation for the Euler–Bernoulli fractal beam

The Lagrangian for the Euler–Bernoulli fractal beams has the form

$$\mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w) = \frac{1}{2} \mu \left(D_t^1 w(x, t) \right)^2 + \frac{1}{2} (EI^{(d)}) \left(D_x^2 w(x, t) \right)^2 - \frac{1}{2} (EI^{(d)}) I_F^2(d) \left(D_x^3 w(x, t) \right)^2 - q(x, t) w(x, t). \quad (201)$$

The first term represents the kinetic energy, where $\mu = \rho A$ is the mass per unit length; the second one represents the potential energy due to an internal forces (when considered with a negative sign); and the third term represents the potential energy due to the external load $q(x, t)$. Note that (x, y, z) are dimensionless variables, and $I_F^2(d)$ is a dimensionless parameter.

The Lagrangian looks similar the usual Lagrangian for an Euler–Bernoulli gradient elastic beam. A difference is in the presence of the moment of inertia $I^{(d)}$ of the fractal material only. In the Lagrangian we used the second moment of area ($I^{(d)} = I_z^{(d)}$) of the fractal beam's cross-section defined by

$$I^{(d)} = I_z^{(d)} = \iint_A y^2 dA_x(d), \quad (202)$$

where we take into account the density of states $c_2(d, y, z)$ in the expression of a fractal surface differential element, i.e.

$$dA_x(d) = c_2(d, y, z) dA_x.$$

In [40,41] it has been suggested to use the derivatives $c_1^{-1}(\alpha_x, x) D_x^1$ instead of the usual derivatives D_x^1 for fractal materials. If we use the derivatives $c_1^{-1}(\alpha_x, x) D_x^1$ instead of D_x^1 for fractal materials according to [40,41], then the Lagrangian for Euler–Bernoulli fractal beams takes the following form

$$\begin{aligned} \mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w) &= \frac{1}{2} \mu \left(D_t^1 w(x, t) \right)^2 + \frac{1}{2} (EI^{(d)}) \left((c_1^{-1}(\alpha_x, x) D_x^1 w(x, t) \right)^2 - \frac{1}{2} \\ &\quad \times (EI^{(d)}) I_F^2(d) \left((c_1^{-1}(\alpha_x, x) D_x^1 w(x, t) \right)^3 - q(x, t) w(x, t). \end{aligned} \quad (203)$$

Using the Lagrangian (201), the corresponding terms in the relevant Euler–Lagrange equation, i.e. Eq. (200), are

$$\frac{\partial \mathcal{L}}{\partial w} = -q(x, t), \quad \frac{\partial \mathcal{L}}{\partial D_t^1 w(x, t)} = \mu D_t^1 w(x, t), \tag{204}$$

$$\frac{\partial \mathcal{L}}{\partial D_x^2 w(x, t)} = (E I^{(d)}) D_x^2 w(x, t), \quad \frac{\partial \mathcal{L}}{\partial D_x^3 w(x, t)} = (E I^{(d)}) I_F^2(d) D_x^3 w(x, t). \tag{205}$$

Substitution of Eqs. (204) and (205) into Eq. (200) gives

$$\mu D_t^2 w + c_1^{-1}(\alpha_x, x) D_x^2 \left(c_1(\alpha_x, x) (E I^{(d)}) (D_x^2 w) \right) - I_F^2(d) c_1^{-1}(\alpha_x, x) D_x^3 \left(c_1(\alpha_x, x) (E I^{(d)}) D_x^3 w \right) - q(x, t) = 0, \tag{206}$$

which is the governing equation of motion for a fractal Euler–Bernoulli beam. For a non-fractal beam, we have $\alpha_x = 1, c_1^{-1}(\alpha_x, x) = 1$, and the standard gradient elasticity Euler–Bernoulli beam equation is recovered

$$\mu D_t^2 w + D_x^2 \left((E I) (D_x^2 w) \right) - I_F^2 D_x^3 \left((E I) D_x^3 w \right) - q(x, t) = 0, \tag{207}$$

where the beam can be non-homogeneous, and E and I may depend on x .

If the fractal beam is homogeneous (see Section 4.2), then E and $I^{(d)}$ are independent of x , and the beam equation has a simpler form

$$\mu D_t^2 w + (E I^{(d)}) c_1^{-1}(\alpha_x, x) D_x^2 \left(c_1(\alpha_x, x) D_x^2 w \right) - I_F^2(d) (E I^{(d)}) c_1^{-1}(\alpha_x, x) D_x^3 \left(c_1(\alpha_x, x) D_x^3 w \right) - q(x, t) = 0. \tag{208}$$

This equation can be expressed as

$$\mu D_t^2 w + E I^{(d)} \mathbb{D}_{x, \alpha_x}^4 w - I_F^2(d) E I^{(d)} \mathbb{D}_{x, \alpha_x}^6 w - q(x, t) = 0, \tag{209}$$

where we have used the notation

$$\mathbb{D}_{x, \alpha_x}^{2n} = c_1^{-1}(\alpha_x, x) D_x^n c_1(\alpha_x, x) D_x^n. \tag{210}$$

If $\alpha_x = 1$, then $c_1(\alpha_x, x) = 1$ and $\mathbb{D}_{x, \alpha_x}^{2n} = D_x^{2n}$.

Using the Lagrangian given by Eq. (203), the corresponding Euler–Lagrange equation has the form of Eq. (208), where the derivatives $\mathbb{D}_{x, \alpha_x}^{2n}$ are replaced by

$$\partial_{x, \alpha_x}^{2n} = (c^{-1}(\alpha_x, x) D_x^1)^{2n},$$

such that

$$\mu D_t^2 w + E I^{(d)} \partial_{x, \alpha_x}^4 w - I_F^2(d) E I^{(d)} \partial_{x, \alpha_x}^6 w - q(x, t) = 0. \tag{211}$$

For non-fractal materials, we have $\alpha_x = 1$ and Eqs. (209) and (211) have the form

$$\mu D_t^2 w + E I D_x^4 w - E I I_F^2(2) D_x^6 w - q(x, t) = 0. \tag{212}$$

This is the gradient elasticity Euler–Bernoulli beam equation for media without fractional non-locality, memory and fractality [16].

5.3. Second moment of area for fractal beam

In this section, we give an example of computation a second moment of the fractal beam’s cross-section by the method suggested in [34]. Let us consider a homogeneous fractal beam with circular cross-section. The second moment $I_z^{(d)} = I_z^{(d)}$ of the fractal beam’s cross-section is

$$I_z^{(d)} = \iint_A y^2 dA_x(d), \tag{213}$$

where $d = d_{yz}$ is the fractal dimension of the circular cross-section of the beam. In Eq. (213) we take into account the density of states $c_2(d, y, z)$ in the fractal material through the relation $dA_x(d) = c_2(d, y, z) dA_x$, where (x, y, z) are dimensionless variable.

Let us derive the polar moment of inertia $I_p^{(d)}$ for the circular cross-section. By using the equalities

$$I_y^{(d)} = I_z^{(d)}, \quad I_p^{(d)} = I_y^{(d)} + I_z^{(d)},$$

we find the moment of inertia by using the relationship

$$I_y^{(d)} = I_z^{(d)} = \frac{1}{2} I_p^{(d)}. \tag{214}$$

The equation for the polar moment of inertia $I_p^{(2)}$ can be written in the form

$$I_p^{(2)} = \rho_0 \int_A (y^2 + z^2) dA_2, \quad (215)$$

where $dA_2 = dydz$, $(x = x_1, y = x_2, z = x_3)$ are dimensionless Cartesian coordinates, and ρ_0 is the constant surface mass density.

The fractional generalization of Eq. (215) is given by expression

$$I_p^{(d)} = \rho_0 \int_A (y^2 + z^2) dA_d, \quad (216)$$

where

$$dA_d = c(d) (\sqrt{y^2 + z^2})^{d-2} dA_2, \quad c(d) = \frac{2^{2-d}}{\Gamma(d/2)}, \quad 0 < d \leq 2. \quad (217)$$

Substitution of Eq. (217) into Eq. (216) gives

$$I_p^{(d)} = \rho_0 c(d) \int_A (y^2 + z^2)^{d/2} dA_2. \quad (218)$$

In Eq. (216) we use the numerical factor $c(d)$ such that the limits $d \rightarrow (2 - 0)$ give the usual integral formula (215). For $d = 2$, Eq. (216) gives Eq. (215). The parameter $d = d_{yz}$ denotes the fractal mass dimension of the circular cross-section of the beam. This parameter can easily be calculated from the experimental data by using the box counting method for the cross-section of the beam.

Let us now consider the circular region A that is defined by

$$A = \{(y, z) : 0 \leq y^2 + z^2 \leq R^2\}. \quad (219)$$

In polar coordinates (ϕ, r) , we have

$$dA_2 = dydz = r dr d\phi, \quad (y^2 + z^2)^{d/2} = r^d. \quad (220)$$

Substitution of Eq. (220) into Eq. (218) gives

$$I_p^{(d)} = 2\pi \rho_0 c(d) \int_0^R r^{d+1} dr = \frac{2\pi \rho_0 c(d)}{(d+2)} R^{d+2}. \quad (221)$$

This equation defines the second moment of the fractal beam's cross-section. If $d = 2$, we obtain the well-known equation $I_p^{(2)} = (1/2)\pi \rho_0 R^4$.

The mass of the homogeneous fractal beam is

$$M_d = \rho_0 \int_A dA_d, \quad (222)$$

where dA_d is defined by Eq. (217), and ρ_0 is the constant surface mass density. Using the polar coordinates (220), we obtain the following mass expression

$$M_d = 2\pi \rho_0 c(d) \int_0^R r^{d-1} dr = \frac{2\pi \rho_0 c(d)}{d} R^d. \quad (223)$$

Substituting (223) into (221), we get

$$I_p^{(d)} = \frac{d}{d+2} M_d R^2, \quad (224)$$

where d is the fractal mass dimension of the beam's circular cross-section ($1 < d \leq 2$). If $d = 2$, we derive the well-known relation $I_p^{(2)} = (1/2)MR^2$. If we consider a fractal beam with mass and radius that are equal to the mass and radius of a beam with integer mass dimension, then these second moments are connected by the equation

$$I_p^{(d)} = \frac{2d}{d+2} I_p^{(2)}, \quad (225)$$

where $I_p^{(2)}$ is the moment for the homogeneous beam with the integer cross-section mass dimension $d = 2$.

Using the relation (214), we get

$$I_z^{(d)} = I_z^{(2)} = \frac{\pi \rho_0}{4} R^4 = \frac{d}{2(d+2)} M_d R^2 = \frac{2d}{d+2} I_z^{(2)}. \quad (226)$$

This is the second moment of a circular cross-section of the fractal beam with cross-section in the yz -plane and fractal dimension $d = d_{yz}$, which should be determined by experiment.

5.4. Gradient Euler–Bernoulli static equation for fractal beam

The gradient Euler–Bernoulli fractal homogeneous beam equation for the static case ($D_t^1 w = 0$ and $q(x, t) = q(x)$) is obtained from Eq. (208) as

$$D_x^2 \left(c_1(\alpha_x, x) D_x^2 w \right) - I_F^2(d) D_x^3 \left(c_1(\alpha_x, x) D_x^3 w \right) = \frac{c_1(\alpha_x, x)}{EI^{(d)}} q(x). \tag{227}$$

For a non-fractal beam ($\alpha_x = 1$), the static gradient Euler–Bernoulli beam equation takes the form

$$D_x^4 w - I_s^2 D_x^6 w = \frac{1}{EI^{(2)}} q(x). \tag{228}$$

It is noted that Eq. (227) for a fractal beam is analogous to the static case of Eq. (207) for a non-fractal beam ($\alpha_x = 1$ and $c_1(\alpha_x, x) = 1$), which is non-homogeneous such that the product $EI_{eff}^{(2)}$ depends on x as well as $c_1(\alpha_x, x)$, i.e. $EI_{eff}^{(2)} \sim x^{\alpha_x - 1}$ ($0 < \alpha_x < 1$). This effective static equation for a gradient Euler–Bernoulli non-homogeneous beam is expressed by

$$D_x^2 \left((EI_{eff}^{(2)}) (D_x^2 w) \right) - I_s^2 D_x^3 \left((EI_{eff}^{(2)}) D_x^3 w \right) = q_{eff}(x) \tag{229}$$

with the effective external load $q_{eff}(x) = c_1(\alpha_x, x)q(x)$.

For the homogeneous case ($q(x) = 0$), Eq. (227) can be written in the form

$$xD_x^4 w(x) + (\alpha_x - 1)D_x^4 w(x) - I_F^{-2}(d) x D_x^2 w(x) = C_5 x^{2-\alpha} + C_6 x^{3-\alpha}, \tag{230}$$

where we take into account the form of the density of states $c_1(\alpha_x, x) = x^{\alpha_x - 1} / \Gamma(\alpha_x)$ and $x > 0$. Here C_5 and C_6 are constants defined by the boundary conditions for the initial problem given by Eq. (227), which is a differential equation of 6th order. The general solution of Eq. (230) has the form

$$w(x) = C_1 + C_2 x + C_3 {}_1F_2 \left[-1/2; 1/2, \alpha_x/2 - 1; I_F^{-2}(d) x^2 / 4 \right] + C_4 \left(I_F^{-1}(d) x^{2-\alpha_x/2} K_{\alpha_x/2-1}(I_F^{-1}(d)x) + I_F^{\alpha_x/2-2}(d) x I(I_F^{-1}(d)x, \alpha_x) \right), \tag{231}$$

where C_1, C_2, C_3 and C_4 are constants defined by appropriate boundary conditions; ${}_1F_2[a_1; b_1, b_2; c]$ denotes the hypergeometric function; $K_a(x)$ denotes the modified (hyperbolic) Bessel function of the second kind; and $I(x, \alpha)$ is the integral of the Bessel function of the form

$$I(x, \alpha) = \int x^{1-\alpha_x/2} K_{\alpha/2-1}(x) dx. \tag{232}$$

We can also use the fundamental solution for ordinary differential equations (2.105) in Kamke's book [51] for the case $b < 0$ and $0 < a < 2$ where $b = -I_F^{\alpha_x/2-2}(d)$ and $a = \alpha_x - 1$.

5.5. Gradient Timoshenko equations for fractal beam

In this section we consider a gradient generalization of the Timoshenko beam equations for a fractal beam, as suggested in [40,41]. In the Timoshenko beam theory without axial effects, the displacement vector $\mathbf{u}(x, y, z, t)$ of the beam is assumed to be given by

$$u_x(x, y, z, t) = -z\varphi(x, t) \quad u_y(x, y, z, t) = 0, \quad u_z(x, y, t) = w(x, t), \tag{233}$$

where (x, y, z) are the coordinates of a point in the beam, (u_x, u_y, u_z) are the components of the displacement vector \mathbf{u} , $\varphi = \varphi(x, t)$ is the angle of rotation of the normal to the mid-surface of the beam, and $w = w(x, t)$ is the displacement of the mid-surface in the z -direction.

In [40,41] it is suggested to use the derivatives

$$\partial_{x,\alpha} = c_1^{-1}(\alpha_x, x) D_x^1, \quad \partial_{x,\alpha}^n = (\partial_{x,\alpha})^n \quad (n \in \mathbb{N}) \tag{234}$$

instead of the usual derivatives D_x^1 and D_x^n for fractal materials. If we use the derivatives given by Eq. (234) for fractal materials according to [40,41], then the gradient Timoshenko equation for a fractal beam can be derived from the force and moment balance equations

$$\rho A D_t^2 w = \partial_{x,\alpha} Q, \quad \rho I^{(d)} D_t^2 \varphi = Q - \partial_{x,\alpha} M, \tag{235}$$

with the bending moment M given by

$$M = -EI^{(d)} \partial_{x,\alpha} \left(\varphi - I_s^2 \partial_{x,\alpha}^2 \varphi \right), \tag{236}$$

and the shear force Q is

$$Q = kGA(\partial_{x,\alpha} w - \varphi) - I_s^2 kGA \partial_{x,\alpha}^2 (\partial_{x,\alpha} w - \varphi). \quad (237)$$

Then, the gradient Timoshenko equations for a homogeneous fractal beam have the form

$$\rho A D_t^2 w = kGA \partial_{x,\alpha} (\partial_{x,\alpha} w - \varphi) - I_s^2 kGA \partial_{x,\alpha}^3 (\partial_{x,\alpha} w - \varphi), \quad (238)$$

$$\rho I^{(d)} D_t^2 \varphi = kGA (\partial_{x,\alpha} w - \varphi) + EI^{(d)} \partial_{x,\alpha}^2 \varphi - I_s^2 kGA \partial_{x,\alpha}^2 (\partial_{x,\alpha} w - \varphi) - EI^{(d)} I_s^2 \partial_{x,\alpha}^4 \varphi. \quad (239)$$

The gradient Timoshenko fractal beam Eqs. (238) and (239) can also be derived from an appropriate variational principle. The Lagrangian for a Timoshenko fractal beam with gradient non-locality has the form

$$\begin{aligned} \mathcal{L}_{GTFB} = & \frac{1}{2} \rho I^{(d)} (D_t^1 \varphi(x, t))^2 + \frac{1}{2} \rho A (D_t^1 w(x, t))^2 - \frac{1}{2} (kGA) (\partial_{x,\alpha} w(x, t) - \varphi(x, t))^2 - \frac{1}{2} (EI^{(d)}) (\partial_{x,\alpha} \varphi(x, t))^2 \\ & - \frac{1}{2} (kGA) I_s^2 (\partial_{x,\alpha}^2 w(x, t) - \partial_{x,\alpha} \varphi)^2 - \frac{1}{2} (EI^{(d)}) I_s^2 (\partial_{x,\alpha}^2 \varphi(x, t))^2. \end{aligned} \quad (240)$$

Then, the stationary action principle gives the equations

$$\frac{\partial \mathcal{L}}{\partial w} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) - D_x^1 \left(\frac{\partial \mathcal{L}}{\partial D_x^1 w} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial D_x^2 w} \right) = 0, \quad (241)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 \varphi} \right) - D_x^1 \left(\frac{\partial \mathcal{L}}{\partial D_x^1 \varphi} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial D_x^2 \varphi} \right) = 0. \quad (242)$$

Eqs. (241) and (242) are the Euler–Lagrange equations for the fractal beams considered herein, as described by the Lagrangian given by Eq. (240). Substitution of Eq. (240) into (241) and (242) suggests that the gradient Timoshenko fractal beam equations (238) and (239) can be expressed as

$$\rho A D_t^2 w = kGA \partial_{x,\alpha} \left(1 - I_s^2 \partial_{x,\alpha}^2 \right) (\partial_{x,\alpha} w - \varphi), \quad (243)$$

$$\rho I^{(d)} D_t^2 \varphi = kGA \left(1 - I_s^2 \partial_{x,\alpha}^2 \right) (\partial_{x,\alpha} w - \varphi) + EI^{(d)} \partial_{x,\alpha}^2 \left(\varphi - I_s^2 \partial_{x,\alpha}^2 \varphi \right). \quad (244)$$

If $\alpha = 1$, then Eqs. (243) and (244) reduce to the gradient Timoshenko equations for a beam made by a homogeneous non-fractal material.

For the models based on [39–43], solutions of equations for fractal materials can be obtained from solutions of equations for non-fractal materials. Let $w_c(x, t)$ and $\varphi_c(x, t)$ be solutions of Eqs. (243) and (244) with $\alpha = 1$ and $x > 0$, i.e., of the gradient Timoshenko equations for homogeneous non-fractal beams. Then, the solutions $w_F(x, t)$ and $\varphi_F(x, t)$ of Eqs. (243) and (244) for a fractal beam with $0 < \alpha < 1$ can be represented in terms of w_c and φ_c as follows:

$$w_F(x, t) = w_c(x^\alpha / \Gamma(\alpha + 1), t), \quad \varphi_F(x, t) = \varphi_c(x^\alpha / \Gamma(\alpha + 1), t). \quad (245)$$

As an example, we consider the equation for an Euler–Bernoulli homogeneous fractal beam in the absence of a transverse load ($q(x) = 0$),

$$\rho A D_t^2 w(x, t) + EI^{(d)} \partial_{x,\alpha}^4 w(x, t) = 0. \quad (246)$$

This equation can be solved using the Fourier decomposition of the displacement into the sum of harmonic vibrations of the form $w(x, t) = \text{Re}[w(x) \exp(-i\omega t)]$, where ω is the frequency of vibration. Then, for each value of frequency, we can solve the ordinary differential equation

$$-\rho A \omega^2 w(x) + EI^{(d)} \partial_{x,\alpha}^4 w(x) = 0. \quad (247)$$

The boundary conditions for a cantilevered fractal beam of length L fixed at $x = 0$ are

$$w(0) = 0, \quad (\partial_{x,\alpha}^1 w)(0) = 0, \quad (248)$$

$$(\partial_{x,\alpha}^2 w)(L) = 0, \quad (\partial_{x,\alpha}^3 w)(L) = 0. \quad (249)$$

The solution for the Euler–Bernoulli homogeneous fractal beam is defined by

$$w_{F,n}(x) = w_0 (\cosh(k_n x^\alpha) - \cos(k_n x^\alpha) + C_n(\alpha) [\sin(k_n x^\alpha) - \sinh(k_n x^\alpha)]), \quad x \in [0; L], \quad (250)$$

where w_0 is a constant, and

$$C_n(\alpha) = \frac{\cos(k_n L^\alpha) + \cosh(k_n L^\alpha)}{\sin(k_n L^\alpha) + \sinh(k_n L^\alpha)}, \quad k_n = \frac{1}{\Gamma(\alpha + 1)} \left(\frac{\rho A \omega_n^2}{EI^{(d)}} \right)^{1/4}. \quad (251)$$

For the boundary conditions given by Eqs. (248) and (249), the solution (250) exist only if k_n are defined by

$$\cosh(k_n L) \cos(k_n L) + 1 = 0. \tag{252}$$

This trigonometric equation is solved numerically. The corresponding natural frequencies of vibration are $\omega_n = k_n^2 \sqrt{(EI^{(d)})/\rho A}$. For a non-trivial value of the displacement, w_0 is assumed to be arbitrary, and the magnitude of the displacement is taken as unknown for free vibrations. Usually, $w_0 = 1$ is used when plotting mode shapes.

5.6. Combined strain-acceleration gradients for fractal beam

Let us consider a 1D model for a fractal material that is described by the action

$$S[w] = \int dt \int dl_{\alpha_x} \mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w, D_x^2 D_t^1 w), \tag{253}$$

with the Lagrangian

$$\mathcal{L}(x, t, w, D_t^1 w, D_x^2 w, D_x^3 w) = \frac{1}{2} \rho A (D_t^1 w(x, t))^2 + \frac{1}{2} EI^{(d)} (D_x^2 w(x, t))^2 - \frac{1}{2} EI^{(d)} l_f^2(d) (D_x^3 w(x, t))^2 - q(x, t)w(x, t), \tag{254}$$

where $dl_{\alpha_x} = dx c_1(\alpha_x, x)$, takes into account combined strain-acceleration gradients [16]. The stationary action principle $\delta S_F[w] = 0$, gives the Euler–Lagrange equation in the form

$$\frac{\partial \mathcal{L}}{\partial w} - c_1(\alpha_x, x) D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w} \right) + D_x^2 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^2 w} \right) - D_x^3 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^3 w} \right) - D_t^1 D_x^2 \left(c_1(\alpha_x, x) \frac{\partial \mathcal{L}}{\partial D_x^2 D_t^1 w} \right) = 0. \tag{255}$$

For a homogeneous fractal beam, we obtain

$$\rho A D_t^2 w + EI^{(d)} \mathbb{D}_{x, \alpha_x}^4 w - l_f^2(d) EI^{(d)} \mathbb{D}_{x, \alpha_x}^6 w + l_f^2(d) \rho I^{(d)} D_t^2 \mathbb{D}_{x, \alpha_x}^4 w - q(x, t) = 0, \tag{256}$$

where the notation (210) was used.

In we use the fractional continuum model [33,32,13] with some changes suggested in [39–41], we derive the Euler–Lagrange equation in the form of Eq. (256), where the derivatives $\mathbb{D}_{x, \alpha_x}^{2n}$ are replaced by $\partial_{x, \alpha_x}^{2n} = (c^{-1}(\alpha_x, x) D_x^1)^{2n}$ such that

$$\rho A D_t^2 w + EI^{(d)} \partial_{x, \alpha_x}^4 w - l_f^2(d) EI^{(d)} \partial_{x, \alpha_x}^6 w + l_f^2(d) \rho I^{(d)} D_t^2 \partial_{x, \alpha_x}^4 w - q(x, t) = 0. \tag{257}$$

If the beam is non-fractal, then $D = 3$, $\alpha_x = 1$, $c_1(\alpha_x, x) = 1$, and Eqs. (256) and (257) take the form

$$\rho A D_t^2 w + EI^{(2)} D_x^4 w - l_s^2 EI^{(2)} D_x^6 w + l_d^2 \rho I^{(2)} D_t^2 D_x^4 w - q(x, t) = 0. \tag{258}$$

This is the usual combined strain-acceleration gradient beam model [16].

Note that Eq. (256) for a fractal beam is analogous to the equation for the usual combined strain-acceleration gradient non-fractal beam ($\alpha_x = 1$ and $c_1^{-1}(\alpha_x, x) = 1$), which is non-homogeneous, such that the product $E l_{eff}^{(2)}$ depends on x , as well as on $c_1(\alpha_x, x)$; i.e. $E l_{eff}^{(2)} \sim x^{2\alpha_x - 1}$ ($0 < \alpha_x < 1$). Eq. (257) can be solved by the method suggested in Section 5.5 from the solutions of Eq. (258) for non-fractal materials.

6. Conclusions

In this paper, we consider non-standard generalizations of the gradient elasticity theory [17–22] for complex materials with power-law non-locality, long-term memory and fractality. These non-standard generalizations may be important in describing unusual properties of nanomaterials [67,68].

To obtain the governing equations for the new fractional generalizations of gradient elasticity theory for materials with power-law non-locality, we use a new fractional variational principle for Lagrangians with Riesz fractional derivatives. New generalizations can also be obtained through extensions of the traditional variational calculus for Lagrangians by using other types of fractional derivatives [23–28], as well as with Riesz derivatives in the form suggested in [29]. We also assume that new fractional integral elasticity models can be derived by using the variational principle suggested in [30], where the Lagrangian contains fractional integrals instead of fractional derivatives.

The fractional approach, which is suggested in this paper, allows us to obtain exact analytical solutions of the fractional differential equations for models of a wide class of material with fractional gradient non-locality. A characteristic feature of the behavior of a fractional non-local continuum is the appearance of spatial power-tails of non-integer order. The fractional gradient models, which are suggested in this paper to describe complex materials with fractional non-locality, can be characterized by a common or universal spatial behavior of elastic materials in analogy to the universal temporal behavior of low-loss dielectrics [62–65].

The proposed generalization of gradient elasticity theory for fractal materials is based on the fractional continuum models proposed in [32–35] (see also [13,37]). In particular, equations for gradient models of fractal materials are obtained by a frac-

tional integral generalization of the variational principle suggested in [35,36] (see also [13]). In the framework of the fractional integral continuum model for fractal materials, modified variational principles considered in [37,38] can also be used.

We assume that new non-standard generalizations of the gradient elasticity models of fractal materials can be obtained by using the analysis on fractals [69,70], as well as by using the methods of the vector calculus for non-integer-dimensional spaces [47,48], and by also using a generalization of fractal lattice models [71–73].

The approach proposed in this paper is based on fractional integral continuum models and it may have a wide application because of the relatively small numbers of parameters that define fractal media of great complexity and rich structure. The fractional continuum model of fractal elastic materials can be used not only to calculate global values and stationary characteristics, but also to describe dynamical properties of fractal materials.

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