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Short communication

On chain rule for fractional derivatives

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For some types of fractional derivatives, the chain rule is suggested in the form $\mathcal{D}^{\alpha}_x f(g(x)) = (\mathcal{D}^1_g f(g))_{g=g(x)} \mathcal{D}^{\alpha}_x g(x)$. We prove that performing of this chain rule for fractional derivative $\mathcal{D}^{\alpha}_x$ of order α means that this derivative is differential operator of the first order ($\alpha = 1$). By proving three statements, we demonstrate that the modified Riemann–Liouville fractional derivatives cannot be considered as derivatives of non-integer order if the suggested chain rule holds.

Derivatives and integrals of non-integer orders [1–3] have a long history [1,4–8]. and these operators have wide applications in physics and mechanics (for example, see [9] and references therein). There are various types of fractional derivatives and integrals that are suggested by Riemann, Liouville, Riesz, Grünwald, Letnikov, Marchaud, Weyl, Caputo.

These fractional derivatives have a set of unusual properties and rules such as violations of semigroup property, the Leibniz and chain rules. For example, the chain rule for fractional derivative of a composite function (see Eq. 2.209 in Section 2.7.3 of [2]) is

$$\mathcal{D}^{\alpha}_x f(g(x)) = \frac{x^{\alpha} f(g(x))}{\Gamma(1-\alpha)} + \sum_{k=1}^{\infty} C^{\alpha}_k \frac{k!\, x^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} (D^m_g f(g))_{g=g(x)} \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{(D^r_x g)(x)}{r!} \right)^{a_r},$$

where $x > 0$, Σ extends over all combinations of non-negative integer values of $a_1, a_2, \dots, a_k$ such that $\sum_{r=1}^{k} r a_r = k$ and $\sum_r^k a_r = m$. Obviously, this equation is much more complicated than the chain rule for the first order derivative

$$\mathcal{D}^1_x f(g(x)) = (\mathcal{D}^1_g f(g))_{g=g(x)} \mathcal{D}^1_x g(x), \quad (1)$$

where $\mathcal{D}^1_x = d/dx$ is the usual derivative of first order. At the same time, the chain rule for the derivative of integer order $n \in \mathbb{N}$ has the form

$$D^n_x f(g(x)) = n! \sum_{m=1}^{n} (D^m_g f(g))_{g=g(x)} \sum \prod_{r=1}^{n} \frac{1}{a_r!} \left(\frac{D^r_x g(x)}{r!} \right)^{a_r}, \quad (2)$$

that is the Faá di Bruno's formula (for example, see [10] and references therein).

Authors of some papers ([11–18] and [24–28]) suggest new types of fractional derivatives and assume some usual or simplified rules for the suggested derivatives. For example, the chain rule, which is suggested in [11–16], has the form

$$\mathcal{D}^{\alpha}_x f(g(x)) = \left(\mathcal{D}^1_g f(g)\right)_{g=g(x)} \mathcal{D}^{\alpha}_x g(x), \quad (3)$$

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where $\mathcal{D}^{\alpha}$ is the fractional derivative of order α with respect to x. The chain rule in the form (3) has been suggested for the modified Riemann–Liouville fractional derivatives (see Eq. 3.12 of [11], Eq. 3.8 of [12], Eq. 4.4 of [13], Eq. 3.13 of [14], Eq. 4.4 of [15], Eq. 4.4 of [16], Eq. 2.15 of [17], Eq. 48 of [18]).

It is obvious that Eq. (3) cannot give (2) for $\alpha = n \in \mathbb{N}$ for $n > 1$. Therefore Eq. (3) cannot be considered as a correct generalization of the chain rule for fractional-order derivatives. In papers [11–16], the well-known equation for fractional derivative of power function

$$\mathcal{D}^{\alpha}_{x} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad (\alpha > 0, \quad \beta > 0, \quad x > 0) \tag{4}$$

is used (see Eq. 3.10 of [11], Eq. 3.6 of [12], Eq. 4.2 of [13], Eq. 3.11 of [14], Eq. 4.2 of [15], Eq. 4.2 of [16], Eq. 2.13 of [17], Eq. 46 of [18]).

Using the rule (3) and Eq. (4), we prove two following statements.

Statement 1. The chain rule (3) and Eq. (4), cannot be performed together for fractional derivatives of non-integer orders $\alpha \neq 1$.

Proof. To prove this statement, we use the functions $f(g) = g^{\beta}$ and $g(x) = x^{\gamma}$, where $\alpha, \beta, \gamma > 0$. Using (4) for these functions, we get

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = \mathcal{D}^{\alpha}_{x} x^{\beta\gamma} = \frac{\Gamma(\beta\gamma+1)}{\Gamma(\beta\gamma-\alpha+1)} x^{\beta\gamma-\alpha}, \tag{5}$$

$$\mathcal{D}^{\alpha}_{x} g(x) = \mathcal{D}^{\alpha}_{x} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \tag{6}$$

$$(\mathcal{D}^{1}_{g} f(g))_{g=g(x)} = (\mathcal{D}^{1}_{g} g^{\beta})_{g=x^{\gamma}} = (\beta g^{\beta-1})_{g=x^{\gamma}} = \beta x^{\gamma(\beta-1)}. \tag{7}$$

Substitution of (5)–(7) into Eq. (3) gives

$$\left(\frac{\Gamma(\beta\gamma+1)}{\Gamma(\beta\gamma-\alpha+1)} - \frac{\beta\,\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} \right) x^{\beta\gamma-\alpha} = 0, \quad (x > 0). \tag{8}$$

As a result, we have the condition

$$\frac{\Gamma(\beta\gamma+1)}{\Gamma(\beta\gamma-\alpha+1)} - \frac{\beta\,\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} = 0, \quad (\alpha, \beta, \gamma > 0), \tag{9}$$

which must be performed if the chain rule (3) holds. For example, if we use $\beta = 2$ and $\gamma = 1$, then condition (9) can be represented in the form

$$\frac{2\,(\alpha-1)}{\Gamma(3-\alpha)} = 0, \tag{10}$$

that holds for $\alpha = 1$ only. As a result, we prove that the chain rule (3) and Eq. (4) with $x \in \mathbb{R}_{+}$ cannot satisfied together for non-integer $\alpha \neq 1$. □

Statement 2. Performing of the chain rule (3) and Eq. (4) for fractional derivative $\mathcal{D}^{\alpha}_{x}$ of order α allows us represent this derivative as a differential operator of the first order, i.e.

$$\mathcal{D}^{\alpha}_{x} f(x) = a(x) \mathcal{D}^{1}_{x} f(x)$$

with some function $a(x)$.

Proof. The chain rule (3) with $g(x) = x^{\beta}$ has the form

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = \left(\mathcal{D}^{1}_{g} f(g)\right)_{g=x^{\beta}} \mathcal{D}^{\alpha}_{x} x^{\beta}. \tag{11}$$

Using (4), Eq. (11) gives

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = \frac{\Gamma(\beta+1)\, x^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} \frac{df(x)}{d(x^{\beta})}, \tag{12}$$

where $df(x)/d(x^{\beta})$ is the first order derivative of $f(x)$ with respect to x^{β}. The well-known expression

$$\frac{df(x)}{d(x^{\beta})} = \frac{df(x)}{dx} \left(\frac{d(x^{\beta})}{dx} \right)^{-1} = \frac{1}{\beta\, x^{\beta-1}} \frac{df(x)}{dx} \tag{13}$$

and $\Gamma(\beta+1) = \beta\,\Gamma(\beta)$ allows us to represent expression (11) in the form

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = a(x) \frac{df(x)}{dx}, \tag{14}$$

where

$$a(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} x^{1-\alpha}. \tag{15}$$

As a result, we prove that the chain rule (3) together with Eq. (4) for fractional derivative $\mathcal{D}^{\alpha}_{x}$ of order α gives that this derivative is a differential operator of integer order $\alpha = 1$ such that $\mathcal{D}^{\alpha}_{x} = a(x) D^{1}_{x}$. □

In the paper [18] it has been suggested three new simple chain rules for fractional derivatives.

The first chain rule, which is suggested by Eq. (63) in Lemma 12 of [18], has the form

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = (\mathcal{D}^{\alpha}_{g} f(g))_{g=g(x)} \left(\mathcal{D}^{1}_{x} g(x)\right)^{\alpha}. \tag{16}$$

The second chain rule suggested by Eq. (64) in Lemma 13 of [18] is

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = \left(\frac{f(g(x))}{g(x)}\right)^{1-\alpha} \left((\mathcal{D}^{1}_{g} f(g))_{g=g(x)}\right)^{\alpha} \mathcal{D}^{\alpha}_{x} g(x). \tag{17}$$

The third chain rule that is given by Eq. (68) in Lemma 14 of [18] has the form

$$\mathcal{D}^{\alpha}_{x} f(g(x)) = \Gamma(2-\alpha)\, g^{\alpha-1}(x)\, (\mathcal{D}^{\alpha}_{g} f(g))_{g=g(x)}\, \mathcal{D}^{\alpha}_{x} g(x). \tag{18}$$

Using Eq. (4) we can prove the following statement.

Statement 3. The chain rules (16), (17), (18) and Eq. (4) hold for fractional derivatives $\mathcal{D}^{\alpha}_{x}$ of order $\alpha > 0$ only if $\alpha = 1$.

Proof. Using (4) for the functions $f(g) = g^{\beta}$ and $g(x) = x^{\gamma}$ with $\alpha, \beta, \gamma > 0$, chain rules (16), (17), (18) gives the following conditions.

The first chain rule (16) gives

$$\frac{\Gamma(\beta\gamma + 1)}{\Gamma(\beta\gamma + 1 - \alpha)} - \frac{\gamma^{\alpha}\,\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} = 0. \tag{19}$$

The rule (17) gives the condition

$$\frac{\Gamma(\beta\gamma + 1)}{\Gamma(\beta\gamma + 1 - \alpha)} - \frac{\beta^{\alpha}\,\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} = 0. \tag{20}$$

The third chain (18) leads to the condition

$$\frac{\Gamma(\beta\gamma + 1)}{\Gamma(\beta\gamma + 1 - \alpha)} - \frac{\Gamma(2-\alpha)\,\Gamma(\beta + 1)\,\Gamma(\gamma + 1)}{\Gamma(\beta + 1 - \alpha)\,\Gamma(\gamma + 1 - \alpha)} = 0. \tag{21}$$

Let us use $\beta = 3$, $\gamma = 2$ for condition (19) and $\beta = 2$, $\gamma = 3$ for condition (20). Then these conditions can be represented in the form of the equation

$$2^{\alpha}(\alpha - 6)(\alpha - 5)(\alpha - 4) + 120 = 0. \tag{22}$$

This equation has solution for $\alpha = 0$ and $\alpha = 1$ only.

For $\beta = 2$ and $\gamma = 2$, conditions (21) gives

$$\alpha(\alpha - 1) = 0. \tag{23}$$

As a result, we can see that (19), (20) and (20) with $\alpha > 0$ hold only for $\alpha = 1$. Therefore, the chain rules (16), (17), (18) for fractional derivatives of order $\alpha > 0$ cannot hold for $\alpha \neq 1$. □

The possible assumption that (3) holds only for non-differentiable functions $f(x)$ and $g(x)$. is incorrect also. Eq. (3) means that the fractional derivatives $\mathcal{D}^{\alpha}_{x} g(x)$, $\mathcal{D}^{\alpha}_{x} f(x)$ and $\mathcal{D}^{1}_{g} f(g)$ exist, i.e. the functions $f(x)$ and $g(x)$ should be fractional differentiable. Therefore arbitrary non-differentiable functions cannot be considered in the chain rule (3). Using Eq. (4), which is used in ([11–16], we can see that author assumes that the power functions x^{γ} ($\gamma > 0$) are fractional differentiable. Using that power functions are fractional differentiable, we can consider the chain rule (3) for the power functions including integer values of β and γ. As a result, we get that the chain rule (3) holds only for $\alpha = 1$ and derivative $\mathcal{D}^{\alpha}_{x}$ is integer derivative of first order. Simple chain rule (3) should be violated for fractional-order derivatives (see also [23]).

It should be noted that the unviolated Leibniz rule in the form

$$\mathcal{D}^{\alpha}_{x}(f(x)\, g(x)) = (\mathcal{D}^{\alpha}_{x} f(x))\, g(x) + f(x)\, (\mathcal{D}^{\alpha}_{x} g(x)) \tag{24}$$

cannot hold for fractional derivatives of order $\alpha \neq 1$ for sets of differentiable and non-differentiable functions [19–22].

Unusual properties of fractional derivatives of non-integer orders that are represented by deformation of the usual Leibniz rule and the usual chain rule can be considered as the characteristic properties of fractional-order derivatives. Furthermore these properties allow us to describe new unusual type of physical systems and media.

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