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Electromagnetic waves in non-integer dimensional spaces and fractals

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ABSTRACT

Electromagnetic waves in non-integer dimensional spaces are considered in the framework of continuous models of fractal media and fields. Using the recently suggested vector calculus for non-integer dimensional space, we consider electromagnetic fields in isotropic case. This D-dimensional calculus allows us to describe fractal properties by continuous models with non-integer dimensional spaces. We prove that the wave equation for non-integer dimensional space is similar to equation of waves in non-fractal medium with heterogeneity of power-law type. The speed of electromagnetic waves and the effective refractive index of non-integer dimensional spaces and fractals are discussed.

1. Introduction

Fractals are measurable metric sets with non-integer dimensions [1,2]. We can describe fractal media by using methods of “analysis on fractals” [3,4]. At present an application of the “analysis on fractals” to solve differential equations on fractals [4] for real physical problems is limited by a weak development of this area of mathematics. We can consider fractal media as continuous media in non-integer dimensional space. The non-integer dimension does not reflect all properties of the fractal media, but it is a main characteristic of fractal media. For this reason, continuous models with non-integer dimensional spaces can allow us to get some important conclusions about the behavior of the fractal media.

Continuous models for fractal distributions of charges, currents, media and fields have been proposed in [5–9]. These models are based on the notion of power-law density of states [10]. To take into account this density of states, we use the fractional-order integrals that is connected with fractional-dimensional integration [8,10]. It should be noted that fractional-order integrals and derivatives are used to describe fractional nonlocal models, which are based on fractional-order vector calculus [11] in general. The suggested continuous models of fractal media and electromagnetic fields have been developed in works [12–14] and [15–24] to describe anisotropic fractal media and electromagnetic waves in fractional space. Continuous models that are used in [15–24], are based on fractional dimensional generalizations of the scalar Laplace operators, which are proposed in papers [25,26]. It should be noted that the first-order differential vector operators (gradient, divergence, curl), and the vector Laplacian are not considered in [25,26]. This greatly restricts us in application of non-integer dimensional space approach to describe fractal media and fields. For example, the scalar Laplacian cannot be used for the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic fields $\mathbf{B}(\mathbf{r}, t)$ in the framework of continuous models with non-integer dimensional spaces.

An attempt to suggest first-order differential vector operators for non-integer dimensional spaces has been proposed in [18–24]. In these works, the operators are suggested

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only as approximations of the square of the Laplace operator. Recently a generalization of differential vector operators of first orders (grad, div, curl), the scalar and vector Laplace operators for non-integer dimension spaces have been suggested in papers [28–30] without any approximation. This allows us to extend the application area of continuous models with non-integer dimensional spaces. Using this new D-dimensional vector calculus, we can describe isotropic and anisotropic fractal media by using the non-integer dimensional space approach.

In this paper, we use the non-integer dimensional vector calculus, which is proposed in paper [28], to describe electromagnetic waves in non-integer dimensional spaces, fractals and isotropic fractal media. We prove that the wave equations for non-integer dimensional spaces are similar to the equations of waves in usual (non-fractal) media with power-law heterogeneity.

2. Vector differentiation for non-integer dimensional space

In the continuous models of fractal media, it is convenient to work with the physically dimensionless variables $x/R_0 \to x$, $y/R_0 \to x$, $z/R_0 \to x$, $\mathbf{r}/R_0 \to \mathbf{r}$, where R_0 is a characteristic size of considered model. This yields dimensionless integration and dimensionless differentiation in D-dimensional space. In this case the physical quantities of fractal media have correct physical dimensions.

Let us give some introduction to non-integer dimensional differentiation of integer orders (for details, see [25–29]). The vector differential operators for non-integer dimension have been derived in [28] by analytic continuation in dimension from integer n to non-integer D.

For simplification we will consider spherically symmetric case of fractal media, where scalar field φ and vector fields $\mathbf{E}$, $\mathbf{B}$ are independent of angles

$$\varphi(\mathbf{r},t)=\varphi(r,t), \quad \mathbf{E}(\mathbf{r},t)=E_r(r,t)\,\mathbf{e}_r, \quad \mathbf{B}(\mathbf{r},t)=B_r(r,t)\,\mathbf{e}_r,$$

where $\mathbf{e}_r=\mathbf{r}/r$, $r=|\mathbf{r}|$. Here $E_r=E_r(r)$ and $B_r=B_r(r)$ are the radial component of $\mathbf{E}$ and $\mathbf{B}$. In this case, we will work with rotationally covariant functions only. This simplification is analogous to the simplification of integration over non-integer dimensional space suggested in [27]. One of the main our simplification is that the electromagnetic components are radial functions. We note that for random fractals, this assumption is natural [38,39].

In general, the dimension D of the region V_D of fractal media and the dimension d of boundary $S_d=\partial V_D$ of this region are not related by the equation $d=D-1$, i.e.,

$$\dim(\partial V_D) \neq \dim(V_D)-1, \tag{1}$$

where $\dim(V_D)=D$ and $\dim(\partial V_D)=d$. We will use the parameter

$$\alpha_r = D-d, \tag{2}$$

which is a dimension of fractal medium along the radial direction.

In [28], the differential operators for non-integer D have been proposed in the following forms.

For non-integer dimensional space, the divergence operator for the vector field $\mathbf{E}=\mathbf{E}(r)$ can be represented [28] in the form

$$\mathrm{Div}_r^{D,d}\,\mathbf{E} = \pi^{(1-\alpha_r)/2}\,\frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \times\left(\frac{1}{r^{\alpha_r-1}}\frac{\partial E_r(r)}{\partial r}+\frac{d}{r^{\alpha_r}}E_r(r)\right). \tag{3}$$

This is (D, d)-dimensional divergence operator for fractal media with $d\neq D-1$. For $\alpha_r=1$, i.e. $d=D-1$, Eq. (3) gives

$$\mathrm{Div}_r^{D}\,\mathbf{E} = \frac{\partial E_r(r)}{\partial r}+\frac{D-1}{r}E_r(r). \tag{4}$$

The gradient for the scalar field $\varphi(\mathbf{r})=\varphi(r)$ depends on the radial dimension α_r [28] in the form

$$\mathrm{Grad}_r^{D,d}\,\varphi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\,r^{\alpha_r-1}}\,\frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r. \tag{5}$$

For $\alpha_r=1$, i.e. $d=D-1$, the gradient in non-integer dimensional space is

$$\mathrm{Grad}_r^{D}\,\varphi = \frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r. \tag{6}$$

The curl operator for the vector field $\mathbf{E}=\mathbf{E}(r)$ is equal to zero, $\mathrm{Curl}_r^D\,\mathbf{E}=0$.

Using the operators (3) and (5) for the fields $\varphi=\varphi(r)$ and $\mathbf{E}=E(r)\,\mathbf{e}_r$, in paper [28] we obtain the scalar and vector Laplace operators for the case $d\neq D-1$ by the equation

$${}^S\Delta_r^{D,d}\varphi = \mathrm{Div}_r^{D,d}\,\mathrm{Grad}_r^{D,d}\,\varphi, \quad {}^V\Delta_r^{D,d}\mathbf{E} = \mathrm{Grad}_r^{D,d}\,\mathrm{Div}_r^{D,d}\,\mathbf{E}. \tag{7}$$

Then the scalar Laplacian for $d\neq D-1$ for the field $\varphi=\varphi(r)$ is

$${}^S\Delta_r^{D,d}\varphi = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)} \times\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2\varphi}{\partial r^2}+\frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial\varphi}{\partial r}\right), \tag{8}$$

For $\alpha_r=1$, i.e. $d=D-1$, Eq. (8) gives

$$\Delta_r^D\varphi = \mathrm{Div}_r^D\mathrm{Grad}_r^D\varphi = \frac{\partial^2\varphi}{\partial r^2}+\frac{D-1}{r}\frac{\partial\varphi}{\partial r}S, \tag{9}$$

where we use $\Gamma(1/2)=\sqrt{\pi}$.

The vector Laplacian in non-integer dimensional space with $d\neq D-1$ and the field $\mathbf{E}=E_r(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^{D,d}\mathbf{E} = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2E_r(r)}{\partial r^2} +\frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial E_r(r)}{\partial r}-\frac{d\alpha_r}{r^{2\alpha_r}}E_r(r)\right)\mathbf{e}_r. \tag{10}$$

For $\alpha_r=1$, i.e. $d=D-1$, Eq. (10) gives

$${}^V\Delta_r^{D}\mathbf{E} = \mathrm{Grad}_r^D\,\mathrm{Div}_r^D\,\mathbf{E} = \left(\frac{\partial^2E_r(r)}{\partial r^2}+\frac{D-1}{r}\frac{\partial E_r(r)}{\partial r}-\frac{D-1}{r^2}E_r(r)\right)\mathbf{e}_r. \tag{11}$$

For $D=3$ Eqs. (4)–(11) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\varphi=\varphi(r)$ and $\mathbf{E}(r)=E_r(r)\,\mathbf{e}_r$.

The vector differential operators (3), (5), (8) and (10), which are suggested in [28], allow us to describe complex fractal media with the boundary dimension $d\neq D-1$ by the non-integer dimensional space approach.

The suggested operators allow us to reduce D-dimensional vector differentiations to usual derivatives of integer orders with respect to $r = |\mathbf{r}|$. As a result, we can reduce spatial partial differential equations for fields in non-integer dimensional space to ordinary differential equations with respect to r.

We should note that Laplacian, which is suggested in [25], can be applied only for scalar fields and it cannot be used to describe vector fields $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ and $\mathbf{B} = B_r(r)\,\mathbf{e}_r$ since the Stillinger's Laplacian for $D = 3$ is not equal to the usual vector Laplacian for $\mathbb{R}^3$. For the electric and magnetic vector fields $\mathbf{E}, \mathbf{B}$ of isotopic fractal case, we should use the vector Laplace operator (11), which is proposed in [28]. Note that the gradient, divergence, curl operator and vector Laplacian are not considered in [25].

3. Wave equation for electric potential in fractal electrodynamics

Let us consider a spherically symmetric case, where scalar field φ is independent of angles $\varphi(\mathbf{r}, t) = \varphi(r, t)$. In the absence of charges and currents the wave equation for electric potential $\varphi(r, t)$. for non-integer dimensional space has the form

$$ {}^{S}\Delta_r^{D,d}\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{12}$$

where c is the speed of light in vacuum (299792458 meters per second) or in the non-fractal medium, and ${}^{S}\Delta_r^{D,d}$ is the scalar Laplacian. Using (8), the wave Eq. (12) can be written as

$$\frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2\varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial\varphi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0. \tag{13}$$

If $\alpha_r = 1$ (i.e. $d = D - 1$), then we have the equation

$$\frac{\partial^2\varphi(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial\varphi(r)}{\partial r} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0. \tag{14}$$

If $D = 3$ and $d = 2$, then $\alpha_r = 1$ and Eqs. (13) and (14) have the form of the usual (non-fractal) wave equations.

Eq. (13) can be represented in the form

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r}\frac{\partial\varphi}{\partial r} - \frac{1}{c^2_{eff}(d,\alpha_r,r)}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{15}$$

where c_{eff} the effective speed of electromagnetic wave in non-integer dimensional space

$$c_{eff}(d,\alpha_r,r) = c\sqrt{\frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}}\, r^{1-\alpha_r} = \frac{c}{n_{eff}(d,\alpha_r,r)}. \tag{16}$$

Here we use $n_{eff}(d, \alpha_r, r)$ that is the refractive index of non-integer dimensional space (or fractal), which is defined by the equation

$$n_{eff}(d,\alpha_r,r) = \sqrt{\frac{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}}\, r^{\alpha_r-1}. \tag{17}$$

Let us consider a solution of the wave equation in the form

$$\varphi(r,t) = r^{\beta} f(r,t). \tag{18}$$

Substitution of (18) into (15) gives

$$\frac{\partial^2 f}{\partial r^2} + \frac{2\beta+d+1-\alpha_r}{r}\frac{\partial f}{\partial r} + \frac{\beta+d-\alpha_r}{r^2} f - \frac{1}{c^2_{eff}(d,\alpha_r,r)}\frac{\partial^2 f}{\partial t^2} = 0. \tag{19}$$

The second and third terms vanish if

$$\beta = -1, \quad d = \alpha_r + 1. \tag{20}$$

Note that we have the condition on space and boundary dimensions in the form $D - d = \alpha_r = d - 1$ in addition to $\beta = -1$. In this case, Eq. (19) has the form

$$\frac{\partial^2 f}{\partial r^2} - \frac{1}{c^2_{eff}(d,\alpha_r,r)}\frac{\partial^2 f}{\partial t^2} = 0. \tag{21}$$

Note that we get this equation only if the dimensions d and α_r are connected by the relation $d = \alpha_r + 1$. Eq. (21) for non-integer dimensional space with $d = \alpha_r + 1$ is similar to equation of propagation of waves in non-fractal medium with heterogeneity of power type.

If the condition $d = \alpha_r + 1$ holds, then we can represent the effective refractive index $n_{eff}(d, \alpha_r, r)$ in the form

$$n_{eff}(d,\alpha_r,r) = \sqrt{\frac{\pi^{\alpha_r-1/2}\,\alpha_r}{2\,\Gamma(\alpha_r+1/2)}}\, r^{\alpha_r-1}, \tag{22}$$

where we use $\Gamma(\alpha/2+1) = (\alpha/2)\,\Gamma(\alpha/2)$.

For the case $\alpha_r > 1$, i.e. $D > d + 1$, we can get a solution of Eq. (21). Let us rewrite Eq. (21) in the form

$$x^{-2(\alpha_r-1)}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0, \tag{23}$$

where $t \in \mathbb{R}$ and the variable x is defined by

$$x = r\left(\frac{\pi^{\alpha_r-1/2}\,\alpha_r}{2\,\Gamma(\alpha_r+1/2)}\right)^{1/\alpha_r}. \tag{24}$$

If we assume that

$$f(0,t) = f_0(t), \quad \left(\frac{\partial f}{\partial x}\right)_{x=0} = g_0(t), \tag{25}$$

then Eq. (23) has the solution (see Section 5.3.4.13 of [31]) in the form

$$\begin{aligned} f(x,t) = {} & \frac{\Gamma(2\beta_r)}{\Gamma^2(\beta_r)}\int_0^1 f_0\Big(t + \frac{1}{\alpha_r}x^{\alpha_r}(2\xi-1)\Big) \times (\xi\,(1-\xi))^{\beta_r-1}\,d\xi \\ & + \frac{\Gamma(2-2\beta_r)}{\Gamma^2(1-\beta_r)}\int_0^1 g_0\Big(t + \frac{1}{\alpha_r}x^{\alpha_r}(2\xi-1)\Big) \times (\xi\,(1-\xi))^{-\beta_r}\,d\xi, \end{aligned} \tag{26}$$

where

$$\beta_r = \frac{\alpha_r - 1}{2\,\alpha_r}. \tag{27}$$

Let us discuss the effective refractive index $n_{eff}(d, \alpha_r, r)$, which is defined by Eqs. (17) and (22), and the corresponding effective speed of electromagnetic waves c_{eff} for non-integer dimensional spaces and fractal sets.

Using Eq. (22), it is easy to see that $n_{eff}(d, \alpha_r, r) = 1$ and $c_{eff} = c$ if $\alpha_r = 1$, since $\Gamma(1/2) = \sqrt{\pi}$.

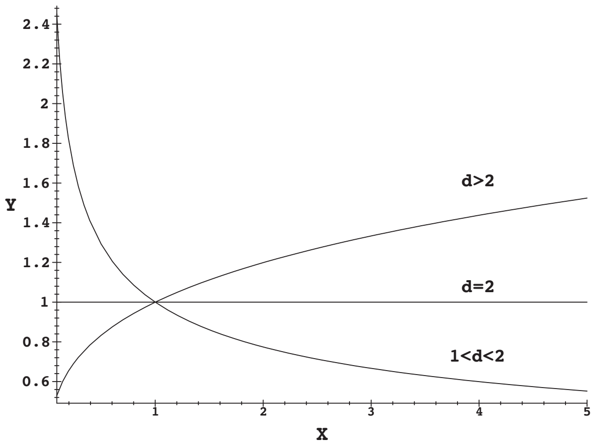

Fig. 1. Plots of the effective refractive index $y = n_{eff}(d, \alpha_r, r)$, where $x = r/r_{eff}(d, \alpha_r)$ and $d = \alpha_r + 1$ with $\alpha_r = \ln(2)/\ln(3) < 1$ (Cantor dust), $\alpha_r = 1$ (non-fractal line) and $\alpha_r = \ln(4)/\ln(3) > 1$ (Koch curve).

For all $\alpha_r > 1$ and $d = \alpha_r + 1$, we have $n_{eff}(d, \alpha_r, r) \geq 1$ and $c_{eff} \leq c$ if $r \geq r_{eff}(d, \alpha_r)$, where

$$r_{eff}(d, \alpha_r) = \left(\frac{2\,\Gamma(\alpha_r + 1/2)}{\pi^{\alpha_r - 1/2}\,\alpha_r} \right)^{1/(2-2\alpha_r)}. \tag{28}$$

For all $0 < \alpha_r < 1$ and $d = \alpha_r + 1$, we have $n_{eff}(d, \alpha_r, r) \geq 1$ and $c_{eff} \leq c$ if $r \leq r_{eff}(d, \alpha_r)$. In Fig. 1, we presented plots of the dependence of the effective refractive index $y = n_{eff}(d, \alpha_r, r)$ on the variable $x = r/r_{eff}(d, \alpha_r)$ and dimension $d = \alpha_r + 1$.

For the cases: (a) $\alpha_r > 1$, $d = \alpha_r + 1$ and $r \leq r_{eff}(d, \alpha_r)$, (b) $0 < \alpha_r < 1$, $d = \alpha_r + 1$ and $r \geq r_{eff}(d, \alpha_r)$, we have $n_{eff}(d, \alpha_r, r) \leq 1$ and $c_{eff} \geq c$ that can be considered as an effective tachyon mode caused by non-integer dimensionality of space or fractality.

Note that it should not be confused between the effective tachyon mode and the tachyon . The tachyon is a hypothetical particle that always moves faster than light. The effective tachyon mode is the light (electromagnetic wave), the speed of which seems more than c for an outside observer. The effective tachyon mode corresponds to fractal properties of space. For example, if the light overcomes some region of space with size R faster than $t < R/c$, then we can state that this space region has fractal properties.

The question arises whether the tachyon modes are a disadvantage of models with non-integer dimension of space or it is characteristic property of fractal models.

Let us note some basic properties of fractal media (for details, see [1,10]). For simplification, we can consider a uniform mass distribution along a line L. For many cases, we can write the asymptotic form for the relation between the mass $M(L)$ of the line segment L, and the radius r containing this mass as follows:

$$M_\alpha = M_0 \left(\frac{r}{r_0} \right)^\alpha \tag{29}$$

for $r/r_0 \gg 1$. The constant M_0 depends on how the sphere of radius r_0 are packed. The parameter α does not depend on the shape of L, or on whether the packing of sphere of radius r_0 is close packing, a random packing or a porous packing with a uniform distribution of holes. The number α is called the mass dimension. The fractal mass dimension is a measure of how the medium fills the space (line) it occupies. The fractality of the distribution of particles means that the mass M_α of the line segment L increase more slowly than the one-dimensional length of this region. The equation can be rewritten in the form

$$M_\alpha = \rho_0(\alpha)\, L_\alpha(r), \tag{30}$$

where $L_\alpha(r)$ is an effective length of the fractal line,

$$L_\alpha(r) = \omega(\alpha)\, r^\alpha, \tag{31}$$

and $\rho_0(\alpha)$ is an effective line density of mass, $\omega(\alpha)$ is numerical coefficient [10]. Note that we use the dimensionless coordinate variables r and r_0. Therefore SI units of the mass density $\rho_0(\alpha)$ is kg. Usually the numerical coefficient is defined as a volume of α-dimensional unit ball

$$\omega(\alpha) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2 + 1)}. \tag{32}$$

As examples, we can consider a uniform mass distribution along the one-dimensional lines that are used to construct two well-known fractal sets such as the Cantor dust and the Koch curve [1,2].

Cantor dust. The Cantor set is defined by repeatedly removing the middle thirds of line segments. The fractal dimension of the Cantor dust is

$$\alpha = \ln(2)/\ln(3) = \log_3(2) \approx 0.631 < 1. \tag{33}$$

Koch curve. The Koch curve is defined by repeatedly removing the middle thirds of line segments and then replacing this interval by equilateral triangle without this segment. The fractal dimension of the Koch curve is

$$\alpha = \ln(4)/\ln(3) = \log_3(4) \approx 1.262 > 1. \tag{34}$$

Using the effective length (31) and the equation for effective speed $c_{eff} L_\alpha(r) = c\, L_1(r)$, we can represent the effective refractive index in the form

$$n_{eff}(d, \alpha_r, r) = \frac{c}{c_{eff}} = \frac{L_\alpha(r)}{L_1(r)} = \frac{\omega(\alpha)}{\omega(1)} \frac{r^\alpha}{r} = \frac{\omega(\alpha)}{\omega(1)}\, r^{\alpha-1}. \tag{35}$$

As a result, we see that the existence of the tachyon modes is characteristic property of fractal models and models with non-integer dimension of space if we assume that fields exist in fractal set only. Obviously the effective tachyon mode cased by non-integer dimensionality of space does not exist if the electromagnetic waves can travel not only in fractal set. The tachyon modes can be used to locate fractal areas of space-time in the microcosm or cosmic space: if the light traverses a path between two spatial points A and B in a time $t < |AB|/c$, then this means that there is a fractal region between these points.

4. Conclusion

The vector calculus for non-integer dimensional space has been suggested in recent papers [28,29]. This D-dimensional calculus includes generalizations of differential vector operators of first orders (gradient, divergence, curl operators), the scalar and vector Laplace operators. It allows us to describe fractal media and fields for isotropic and anisotropic cases by using continuous models with non-integer dimensional spaces. In this paper, we use the vector calculus for non-integer dimensional space, which is proposed in paper [28], to describe electromagnetic waves in non-integer dimensional spaces and fractals.

It should be noted that the D-dimensional vector calculus of [28] cannot be applied for fractal media and fields in the anisotropic case. For anisotropic fractal media the gradient, divergence, and curl operators have been proposed in papers [18–22] and book [23]. In these works the first-order differential vector operators are defined only as approximations of the square of Laplace operator suggested in [26]. New approach to generalize grad, div, curl operators and the corresponding scalar and vector Laplace operators for anisotropic fractal case have been proposed in paper [29] without any approximations. This approach can be used for more rigorous description of anisotropic fractal media and fields by continuous models with non-integer dimensional spaces.

Let us note the Lorentz invariance in the suggested non-integer dimensional space approach to fractal electrodynamics. It is well-known that non-integer dimensions are widely used in quantum electrodynamics and theory of quantum fields to remove the ultraviolet divergences by renormalization. The dimensional regularization [27,32] is the best renormalization of quantum field theories. The main advantage of this regularization is that it preserves the Lorentz, Poincare and gauge invariances. This is a main motivation to apply the dimensional regularization in quantum field theories. As a result, we can state that the non-integer dimensional space approach allows us to preserve the Lorentz invariance, the Poincare invariance and the gauge invariance in the fractal electrodynamics [33].

We assume that the suggested approach can be used in engineering [34,35] to describe properties of fractal antennas, apertures and arrays. The proposed approach, which is based on non-integer dimensional spaces, can be applied in cosmic electrodynamics [36,37] to describe fractal distribution of charges, currents and fields. Fractal electrodynamics [33] can be important for astrophysics and cosmology. We assume that the suggested concept of tachyon modes can allow us to locate fractal areas of space-time in the Universe and the surrounding cosmic space.

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