

# Fractional generalization of Liouville equations

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In this paper fractional generalization of Liouville equation is considered. We derive fractional analog of normalization condition for distribution function. Fractional generalization of the Liouville equation for dissipative and Hamiltonian systems was derived from the fractional normalization condition. This condition is considered as a normalization condition for systems in fractional phase space. The interpretation of the fractional space is discussed. © 2004 American Institute of Physics. [DOI: 10.1063/1.1633491]

**We call a fractional equation a differential equation that uses fractional derivatives or integrals. Fractional derivatives and integrals have found many applications in recent studies of scaling phenomena. We formulate fractional analog of main integro-differential equation to describe some scaling process—Liouville equation. Usually used for scaling phenomena Fokker–Planck equation can be derived from Liouville equation. Therefore it is interesting to consider a fractional generalization of the Liouville equation. To derive fractional equation for a distribution function we must consider a fractional analog of the normalization condition for distribution function. Most of the fractional equations for distribution function does not use correspondent normalization condition. Therefore these equations (with fractional coordinate derivatives) can be incorrect equations. In this paper fractional Liouville equation for dissipative systems is derived from the normalization condition. The coordinate fractional integration for this normalization condition is used.**

## I. INTRODUCTION

Fractional derivatives and integrals have found many applications in recent studies of scaling phenomena.<sup>1–5</sup> The main aim of most of these papers is to formulate fractional integro-differential equations to describe some scaling process. Modifications of equations governing physical processes such as the Langevin equation,<sup>6</sup> diffusion equations, and Fokker–Planck equation have been suggested<sup>7–13</sup> which incorporate fractional derivatives with respect to time. It was shown in Ref. 14 that the chaotic Hamiltonian dynamics of particles can be described by using fractional generalization of the Fokker–Planck–Kolmogorov equation. In Ref. 14, coordinate fractional derivatives in the Fokker–Planck equation were used.

It is known that Fokker–Planck equation can be derived from Liouville equation.<sup>15,16</sup> Therefore it is interesting to consider a fractional generalization of Liouville equation and Bogoliubov hierarchy equations. We call a fractional equa-

tion a differential equation that uses fractional derivatives or integrals.

To derive fractional equations for a distribution function we must consider a fractional analog of the normalization condition for distribution function. Fractional Liouville equation for dissipative systems is derived from the normalization condition. In this paper, the coordinate fractional integration for normalization condition is used. This condition is considered as a normalization condition for systems in fractional phase space. If any fractional equation for distribution function does not use correspondent normalization condition, then this equation (with fractional coordinate derivatives) can be incorrect.

In Sec. II the normalization condition for distribution function and notations are considered. In Sec. III we derive the Liouville equation from the normalization condition. In Sec. IV, the physical interpretation of fractional normalization condition is considered. Finally, a short conclusion is given in Sec. V.

## II. NORMALIZATION CONDITION

Let us consider a distribution function  $\rho(x,t)$  for  $x \in R^1$ . Let  $\rho(x,t) \in L_1(R^1)$ , where  $t$  is a parameter. Normalization condition has the form

$$\int_{-\infty}^{+\infty} \rho(x,t) dx = 1.$$

This condition can be rewritten in the form

$$\int_{-\infty}^y \rho(x,t) dx + \int_y^{+\infty} \rho(x,t) dx = 1, \quad (1)$$

where  $y \in (-\infty, +\infty)$ .

Let  $\rho(x,t) \in L_p(R^1)$ , where  $1 < p < 1/\alpha$ . Fractional integrations on  $(-\infty, y)$  and  $(y, +\infty)$  are defined<sup>17</sup> by

$$(I_+^\alpha \rho)(y,t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^y \frac{\rho(x,t) dx}{(y-x)^{1-\alpha}}, \quad (2)$$

$$(I_-^\alpha \rho)(y,t) = \frac{1}{\Gamma(\alpha)} \int_y^{+\infty} \frac{\rho(x,t) dx}{(x-y)^{1-\alpha}}. \quad (3)$$

Using these notations, Eq. (1) has the form

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$$(I_+^1 \rho)(y, t) + (I_-^1 \rho)(y, t) = 1.$$

Using definitions (2) and (3) we can get the fractional analog of normalization condition (1),

$$(I_+^\alpha \rho)(y, t) + (I_-^\alpha \rho)(y, t) = 1.$$

Equations (2) and (3) can be rewritten in the form

$$(I_\pm^\alpha \rho)(y, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \rho(y \mp x, t) dx. \tag{4}$$

This leads to the normalization condition

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} [\rho(y-x, t) + \rho(y+x, t)] dx = 1. \tag{5}$$

If we denote

$$\tilde{\rho}(x, t) = \rho(y-x, t) + \rho(y+x, t) \tag{6}$$

and

$$d\mu_\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx, \tag{7}$$

then condition (1) has the form

$$\int_0^\infty \tilde{\rho}(x, t) d\mu_\alpha(x) = 1. \tag{8}$$

Note that substituting  $y=ct$  in (6), we get the sum

$$\tilde{\rho}(x_t, t) = \rho(ct-x_t, t) + \rho(ct+x_t, t).$$

This sum can be considered as a sum of right and back waves of the distribution functions.

### III. LIOUVILLE EQUATION

Let us consider a domain  $B_0$  for the time  $t=0$ . In the Hamilton picture we have

$$\int_{B_t} \tilde{\rho}(x_t, t) d\mu_\alpha(x_t) = \int_{B_0} \tilde{\rho}(x_0, 0) d\mu_\alpha(x_0).$$

Using the replacement of variables  $x_t = x_t(x_0)$ , where  $x_0$  is a Lagrangian variable, we get

$$\int_{B_0} \tilde{\rho}(x_t, t) x_t^{\alpha-1} \frac{\partial x_t}{\partial x_0} dx_0 = \int_{B_0} \tilde{\rho}(x_0, 0) x_0^{\alpha-1} dx_0.$$

Since  $B_0$  is an arbitrary domain we have

$$\tilde{\rho}(x_t, t) d\mu_\alpha(x_t) = \tilde{\rho}(x_0, 0) d\mu_\alpha(x_0),$$

or

$$\tilde{\rho}(x_t, t) x_t^{\alpha-1} \frac{\partial x_t}{\partial x_0} = \tilde{\rho}(x_0, 0) x_0^{\alpha-1}.$$

Differentiating this equation in time  $t$ , we obtain

$$\frac{d\tilde{\rho}(x_t, t)}{dt} x_t^{\alpha-1} \frac{\partial x_t}{\partial x_0} + \tilde{\rho}(x_t, t) \frac{d}{dt} \left( x_t^{\alpha-1} \frac{\partial x_t}{\partial x_0} \right) = 0,$$

or

$$\frac{d\tilde{\rho}(x_t, t)}{dt} + \Omega_\alpha(x_t, t) \tilde{\rho}(x_t, t) = 0, \tag{9}$$

where  $d/dt$  is a total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + F_t \frac{\partial}{\partial x_t}.$$

The function

$$\Omega_\alpha(x_t, t) = \frac{d}{dt} \ln \left( x_t^{\alpha-1} \frac{\partial x_t}{\partial x_0} \right)$$

describes the velocity of (phase space) volume change. Equation (9) is a fractional Liouville equation in the Hamilton picture. If the equation of motion has the form

$$\frac{dx_t}{dt} = F_t(x),$$

then the function  $\Omega$  is defined by

$$\begin{aligned} \Omega_\alpha(x_t, t) &= \frac{d}{dt} \left( \ln x_t^{\alpha-1} + \ln \frac{\partial x_t}{\partial x_0} \right) \\ &= (\alpha-1) \frac{1}{x_t} \frac{dx_t}{dt} + \frac{\partial}{\partial x_t} \frac{dx_t}{dt}. \end{aligned}$$

As the result we have

$$\Omega_\alpha(x_t, t) = \frac{(\alpha-1)F_t}{x_t} + \frac{\partial F_t}{\partial x_t}.$$

The normalization in the phase space is derived by analogy with a normalization in the configuration space. The fractional normalization condition in the phase space

$$\int_0^\infty \int_0^\infty \tilde{\rho}(q, p, t) d\mu_\alpha(q, p) = 1, \tag{10}$$

where  $d\mu_\alpha(q, p)$  has the form

$$\begin{aligned} d\mu_\alpha(q, p) &= d\mu_\alpha(q) \wedge d\mu_\alpha(p) = \frac{dq^\alpha \wedge dp^\alpha}{(\alpha\Gamma(\alpha))^2} \\ &= \frac{(qp)^{\alpha-1}}{\Gamma^2(\alpha)} dq \wedge dp. \end{aligned} \tag{11}$$

The distribution function  $\tilde{\rho}(q, p, t)$  in the phase space is defined by

$$\begin{aligned} \tilde{\rho}(q, p, t) &= \rho(q' - q, p' - p, t) + \rho(q' + q, p' - p, t) \\ &\quad + \rho(q' - q, p' + p, t) + \rho(q' + q, p' + p, t). \end{aligned}$$

Let us use the well-known transformation

$$dq_t \wedge dp_t = \{q_t, p_t\}_0 dq_0 \wedge dp_0, \tag{12}$$

where  $\{q_t, p_t\}_0$  is Jacobian which is defined by the determinant

$$\{q_t, p_t\}_0 = \det \frac{\partial(q_t, p_t)}{\partial(q_0, p_0)} = \det \begin{pmatrix} \partial q_{kt} / \partial q_{k0} & \partial q_{kt} / \partial p_{l0} \\ \partial p_{kt} / \partial q_{l0} & \partial p_{kt} / \partial p_{l0} \end{pmatrix}.$$

Using  $\tilde{\rho}_t d\mu_\alpha(q_t, p_t) = \tilde{\rho}_0 d\mu_\alpha(q_0, p_0)$ , we get the relation

$$\tilde{\rho}_t \frac{(q_t p_t)^{\alpha-1}}{\Gamma^2(\alpha)} dq_t \wedge dp_t = \tilde{\rho}_0 \frac{(q_0 p_0)^{\alpha-1}}{\Gamma^2(\alpha)} dq_0 \wedge dp_0. \tag{13}$$

Using (12), we have condition (13) in the form

$$\tilde{\rho}_t(q_t, p_t)^{\alpha-1} \{q_t, p_t\}_0 = (q_0 p_0)^{\alpha-1} \tilde{\rho}_0. \tag{14}$$

Let us write condition (14) in more simple form

$$\tilde{\rho}_t \{q_t^\alpha, p_t^\alpha\}_0 = \alpha^2 (q_0 p_0)^{\alpha-1} \tilde{\rho}_0. \tag{15}$$

The time derivatives of this equation lead to the fractional Liouville equation

$$\frac{d\tilde{\rho}(q_t, p_t, t)}{dt} + \Omega_\alpha(q_t, p_t, t) \tilde{\rho}(q_t, p_t, t) = 0, \tag{16}$$

where the function  $\Omega_\alpha$  is defined by

$$\Omega_\alpha(q_t, p_t, t) = \{q_t^\alpha, p_t^\alpha\}_0^{-1} \frac{d}{dt} \{q_t^\alpha, p_t^\alpha\}_0 = \frac{d}{dt} \ln \{q_t^\alpha, p_t^\alpha\}_0. \tag{17}$$

In the usual notations we have

$$\Omega_\alpha(q_t, p_t, t) = \frac{d}{dt} \ln \det \frac{\partial(q_t^\alpha, p_t^\alpha)}{\partial(q_0, p_0)}. \tag{18}$$

Using well-known relation  $\ln \det A = Sp \ln A$ , we can write the  $\alpha$ -omega function in the form

$$\Omega_\alpha = \left\{ \frac{dq_t^\alpha}{dt}, p_t^\alpha \right\}_\alpha + \left\{ q_t^\alpha, \frac{dp_t^\alpha}{dt} \right\}_\alpha,$$

where

$$\{A, B\}_\alpha = \frac{\partial A}{\partial q^\alpha} \frac{\partial B}{\partial p^\alpha} - \frac{\partial A}{\partial p^\alpha} \frac{\partial B}{\partial q^\alpha}.$$

In the general case ( $\alpha \neq 1$ ) the function  $\Omega_\alpha$  is not equal to zero ( $\Omega_\alpha \neq 0$ ) for Hamiltonian systems. If  $\alpha = 1$ , we have  $\Omega_\alpha \neq 0$  only for non-Hamiltonian systems.

It is easy to see that any system which is defined by the equations

$$\frac{dq_t^\alpha}{dt} = \frac{p_t}{m}, \quad \frac{dp_t^\alpha}{dt} = f(q_t), \tag{19}$$

has the  $\alpha$ -omega function equal to zero  $\Omega_\alpha = 0$ . This system can be called a fractional nondissipative system. For example, a fractional oscillator is defined by the equation

$$\frac{dq_t^\alpha}{dt} = \frac{p_t^\alpha}{m}, \quad \frac{dp_t^\alpha}{dt} = -m\omega^2 q_t^\alpha. \tag{20}$$

The  $\alpha$ -omega function can be rewritten in the form

$$\Omega_\alpha(q_t, p_t, t) = (\alpha - 1) \left( q_t^{-1} \frac{dq_t}{dt} + p_t^{-1} \frac{dp_t}{dt} \right) + \left\{ \frac{dq_t}{dt}, p_t \right\}_1 + \left\{ q_t, \frac{dp_t}{dt} \right\}_1, \tag{21}$$

where  $\{.,.\}_1$  is usual Poisson bracket. If the Hamilton equations have the form

$$\frac{dq_t}{dt} = G(q_t, p_t), \quad \frac{dp_t}{dt} = F(q_t, p_t), \tag{22}$$

then the  $\alpha$ -omega function is defined by

$$\Omega_\alpha(q, p) = (\alpha - 1)(q^{-1}G(q, p) + p^{-1}F(q, p)) + \{G, p\}_1 + \{q, F\}_1. \tag{23}$$

This relation allows to derive  $\Omega_\alpha$  for all dynamical systems (22). It is easy to see that the usual nondissipative system

$$\frac{dq_t}{dt} = \frac{p_t}{m}, \quad \frac{dp_t}{dt} = f(q_t), \tag{24}$$

has the  $\alpha$ -omega function

$$\Omega_\alpha(q, p) = (\alpha - 1)(mqp)^{-1}(p^2 + mqf(q))$$

and can be called a fractional dissipative system. For example, the linear harmonic oscillator ( $f(q) = -m\omega^2q$ )

$$\Omega_\alpha(q, p) = (\alpha - 1)(mqp)^{-1}(p^2 - m^2\omega^2q^2),$$

is a fractional dissipative system.

#### IV. INTERPRETATION

The fractional normalization condition can be considered as a normalization condition for the distribution function in a fractional phase space. In order to use this interpretation we must define a fractional phase space.

The first interpretation of the fractional phase space is connected with fractional dimension. This interpretation follows from the well-known formulas for dimensional regularizations:<sup>18</sup>

$$\int \rho(x) d^n x = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \rho(x) x^{n-1} dx. \tag{25}$$

Using Eq. (25), we get that the fractional normalization condition (8) can be considered as a normalization condition in the fractional dimension space

$$\frac{\Gamma(\alpha/2)}{2\pi^{\alpha/2}\Gamma(\alpha)} \int \tilde{\rho}(x, t) d^\alpha x = 1 \tag{26}$$

up to the numerical factor  $\Gamma(\alpha/2)/(2\pi^{\alpha/2}\Gamma(\alpha))$ .

The second interpretation is connected with the fractional measure of phase volume. The parameter  $\alpha$  defines the space with the fractional phase volume

$$V_\alpha = \int_B d\mu_\alpha(q, p).$$

It is easy to prove that the velocity of the fractional phase volume change is defined by

$$\frac{dV_\alpha}{dt} = \int_B \Omega_\alpha(q, p, t) d\mu_\alpha(q, p).$$

Note that the volume element of fractional phase space can be realized by fractional exterior derivatives<sup>19</sup>

$$d^\alpha = \sum_{k=1}^n dq_k^\alpha \frac{\partial^\alpha}{(\partial(q_k - a_k))^\alpha} + \sum_{k=1}^n dp_k^\alpha \frac{\partial^\alpha}{(\partial(p_k - b_k))^\alpha}, \tag{27}$$

in the form

$$dq^\alpha \wedge dp^\alpha = \left( \frac{4}{\Gamma^2(2-\alpha)} + \frac{1}{\Gamma^2(1-\alpha)} \right)^{-1} (qp)^{\alpha-1} d^\alpha q \wedge d^\alpha p. \quad (28)$$

The system can be called a fractional dissipative system if a fractional phase volume changes, i.e.,  $\Omega_\alpha \neq 0$ . The system which is a nondissipative system in the usual phase space, can be a dissipative system in the fractional phase space. The fractional analog of the usual conservative Hamiltonian nondissipative system is defined by the equations

$$\frac{dq_k^\alpha}{dt} = \frac{g_k(p)}{m}, \quad \frac{dp_k^\alpha}{dt} = f_k(q). \quad (29)$$

The usual nondissipative systems (24) are dissipative in the fractional phase space.

In the general case, the fractional system is a system in the fractional phase space. We shall say that a system is called a fractional system if this system can be described by the fractional powers of coordinates and momenta,

$$q_k^\alpha = |q_k|^\alpha, \quad p_k^\alpha = |p_k|^\alpha,$$

where  $k = 1, \dots, n$ .

The fractional systems allow to consider the interpretation of the fractional normalization condition which is used to derive the fractional Liouville equation. The fractional normalization condition for the distribution function can be considered as a normalization condition for the systems in the fractional phase space.

The Hamilton equations for the fractional system have the form

$$\frac{dq_k^\alpha}{dt} = \frac{p_k^\alpha}{m}, \quad \frac{dp_k^\alpha}{dt} = F_k(q^\alpha, p^\alpha). \quad (30)$$

Obviously, that the equation  $dp_k/dt = F_k$  can be rewritten in the fractional form. Multiplying both sides of this equation by  $\alpha p^{\alpha-1}$ , we obtain  $dp_k^\alpha/dt = \alpha p^{\alpha-1} F_k$ . However the equation  $dq_k/dt = p_k/m$  cannot be rewritten in the fractional form  $dq_k^\alpha/dt = p_k^\alpha/m$ .

The fractional conservative Hamiltonian system is described by the equation

$$\frac{dq_k^\alpha}{dt} = \frac{\partial H(q^\alpha, p^\alpha)}{\partial p_k^\alpha}, \quad \frac{dp_k^\alpha}{dt} = - \frac{\partial H(q^\alpha, p^\alpha)}{\partial q_k^\alpha}, \quad (31)$$

or

$$\frac{dq_k^\alpha}{dt} = \{q_k^\alpha, H\}_\alpha, \quad \frac{dp_k^\alpha}{dt} = \{p_k^\alpha, H\}_\alpha. \quad (32)$$

Here  $H = H(q^\alpha, p^\alpha)$  is a fractional analog of the Hamiltonian function

$$H(q^\alpha, p^\alpha) = \sum_{k=1}^n \frac{p_k^{2\alpha}}{2m} + U(q^\alpha). \quad (33)$$

The fractional system can be considered as a nonlinear system with

$$H(q^\alpha, p^\alpha) = \sum_{k=1}^n \frac{1}{2} g_{kl}(q, p) p_k p_l + U(q). \quad (34)$$

Note that the Hamiltonian (34) defines a nonlinear one-dimensional sigma-model<sup>20,21</sup> with metric

$$g_{kl}(q, p) = m^{-1} p_k^{2(\alpha-1)} \delta_{kl}.$$

It is easy to see that fractional systems (33) can lead to the non-Gaussian statistics. The interest in and relevance of fractional kinetic equations is a natural consequence of the realization of the importance of non-Gaussian statistics of many dynamical systems. There is already a substantial literature studying such equations in one or more space dimensions.

Note that the classical (nonlinear) dissipative systems can have canonical Gibbs distribution as a solution of stationary Liouville equations for this dissipative system.<sup>22</sup> Using the methods,<sup>22</sup> it is easy to prove that some of fractional dissipative systems can have fractional canonical Gibbs distribution

$$\rho(q, p) = Z(T) \exp - \frac{H(q^\alpha, p^\alpha)}{kT},$$

as a solution of the fractional Liouville equations

$$\frac{\partial \rho}{\partial t} + \frac{p_k^\alpha}{m} \frac{\partial \rho}{\partial q_k^\alpha} + \frac{\partial}{\partial p_k^\alpha} (F_k(q^\alpha, p^\alpha) \rho) = 0. \quad (35)$$

Here the function  $H(q^\alpha, p^\alpha)$  is defined by (33).

## V. CONCLUSION

Derivatives and integrals of fractional order have found many applications in studies of scaling phenomena.<sup>1-5</sup> In this paper we formulate fractional analog of main integro-differential equation to describe some scaling process—Liouville equation. We consider the fractional analog of the normalization condition for the distribution function. Fractional Liouville equation for dissipative systems is derived from the normalization condition. In this paper, the coordinate fractional integration for the normalization condition is used. This fractional normalization condition can be considered as a simulating unconventional environment for systems with fractional dimensional phase space or phase space with fractional powers of coordinates and momentums. Note that the adoption of fractal formalism yields properties that the ordinary formalism would produce only in the case where the system is made non-Hamiltonian by the presence of an environment, whose influence can be mimicked by means of friction, for instance.

Suggested fractional Liouville equation allows to formulate the fractional equation for quantum dissipative systems<sup>23</sup> by methods suggested in Refs. 24 and 25. In general, we can consider this dissipative quantum systems as quantum computer with mixed states.<sup>26</sup> These dissipative quantum systems can have stationary states.<sup>27</sup> Stationary states of dissipative quantum systems can coincide with stationary states of Hamiltonian systems.<sup>28</sup>

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