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Fractional dynamics of coupled oscillators with long-range interaction

Vasily E. Tarasov
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia and Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

George M. Zaslavsky
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012 and Department of Physics, New York University, New York, New York 10003

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We consider a one-dimensional chain of coupled linear and nonlinear oscillators with long-range powerwise interaction. The corresponding term in dynamical equations is proportional to $1/|n-m|^{\alpha+1}$. It is shown that the equation of motion in the infrared limit can be transformed into the medium equation with the Riesz fractional derivative of order α, when $0<\alpha<2$. We consider a few models of coupled oscillators and show how their synchronization can appear as a result of bifurcation, and how the corresponding solutions depend on α. The presence of a fractional derivative also leads to the occurrence of localized structures. Particular solutions for fractional time-dependent complex Ginzburg-Landau (or nonlinear Schrödinger) equation are derived. These solutions are interpreted as synchronized states and localized structures of the oscillatory medium.
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Although the fractional calculus is known for more than 200 years and its development is an active area of mathematics, appearance and use of it in physical literature is fairly recent and sometimes is considered as exotic. In fact, there are many different areas where fractional equations, i.e., equations with fractional integro-differentiation, describe real processes. Between the most related areas are chaotic dynamics,[1] random walk in fractal space-time,[2] and random processes of the Levy-type.[3–6] The physical reasons for the appearance of fractional equations are intermittancy, dissipation, wave propagation in complex media, long memory, and others. This article deals with long-range interaction that can work in some way as a long memory. A unified approach to the origin of fractional dynamics from the long-range interaction of nonlinear oscillators or other objects permits us to consider such phenomena as synchronization, breathers formation, space-time structures by the same formalism using new tools from the fractional calculus.

I. INTRODUCTION

Collective oscillation and synchronization are the fundamental phenomena in physics, chemistry, biology, and neuroscience, which are actively studied recently,[7–9] having both important theoretical and applied significance. Beginning with the pioneering contributions by Winfree[10] and Kuramoto,[11] studies of synchronization in populations of coupled oscillators became an active field of research in biology and chemistry. An oscillatory medium is an extended system, where each site (element) performs self-sustained oscillations. A good physical and chemical example is the oscillatory Belousov-Zhabotinsky reaction[11–13] in a medium where different sites can oscillate with different periods and phases. Typically, the reaction is accompanied by a color variation of the medium. Complex Ginzburg-Landau equation[51–53] is canonical model for oscillatory systems with local coupling near the Hopf bifurcation. Recently, Tanaka and Kuramoto[14] have shown how, in the vicinity of the bifurcation, the description of an array of nonlocally coupled oscillators can be reduced to the complex Ginzburg-Landau equation. In Ref. 15, a model of population of diffusively coupled oscillators with limit cycles is described by the complex Ginzburg-Landau equation with nonlocal interaction. Nonlocal coupling is considered in Refs. 15–17. The long-range interaction that decreases as $1/|x|^{\alpha+1}$ with $0<\alpha<2$ is considered in Refs. 18–22 with respect to the system's thermodynamics and phase transition. It is also shown in Ref. 23 that using the Fourier transform and limit for the wave number $k\to 0$, the long-range term interaction leads under special conditions to the fractional dynamics.

In the last decade it is found that many physical processes can be adequately described by equations that consist of derivatives of fractional order. In a fairly short period of time the list of such applications becomes long and the area of applications is broad. Even in a concise form, the applications include material sciences,[24–27] chaotic dynamics,[1] quantum theory,[28–31] physical kinetics,[1,3,32,33] fluids and plasma physics,[34,35] and many other physical topics related to wave propagation,[36] long-range dissipation,[37] anomalous diffusion and transport theory (see reviews in Refs. 1, 2, 4, 24, and 38). Some historical comments on the origin of fractional calculus can be found in Ref. 39.

It is known that the appearance of fractional derivatives in equations of motion can be linked to nonlocal properties of dynamics. Fractional Ginzburg-Landau equation has been suggested in Refs. 40–42. In this paper, we consider the synchronization for oscillators with long-range interaction that in continuous limit leads to the fractional complex Ginzburg-Landau equation. We confirm the result obtained in Ref. 23

that the infrared limit (wave number $k \to 0$) of an infinite chain of oscillators with the long-range interaction can be described by equations with the fractional Riesz coordinate derivative of order $\alpha < 2$. This result permits us to apply different tools of the fractional calculus to the considered systems, and to interpret different systems' features in a unified way.

In Sec. II, we consider a systems of oscillators with linear long-range interaction. For infrared behavior of the oscillatory medium, we obtain the equations that have coordinate derivatives of fractional order. In Sec. III, some particular solutions are derived with a constant wave number for the fractional Ginzburg-Landau equation. These solutions are interpreted as synchronization in the oscillatory medium. In Sec. IV, we derive solutions of the fractional Ginzburg-Landau equation near a limit cycle. These solutions are interpreted as coherent structures in the oscillatory medium with long-range interaction. In Sec. V, we consider the nonlinear long-range interaction of oscillators and corresponding equations for the spin field. Finally, discussion of the results and conclusion are given in Sec. VI.

II. LONG-RANGE INTERACTION OF OSCILLATORS

A. Derivation of equation for the continuous oscillatory medium

In this section we consider a simplified version of a chain of N oscillators ($N \to \infty$) that have a long-range interaction of the power type. The corresponding equation of motion can be written as

$$\frac{d}{dt} z_n(t) = F(z_n) + g_0 \sum_{m=-\infty, m\neq n}^{\infty} J_\alpha(n-m)(z_n - z_m), \tag{1}$$

where z_n is the position of the nth oscillator in the complex plane, and F is a force. As an example, for the oscillators with a limit cycle, F can be taken as

$$F(z) = (1+ia)z - (1+ib)|z|^2 z. \tag{2}$$

The nonlocal interaction is given by the power function

$$J_\alpha(n) = |n|^{-\alpha-1}. \tag{3}$$

This coupling in the limit $\alpha \to \infty$ is a nearest-neighbor interaction. This type of interaction was introduced by Dyson[18] to study phase transitions and then was considered in numerous papers related to magnetic systems.[19–22] Power type long-range interaction can appear as an effective interaction in dispersive or complex systems.[26,36,40] The complexity of the system reveals in a noninteger α that is defined by a specific type of the material. Let us provide also two examples from fluid dynamics where the dispersion, and nonlinear properties of the media define the order of fractional derivatives: tracer dynamics in the presence of convective rolls,[43] and the equation for surface wave interaction.[44]

Let us derive the equation for continuous medium limit of system (1) with long-range interaction (3). For this goal it is convenient to introduce the field

$$Z(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \sum_{n=-\infty}^{+\infty} e^{-ikn} z_n(t). \tag{4}$$

Multiplying Eq. (1) by $\exp(-ikn)$, and summing over n from $-\infty$ to $+\infty$, we obtain

$$\frac{\partial y(k,t)}{\partial t} \equiv \sum_{n=-\infty}^{+\infty} e^{-ikn} \frac{d}{dt} z_n(t) = \sum_{n=-\infty}^{+\infty} e^{-ikn} F(z_n) + g_0 \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{(z_n - z_m)}{|n-m|^{\alpha+1}}, \tag{5}$$

where

$$y(k,t) = \sum_{n=-\infty}^{+\infty} e^{-ikn} z_n(t). \tag{6}$$

Using the notation

$$\tilde{J}_\alpha(k) = \sum_{n=-\infty, n\neq 0}^{+\infty} e^{-ikn} J_\alpha(n) = \sum_{n=-\infty, n\neq 0}^{+\infty} e^{-ikn} \frac{1}{|n|^{\alpha+1}}, \tag{7}$$

the interaction term in (5) can be presented as

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} (z_n - z_m) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} z_n - \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} z_m. \tag{8}$$

For the first term in the right-hand side of (8):

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} z_n = \sum_{n=-\infty}^{+\infty} e^{-ikn} z_n \sum_{m'=-\infty, m'\neq 0}^{+\infty} \frac{1}{|m'|^{\alpha+1}} = y(k,t) \tilde{J}_\alpha(0), \tag{9}$$

where

$$\tilde{J}_\alpha(0) = \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{1}{|n|^{\alpha+1}} = 2 \sum_{n=1}^{\infty} \frac{1}{|n|^{\alpha+1}} = 2\zeta(\alpha+1), \tag{10}$$

and $\zeta(z)$ is the Riemann zeta function. For the second term in the RHS of (8):

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty, m\neq n}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} z_m = \sum_{m=-\infty}^{+\infty} z_m \sum_{n=-\infty, n\neq m}^{+\infty} e^{-ikn} \frac{1}{|n-m|^{\alpha+1}} = \sum_{m=-\infty}^{+\infty} z_m e^{-ikm} \sum_{n'=-\infty, n'\neq 0}^{+\infty} e^{-ikn'} \frac{1}{|n'|^{\alpha+1}} = y(k,t) \tilde{J}_\alpha(k). \tag{11}$$

As the result, Eq. (5) yields

$$\frac{\partial}{\partial t}y(k,t) = \mathcal{F}\{F(z_n)\} + g_0[\tilde{J}_\alpha(0) - \tilde{J}_\alpha(k)]y(k,t), \tag{12}$$

where $\mathcal{F}\{F(z_n)\}$ is an operator notation for the Fourier transform of $F(z_n)$:

$$\mathcal{F}\{F(z_n)\} = \sum_{n=-\infty}^{+\infty} e^{-ikn}F(z_n).$$

The function $\tilde{J}_\alpha(k)$ introduced in (7) can be transformed as

$$\begin{aligned}\tilde{J}_\alpha(k) &= \sum_{n=-\infty, n\neq 0}^{+\infty} e^{-ikn}\frac{1}{|n|^{\alpha+1}} \\ &= \sum_{n=1}^{+\infty} e^{-ikn}\frac{1}{|n|^{\alpha+1}} + \sum_{n=-1}^{-\infty} e^{-ikn}\frac{1}{|n|^{\alpha+1}} \\ &= \sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}(e^{-ikn}+e^{ikn}) = Li_{\alpha+1}(e^{ik}) + Li_{\alpha+1}(e^{-ik}),\end{aligned} \tag{13}$$

where $Li_\alpha(z)$ is a polylogarithm function. This presentation was also obtained in Ref. 23, and it plays an important role in the following transition to fractional dynamics. Using the expansion

$$Li_\beta(e^z) = \Gamma(1-\beta)(-z)^{\beta-1} + \sum_{n=0}^{\infty}\frac{\zeta(\beta-n)}{n!}z^n, \quad |z| < 2\pi, \tag{14}$$

we obtain

$$\tilde{J}_\alpha(k) = 2\Gamma(-\alpha)\cos(\pi\alpha/2)|k|^\alpha + 2\sum_{n=0}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}(-k^2)^n, \tag{15}$$

$\tilde{J}_\alpha(0) = 2\zeta(\alpha+1)$.

From (13) we can see that

$$\tilde{J}_\alpha(k+2\pi m) = \tilde{J}_\alpha(k), \tag{16}$$

where m is an integer. For $\alpha=2$, $\tilde{J}_\alpha(k)$ is the Clausen function $Cl_3(k)$.[54] The plots of $\tilde{J}_\alpha(k)$ for $\alpha=1.1$, and $\alpha=1.9$ are presented in Fig. 1.

After substituting (15) into (12), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}y(k,t) &= \mathcal{F}\{F(z_n)\} - g_0 a_\alpha |k|^\alpha y(k,t) \\ &\quad - 2g_0\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}(-k^2)^n y(k,t),\end{aligned} \tag{17}$$

where

$$a_\alpha = 2\Gamma(-\alpha)\cos(\pi\alpha/2) \quad (0<\alpha<2, \alpha\neq 1). \tag{18}$$

To derive the equation for field (4), we can use definition (6)

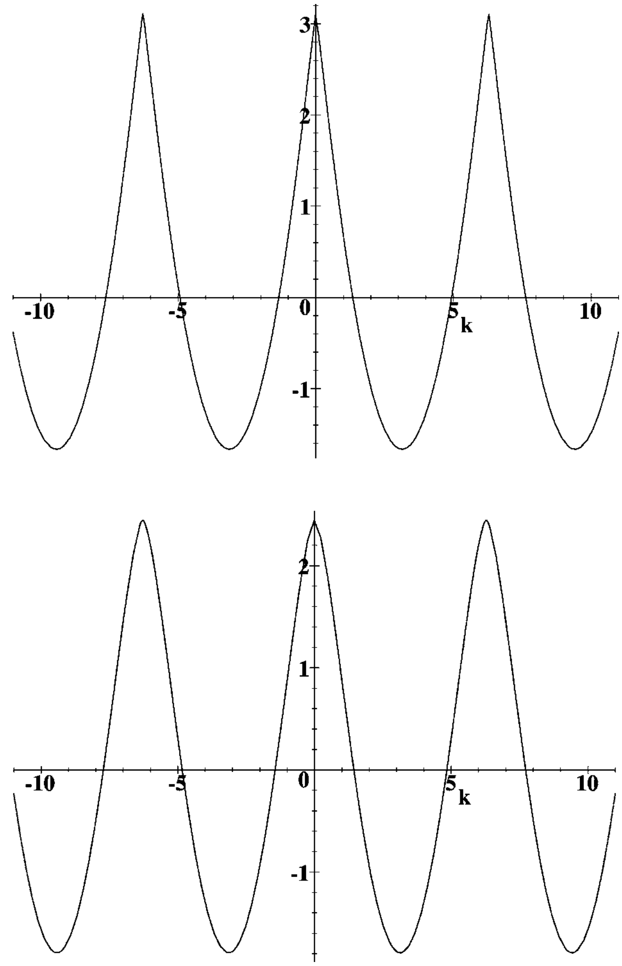

FIG. 1. The function $\tilde{J}_\alpha(k)$ for orders $\alpha=1.1$, and $\alpha=1.9$.

$$Z(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}y(k,t)dk, \tag{19}$$

and the connection between Riesz fractional derivative and its Fourier transform:[45]

$$|k|^\alpha \leftrightarrow -\frac{\partial^\alpha}{\partial|x|^\alpha}, \quad k^2 \leftrightarrow -\frac{\partial^2}{\partial|x|^2}. \tag{20}$$

The properties of the Riesz derivative can be found in Refs. 45–48. Another expression is

$$\frac{\partial^\alpha}{\partial|x|^\alpha}Z(x,t) = -\frac{1}{2\cos(\pi\alpha/2)}(\mathcal{D}_+^\alpha Z(x,t) + \mathcal{D}_-^\alpha Z(x,t)), \tag{21}$$

where $\alpha\neq 1, 3, 5, \ldots$, and $\mathcal{D}_\pm^\alpha$ are Riemann-Liouville left and right fractional derivatives

$$\begin{aligned}\mathcal{D}_+^\alpha Z(x,t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{Z(\xi,t)d\xi}{(x-\xi)^{\alpha-n+1}}, \\ \mathcal{D}_-^\alpha Z(x,t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{x}^{\infty}\frac{Z(\xi,t)d\xi}{(\xi-x)^{\alpha-n+1}}.\end{aligned} \tag{22}$$

Substitution of Eqs. (22) into Eq. (21) gives

$$\frac{\partial^\alpha}{\partial|x|^\alpha}Z(x,t)=\frac{-1}{2\cos(\pi\alpha/2)\Gamma(n-\alpha)}\times\frac{\partial^n}{\partial x^n}\left(\int_{-\infty}^{x}\frac{Z(\xi,t)d\xi}{(x-\xi)^{\alpha-n+1}}+\int_{x}^{\infty}\frac{(-1)^n Z(\xi,t)d\xi}{(\xi-x)^{\alpha-n+1}}\right). \quad (23)$$

Multiplying Eq. (17) on $\exp(ikx)$, and integrating over k from $-\infty$ to $+\infty$, we obtain

$$\frac{\partial}{\partial t}Z=\tilde{F}(Z)+g_0 a_\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}Z-2g_0\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}Z, \quad (24)$$

$$Z=Z(x,t)\quad(\alpha\neq 0,\ 1,\ 2,\ \ldots),$$

where $\tilde{F}(Z)$ is the inverse Fourier transform of $\mathcal{F}\{F(z_n)\}$:

$$\tilde{F}(Z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk e^{ikx}\mathcal{F}\{F(z_n)\}.$$

For $x=n$ $(\forall n)$ one can see that

$$\tilde{F}(Z(x,t))=F(Z(n,t))=F(z_n(t)). \quad (25)$$

This is a standard procedure for the replacement of a discrete chain by the continuous one and in the following we will write $F(Z)$ instead of $\tilde{F}(Z)$.

The first term $(n=1)$ of the sum is $\zeta(\alpha-1)\partial_x^2 Z$. Let us compare the coefficients of terms with fractional and second derivatives in Eq. (24). For $\alpha\rightarrow 2$, one can use the asymptotics

$$\zeta(\alpha-1)\approx\frac{1}{\alpha-2}+O(1),\quad a_\alpha\approx\frac{1}{\alpha-2}+O(1)\quad(\alpha\neq 2).$$

As an example, for $\alpha=1.99$,

$$\zeta(\alpha-1)\approx-99.423\,51,\quad a_\alpha\approx-100.929\,21.$$

Therefore $\zeta(\alpha-1)/a_\alpha\sim 1$ for $2-\alpha\ll 1$.

B. Infrared approximation

In this section, we derive the main relation that permits us to transfer the system of discrete oscillators into a fractional differential equation. This transform will be called the infrared limit. For $0<\alpha<2$, $\alpha\neq 1$, and $k\rightarrow 0$, the fractional power of $|k|$ is a leading asymptotic term in Eq. (17), and

$$[\tilde{J}_\alpha(0)-\tilde{J}_\alpha(k)]\approx a_\alpha|k|^\alpha\quad(0<\alpha<2,\ \alpha\neq 1). \quad (26)$$

Equation (26) can be considered as an infrared approximation of (17). Substitution of (26) into (12) gives

$$\frac{\partial}{\partial t}y(k,t)=\mathcal{F}\{F(z_n)\}-g_0 a_\alpha|k|^\alpha y(k,t)\quad(0<\alpha<2,\ \alpha\neq 1). \quad (27)$$

Then

$$\frac{\partial}{\partial t}Z=F(Z)+g_0 a_\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}Z\quad(0<\alpha<2,\ \alpha\neq 1). \quad (28)$$

Equation (28) can be considered an equation for continuous oscillatory medium with $\alpha<2$ in the infrared $(k\rightarrow 0)$ approximation.

As an example, for $F(z)=0$, Eq. (28) gives the fractional kinetic equation

$$\frac{\partial}{\partial t}Z=g_0 a_\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}Z\quad(0<\alpha<2,\ \alpha\neq 1) \quad (29)$$

that describes the fractional superdiffusion.[3,4,32] For $F(z)$ defined by (2), Eq. (28) is a fractional Ginzburg-Landau equation that has been suggested in Ref. 40 (see also Refs. 41 and 42), and will be considered in Sec. III. For $\alpha>2$ and $k\rightarrow 0$ the main term in (15) is proportional to k^2 and in (28) and (29), we have a second derivative instead of the fractional one. The existence of the critical value $\alpha=2$ was obtained in Ref. 23.

III. FRACTIONAL GINZBURG-LANDAU EQUATION

A. Synchronized states for the Ginzburg-Landau equation

The one-dimensional lattice of weakly coupled nonlinear oscillators is described by

$$\frac{d}{dt}z_n(t)=(1+ia)z_n-(1+ib)|z_n|^2 z_n+(c_1+ic_2)(z_{n+1}-2z_n+z_{n-1}), \quad (30)$$

where we assume that all oscillators have the same parameters. A transition to the continuous medium assumes[8] that the difference $z_{n+1}-z_n$ is of the order Δx, and the interaction constants c_1 and c_2 are large. Setting $c_1=g(\Delta x)^{-2}$, and $c_2=gc(\Delta x)^{-2}$, we get

$$\frac{\partial}{\partial t}Z=(1+ia)Z-(1+ib)|Z|^2 Z+g(1+ic)\frac{\partial^2}{\partial x^2}Z, \quad (31)$$

which is a complex time-dependent Ginzburg-Landau equation.[51–53] Here $Z(n\Delta x,t)$ coincides with (4) if we put $\Delta x=1$. The simplest coherent structures for this equation are plane-wave solutions,[8]

$$Z(x,t)=R(K)\exp[iKx-i\omega(K)t+\theta_0], \quad (32)$$

where

$$R(K)=(1-gK^2)^{1/2},\quad\omega(K)=(b-a)+(c-b)gK^2, \quad (33)$$

and θ_0 is an arbitrary constant phase. These solutions exist for

$$gK^2<1. \quad (34)$$

The solution (32) can be interpreted as a synchronized state.[8]

B. Particular solution for the fractional Ginzburg-Landau equation

Let us come back to the equation for nonlinear oscillators (1) with $F(z)$ in Eq. (2) and long-range coupling (3),

$$\frac{d}{dt}z_n = (1+ia)z_n - (1+ib)|z_n|^2 z_n + g_0 \sum_{m\neq n} \frac{1}{|n-m|^{\alpha+1}}(z_n - z_m), \tag{35}$$

where $z_n = z_n(t)$ is the position of the nth oscillator in the complex plane, $1<\alpha<2$. The corresponding equation in the continuous limit and infrared approximation can be obtained in the same way as (28)

$$\frac{\partial}{\partial t}Z = (1+ia)Z - (1+ib)|Z|^2 Z + g(1+ic)\frac{\partial^\alpha}{\partial|x|^\alpha}Z, \tag{36}$$

where $g(1+ic) = g_0 a_\alpha$, and $1<\alpha<2$. Equation (36) is a fractional generalization of the complex time-dependent Ginzburg-Landau equation (31) [compare to (28)]. Here, this equation is derived in a specific approximation for the oscillatory medium.

We seek a particular solution of (36) in the form

$$Z(x,t) = A(K,t)e^{iKx}, \tag{37}$$

which allows us to use

$$\frac{\partial^\alpha}{\partial|x|^\alpha}e^{iKx} = -|K|^\alpha e^{iKx}. \tag{38}$$

Equation (37) represents a particular solution of (36) with a fixed wave number K.

The substitution of (37) into (36) gives

$$\frac{\partial}{\partial t}A(K,t) = (1+ia)A - (1+ib)|A|^2 A - g(1+ic)|K|^\alpha A. \tag{39}$$

Rewriting this equation in polar coordinates,

$$A(K,t) = R(K,t)e^{i\theta(K,t)}, \tag{40}$$

we obtain

$$\frac{dR}{dt} = (1-g|K|^\alpha)R - R^3, \qquad \frac{d\theta}{dt} = (a - cg|K|^\alpha) - bR^2. \tag{41}$$

The limit cycle here is a circle with the radius

$$R = (1-g|K|^\alpha)^{1/2}, \quad g|K|^\alpha < 1. \tag{42}$$

The solution of (41) with arbitrary initial conditions

$$R(K,0) = R_0, \quad \theta(K,0) = \theta_0 \tag{43}$$

is

$$R(t) = R_0(1-g|K|^\alpha)^{1/2}(R_0^2 + (1-g|K|^\alpha - R_0^2) \times e^{-2(1-g|K|^\alpha)t})^{-1/2}, \tag{44}$$

$$\theta(t) = -\frac{b}{2}\ln[(1-g|K|^\alpha)^{-1}(R_0^2 + (1-g|K|^\alpha - R_0^2)e^{-2at})] - \omega_\alpha(K)t + \theta_0, \tag{45}$$

where

$$\omega_\alpha(K) = (b-a) + (c-b)g|K|^\alpha, \quad 1-g|K|^\alpha > 0. \tag{46}$$

This solution can be interpreted as a coherent structure in nonlinear oscillatory medium with long-range interaction.

If

$$R_0^2 = 1 - g|K|^\alpha, \quad g|K|^\alpha < 1,$$

then Eqs. (44) and (45) give

$$R(t) = R_0, \quad \theta(t) = -\omega_\alpha(K)t + \theta_0. \tag{47}$$

Solution (47) means that on the limit cycle (42) the angle variable θ rotates with a constant velocity $\omega_\alpha(K)$. As the result, we have the plane-wave solution

$$Z(x,t) = (1-g|K|^\alpha)^{1/2}e^{iKx - i\omega_\alpha(K)t + i\theta_0}, \quad 1-g|K|^\alpha > 0, \tag{48}$$

which can be interpreted as a synchronized state of the oscillatory medium.

For initial amplitude that deviates from (42), i.e., $R_0^2 \neq 1-g|K|^\alpha$, an additional phase shift occurs due to the term which is proportional to b in (45). The oscillatory medium can be characterized by a single generalized phase variable. To define it, let us rewrite (41) as

$$\frac{d}{dt}\ln R = (1-g|K|^\alpha) - R^2, \tag{49}$$

$$\frac{d}{dt}\theta = (a - cg|K|^\alpha) - bR^2. \tag{50}$$

Substitution of R^2 from (49) into (50) gives

$$\frac{d}{dt}(\theta - b\ln R) = (a - cg|K|^\alpha) - b(1-g|K|^\alpha). \tag{51}$$

Thus, the generalized phase[8] can be defined by

$$\phi(R,\theta) = \theta - b\ln R. \tag{52}$$

From (51), we get

$$\frac{d}{dt}\phi = -\omega_\alpha(K). \tag{53}$$

This equation means that generalized phase $\phi(R,\theta)$ rotates uniformly with constant velocity. For $g|K|^\alpha = (b-a)/(b-c) < 1$, we have the lines of the constant generalized phase. On the (R,θ) plane these lines are logarithmic spirals $\theta - b\ln R = \text{const}$. The decrease of α corresponds to the increase of K. For the case $b=0$ instead of spirals we have straight lines $\phi = \theta$.

C. Group and phase velocity of plane waves

Energy propagation can be characterized by the group velocity

$$v_{\alpha,g} = \frac{\partial\omega_\alpha(K)}{\partial K}. \tag{54}$$

From Eq. (46), we obtain

$$v_{\alpha,g} = \alpha(c-b)g|K|^{\alpha-1}. \tag{55}$$

For

$$|K| < K_1 = (\alpha/2)^{2-\alpha}, \tag{56}$$

we get

$$|v_{\alpha,g}| > |v_{2,g}|. \tag{57}$$

The phase velocity is

$$v_{\alpha,\mathrm{ph}} = \omega_\alpha(K)/K = (c-b)g|K|^{\alpha-1}. \tag{58}$$

For

$$|K| < K_2 = 2^{\alpha-2}, \tag{59}$$

we have

$$|v_{\alpha,\mathrm{ph}}| > |v_{2,\mathrm{ph}}|. \tag{60}$$

Therefore, the long-range interaction decreases as $|x|^{-(\alpha+1)}$ with $1<\alpha<2$ leads to an increase in the group and phase velocities for small wave numbers ($K \to 0$). Note that the ratio $v_{\alpha,g}/v_{\alpha,\mathrm{ph}}$ between the group and phase velocities of plane waves is equal to α.

D. Stability of the plane wave solution

The solution of (48) can be presented as

$$X = R(K,t)\cos(\theta(K,t) + Kx),$$
$$Y = R(K,t)\sin(\theta(K,t) + Kx), \tag{61}$$

where $X = X(K,t) = \mathrm{Re}\, Z(x,t)$ and $Y = Y(K,t) = \mathrm{Im}\, Z(x,t)$, and $R(K,t)$ and $\theta(K,t)$ are defined by (44) and (45). For the plane waves

$$X_0(x,t) = (1 - g|K|^\alpha)^{1/2}\cos(Kx - \omega_\alpha(K)t + \theta_0),$$
$$Y_0(x,t) = (1 - g|K|^\alpha)^{1/2}\sin(Kx - \omega_\alpha(K)t + \theta_0),$$
$$1 - g|K|^\alpha > 0. \tag{62}$$

Not all of the plane waves are stable. To obtain the stability condition, consider the variation of (39) near the solution (62)

$$\frac{d}{dt}\delta X(K,t) = A_{11}\delta X + A_{12}\delta Y, \quad \frac{d}{dt}\delta Y(K,t) = A_{21}\delta X + A_{22}\delta Y, \tag{63}$$

where δX and δY are small variations of X and Y, and

$$A_{11} = 1 - g|K|^\alpha - 2X_0(X_0 - bY_0) - (X_0^2 + Y_0^2),$$
$$A_{12} = -a + gc|K|^\alpha - 2Y_0(X_0 - bY_0) + b(X_0^2 + Y_0^2),$$
$$A_{21} = a - gc|K|^\alpha - 2X_0(Y_0 + bX_0) - b(X_0^2 + Y_0^2),$$
$$A_{22} = 1 - g|K|^\alpha - 2Y_0(Y_0 + bX_0) - (X_0^2 + Y_0^2). \tag{64}$$

The conditions of asymptotic stability for (63) are

$$A_{11} + A_{22} < 0, \quad A_{11}A_{22} - A_{12}A_{21} < 0. \tag{65}$$

From Eqs. (62) and (64), we get

$$A_{11} + A_{22} = -2(1 - g|K|^\alpha), \quad 1 - g|K|^\alpha > 0, \tag{66}$$

and the first condition of (65) is valid. Substitution of Eqs. (62) and (64) into (65) gives

$$A_{11}A_{22} - A_{12}A_{21} = (b(1 - g|K|^\alpha) - (a - gc|K|^\alpha))$$
$$\times (3b(1 - g|K|^\alpha) - (a - gc|K|^\alpha)). \tag{67}$$

Then the second condition of (65) has the form

$$(V-1)(V-3) < 0, \tag{68}$$

where

$$V = \frac{a - gc|K|^\alpha}{b(1 - g|K|^\alpha)}. \tag{69}$$

As the result, we obtain

$$0 < 1 - g|K|^\alpha < a/b - (c/b)g|K|^\alpha < 3(1 - g|K|^\alpha), \tag{70}$$

i.e., the plane wave solution (48) is stable if parameters a, b, c and g satisfy (70). Condition (70) defines the region of parameters for plane waves where the synchronization exists.

E. Forced fractional Ginzburg-Landau equation for the isochronous case

In this section, we consider the fractional Ginzburg-Landau (FGL) equation (39) forced by a constant E [the so-called forced isochronous case ($b=0$) (Ref. 8)]

$$\frac{\partial}{\partial t}A = (1 + ia)A - |A|^2 A - g(1 + ic)|K|^\alpha A - iE \quad (\mathrm{Im}\, E = 0), \tag{71}$$

where $A = A(K,t)$, and we put for simplicity $b=0$, and K is a fixed wave number. Our main goal will be transition to synchronized states and its dependence on the order α of the long-range interaction. The system of real equations is

$$\frac{d}{dt}X = (1 - g|K|^\alpha)X - (a - gc|K|^\alpha)Y - (X^2 + Y^2)X,$$
$$\frac{d}{dt}Y = (1 - g|K|^\alpha)Y + (a - gc|K|^\alpha)X - (X^2 + Y^2)Y - E, \tag{72}$$

where $X = X(K,t)$ is real and $Y = Y(K,t)$ are imaginary parts of $A(K,t)$.

In the simulation of Eq. (72), we will take the parameters close to the selected ones in Ref. 8, where the parameters a, g, c, e, K were selected to demonstrate the existence of the Hopf-type bifurcation and the appearance of synchronization. Some differences in our case are due to the fractional value of the interaction exponent $\alpha<2$, while in Ref. 8 it was $\alpha=2$.

A numerical solution of Eq. (72) was performed with parameters $a=1$, $g=1$, $c=70$, $E=0.9$, $K=0.1$, for α within the interval $\alpha \in (1;2)$. The results are presented in Fig. 2, and Fig. 3. For $\alpha_0 < \alpha < 2$, where $\alpha_0 \approx 1.51$, …, the only stable solution is a stable fixed point. This region is of perfect synchronization (phase locking), where the synchronous oscillations have a constant amplitude and a constant phase shift with respect to the external force. For $\alpha < \alpha_0$ the global

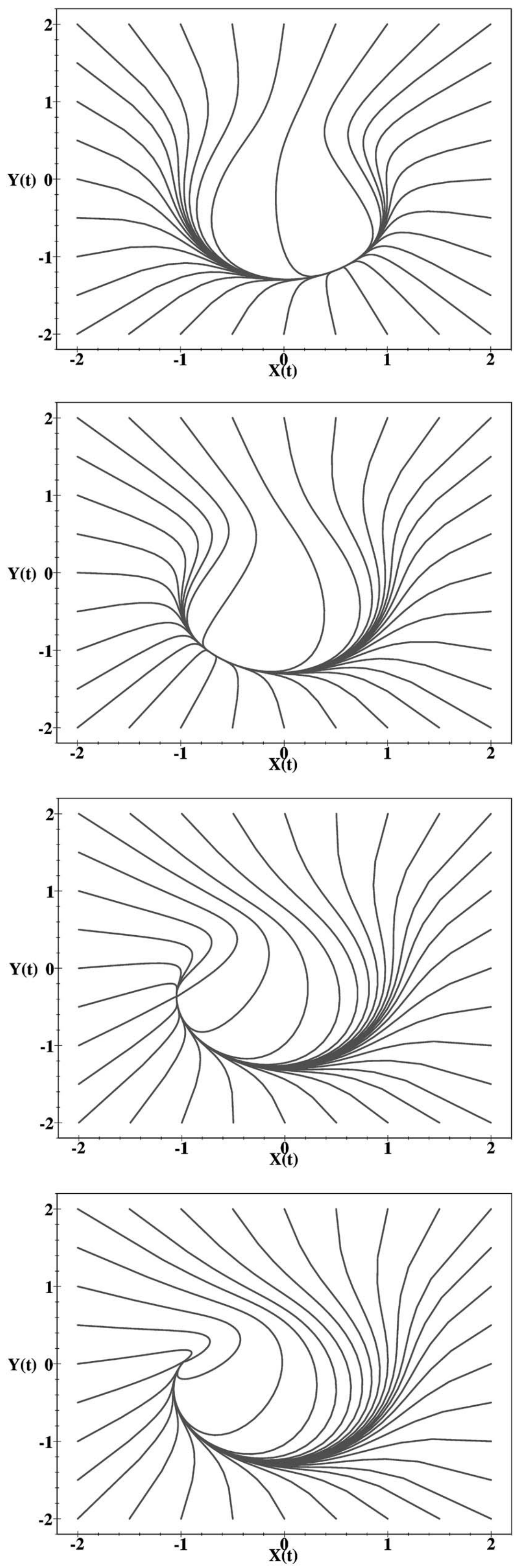

FIG. 2. Approaching the bifurcation point $\alpha=\alpha_0=1.51\ldots$ of the solution of the forced FGL equation for the isochronous case with fixed wave number $K=0.1$ is represented by real $X(K,t)$ and imaginary $Y(K,t)$ parts of $A(K,t)$. The plots for orders $\alpha=2.00$, $\alpha=1.70$, $\alpha=1.60$, $\alpha=1.56$.

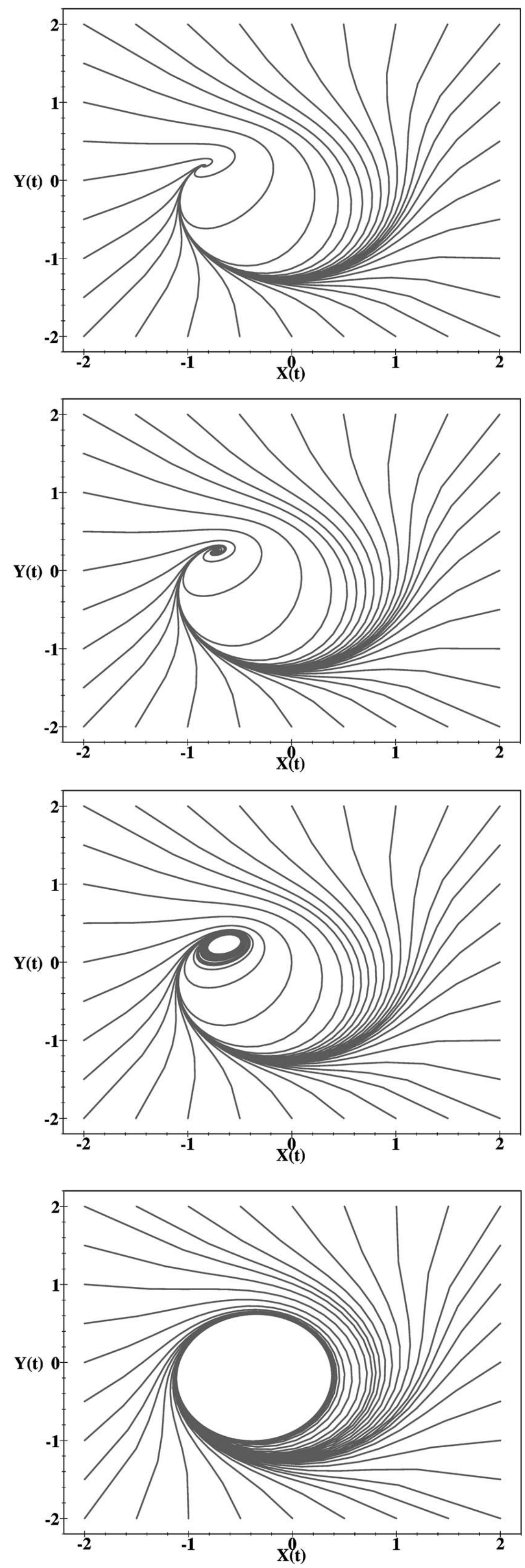

FIG. 3. Transformation to the limit cycle of the solution of the forced FGL equation for the isochronous case with fixed wave number $K=0.1$ is represented by real $X(K,t)$ and imaginary $Y(K,t)$ parts of $A(K,t)$. The plots for orders $\alpha=1.54$, $\alpha=1.52$, $\alpha=1.50$, $\alpha=1.40$.

attractor for (72) is a limit cycle. Here, the motion of the forced system is quasiperiodic. For $\alpha=2$ there is a stable node. When α decreases, the stable mode transfers into a stable focus. At the transition point it loses stability, and a stable limit cycle appears. As the result, we have the decrease of order α from 2 to 1 leads to the loss of synchronization (see Figs. 2 and 3).

The value of $\alpha \sim 1$, when the bifurcation and synchronization appears in our case can be easily understood from the results of Ref. 63, where it was shown that the fractional derivative in a nonlinear oscillations model leads to a dissipation with a decrement of the order $\cos(\pi/\alpha)$ for $1<\alpha<2$. Our results show that the fractional derivative in Eq. (36) does not change the qualitative pattern of synchronization but, instead, brings a new parameter to control the process under consideration. Evidently, synchronization and bifurcation in the following simulations are at the dissipation parameter value of order one since the dissipation, frequency, and nonlinearity terms in (72) are all of order one. The choice of the wave number K can be arbitrary but we select it to be small in order to satisfy the infrared approximation.

In Fig. 2 ($\alpha=2.00$, $\alpha=1.70$, and $\alpha=1.60$, $\alpha=1.56$), we see that in the synchronization region all trajectories are attracted to a stable node.

In Fig. 3 ($\alpha=1.54$, $\alpha=1.52$, and $\alpha=1.50$, $\alpha=1.40$), a stable limit cycle appears via the Hopf bifurcation. For $\alpha=1.54$, and $\alpha=1.52$, near the boundary of synchronization the fixed point is a focus. For $\alpha=1.4$, the amplitude of the limit cycle grows, and synchronization breaks down.

F. Phase and amplitude for the forced FGL equation

The oscillator medium can be characterized by a single generalized phase variable (52). We can rewrite (52) as

$$\phi(X,Y) = \arctan(Y/X) - \frac{b}{2}\ln(X^2+Y^2), \tag{73}$$

where X and Y are defined by (61). For $E=0$, the phase rotates uniformly

$$\frac{d}{dt}\phi = -\omega_\alpha(K) = a - gc|K|^\alpha, \tag{74}$$

where $\omega_\alpha(K)$ is given by (46) with $b=0$, and can be considered as a frequency of natural oscillations. For $E \neq 0$, Eqs. (72) and (73) give

$$\frac{d}{dt}\phi = -\omega_\alpha(K) - E\cos\phi. \tag{75}$$

This equation has an integral of motion. The integral is

$$\begin{aligned} I_1 = {} & 2(\omega^2-E^2)^{-1/2}\arctan((\omega-E) \\ & \times(\omega^2-E^2)^{-1/2}\tan(\phi(t)/2)) + t, \quad \omega^2 > E^2, \end{aligned} \tag{76}$$

$$\begin{aligned} I_2 = {} & 2(E^2-\omega^2)^{-1/2}\operatorname{arctanh}((E-\omega) \\ & \times(E^2-\omega^2)^{-1/2}\tan(\phi(t)/2)) + t, \quad \omega^2 < E^2. \end{aligned} \tag{77}$$

These expressions help to obtain the solution in the form (40) for the forced case (71) keeping the same notations as in (40). For polar coordinates we get

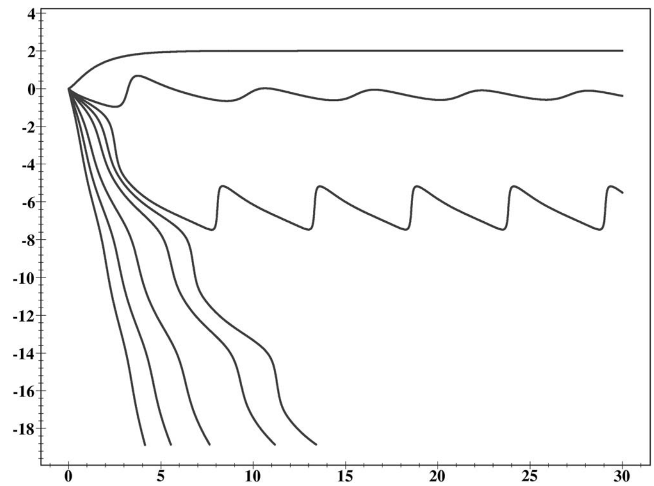

FIG. 4. Phase $\theta(K,t)$ for $K=0.1$ and $\alpha=2.00$, $\alpha=1.50$, $\alpha=1.47$, $\alpha=1.44$, $\alpha=1.40$, $\alpha=1.30$, $\alpha=1.20$, $\alpha=1.10$. The decrease of order α corresponds to the clockwise rotation of curves. For the upper curve $\alpha=2$. For the most vertical curve $\alpha=1.1$.

$$\begin{aligned} \frac{dR}{dt} &= (1-g|K|^\alpha)R - R^3 - E\sin\theta, \\ \frac{d\theta}{dt} &= (a-cg|K|^\alpha) - \frac{E\cos\theta}{R}. \end{aligned} \tag{78}$$

The numerical solution of (78) was performed with the same parameters as for Eq. (72), i.e., $a=1$, $g=1$, $c=70$, $E=0.9$, $K=0.1$, and α within the interval $\alpha \in (1,2)$. The results are presented in Figs. 4 and 5.

The time evolution of phase $\theta(K,t)$ is given in Fig. 4 for $\alpha=2.00$, $\alpha=1.50$, $\alpha=1.47$, $\alpha=1.44$, $\alpha=1.40$, $\alpha=1.30$, $\alpha=1.20$, $\alpha=1.10$. The decrease of α from 2 to 1 leads to the oscillations of the phase $\theta(K,t)$ after the Hopf bifurcation at $\alpha_0=1.51,\ldots$, then the amplitude of phase oscillation decreases and the velocity of phase rotations increases.

The amplitude $R(K,t)$ is shown in Fig. 5 for $\alpha=1.6$, $\alpha=1.55$, $\alpha=1.55$, $\alpha=1.51$, $\alpha=1.50$, $\alpha=1.45$, $\alpha=1.2$. The appearance of oscillations in the plots means the loss of synchronization.

IV. SPACE-STRUCTURES FROM THE FGL EQUATION

In previous sections, we considered mainly time evolution and "time structures" as solutions for the FGL equation. Particularly, the synchronization process was an example of the solution that converged to a time-coherent structure. Here, we focus on the space structures for the solution of the FGL equation (36) with $b=c=0$ and the constants a_1 and a_2 ahead of the linear term,

$$\frac{\partial}{\partial t}Z = (a_1+ia_2)Z - |Z|^2 Z + g\frac{\partial^\alpha}{\partial|x|^\alpha}Z. \tag{79}$$

Let us seek a particular solution of (79) in the form

$$Z(x,t) = R(x,t)e^{i\theta(t)}, \quad R^*(x,t) = R(x,t), \quad \theta^*(t) = \theta(t). \tag{80}$$

Substitution of (80) into (79) gives

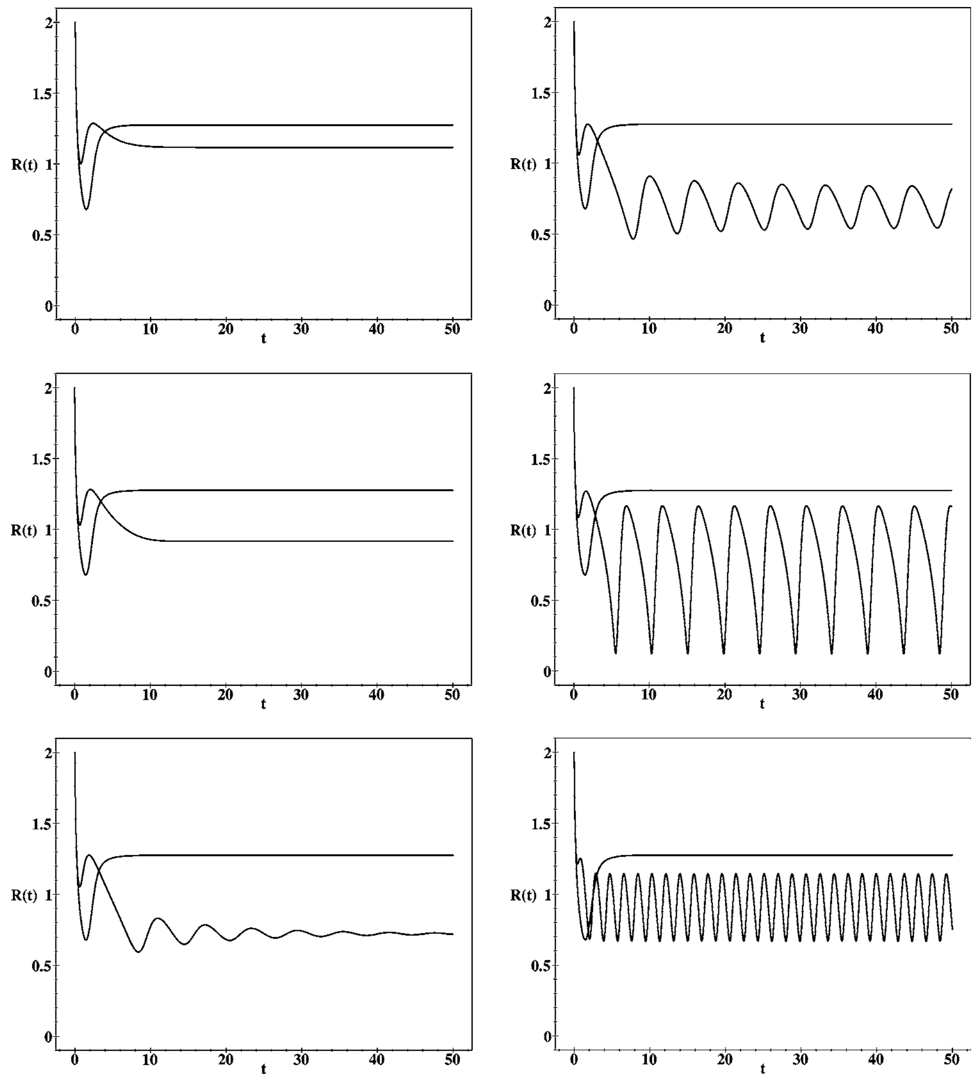

FIG. 5. Amplitude $R(K,t)$. The upper curve corresponds to $\alpha=2$ for all plots. The lower curves correspond to $\alpha=1.6$, $\alpha=1.55$, $\alpha=1.51$, $\alpha=1.50$, $\alpha=1.45$, $\alpha=1.2$. The appearance of oscillations on the plots means the loss of synchronization.

$$\frac{\partial}{\partial t}R = a_1 R - R^3 - g\frac{\partial^\alpha}{\partial|x|^\alpha}R, \quad \frac{\partial}{\partial t}\theta(t) = a_2. \tag{81}$$

Using $\theta(t)=a_2 t+\theta(0)$, we arrive at the existence of a limit cycle with $R_0=a_1^{1/2}$.

A particular solution of (81) in the vicinity of the limit cycle can be found as an expansion

$$R(x,t) = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ \dots \quad (\varepsilon \ll 1). \tag{82}$$

Zero approximation $R_0=a_1^{1/2}$ satisfies (81) since $\partial^\alpha/\partial|x|^\alpha 1 = 0$, and for $R_1=R_1(x,t)$, we have

$$\frac{\partial}{\partial t}R_1 = -2a_1 R_1 + g\frac{\partial^\alpha}{\partial|x|^\alpha}R_1. \tag{83}$$

Consider the Cauchy problem for (83) with an initial condition

$$R_1(x,0) = \varphi(x), \tag{84}$$

and the Green function $G(x,t)$ such that

$$R_1(x,t) = \int_{-\infty}^{+\infty} G(x',t)\varphi(x-x')dx'. \tag{85}$$

Let us apply the Laplace transform for t and the Fourier transform for x,

$$\tilde{G}(k,s) = \int_0^\infty dt \int_{-\infty}^{+\infty} dx e^{-st+ikx}G(x,t). \tag{86}$$

By the definition of the Riesz derivative,

$$\frac{\partial^\alpha}{\partial|x|^\alpha}G(x,t) \leftrightarrow -|k|^\alpha \tilde{G}(k,s), \tag{87}$$

and for the Laplace transform with respect to time

$$\frac{\partial}{\partial t}G(x,t) \leftrightarrow s\tilde{G}(k,s) - 1. \tag{88}$$

Applying (86)–(88) to (83), we obtain

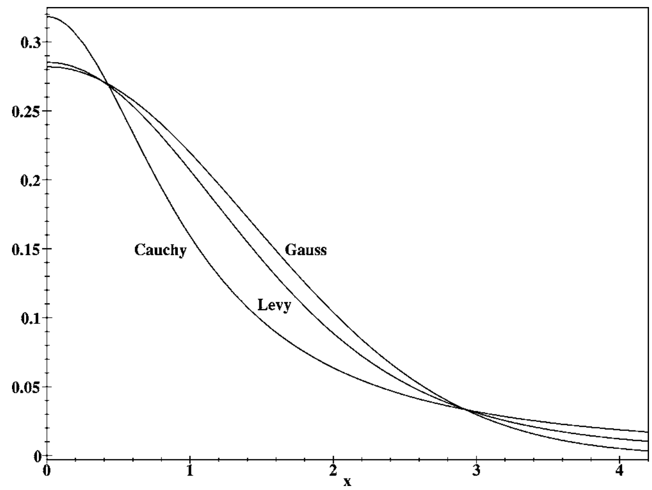

FIG. 6. Gauss PDF ($\alpha=2$), Levy PDF ($\alpha=1.6$), and Cauchy PDF ($\alpha=1.0$). Levy for $\alpha=1.6$ lies between the Cauchy and Gauss PDF. In the asymptotic $x\to\infty$ and $x>3$ on the plot, the upper curve is the Cauchy PDF, and the lower curve is the Gauss PDF.

$$s\widetilde{G}(k,s)-1=-2a_1\widetilde{G}(k,s)-g|k|^{\alpha}\widetilde{G}(k,s) \tag{89}$$

or

$$\widetilde{G}(k,s)=\frac{1}{s+2a_1+g|k|^{\alpha}}. \tag{90}$$

Let us first invert the Laplace transform in (90). Then, the Fourier transform of the Green function

$$\hat{G}(k,t)=\int_{-\infty}^{+\infty}dx e^{ikx}G(x,t)=e^{-(2a_1+g|k|^{\alpha})t}=e^{-2a_1t}e^{-g|k|^{\alpha}t}. \tag{91}$$

As the result, we get

$$G(x,t)=(gt)^{-1/\alpha}e^{-2a_1t}L_{\alpha}(x(gt)^{-1/\alpha}), \tag{92}$$

where

$$L_{\alpha}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk e^{-ikx}e^{-a|k|^{\alpha}} \tag{93}$$

is the Levy stable PDF (Ref. 55). The PDF $L_{\alpha}(x)$ for $\alpha=2.0$, $\alpha=1.6$, and $\alpha=1.0$ are shown in Fig. 6.

As an example, for $\alpha=1$ we have the Cauchy distribution with respect to the coordinate

$$e^{-|k|}\leftrightarrow L_1(x)=\frac{1}{\pi}\frac{1}{x^2+1} \tag{94}$$

and

$$G(x,t)=\frac{1}{\pi}\frac{(gt)^{-1}e^{-2a_1t}}{x^2(gt)^{-2}+1}. \tag{95}$$

For $\alpha=2$, we get the Gauss distribution

$$e^{-k^2}\leftrightarrow L_2(x)=\frac{1}{2\sqrt{\pi}}e^{-x^2/4} \tag{96}$$

and

$$G(x,t)=(gt)^{-1/2}e^{-2a_1t}\frac{1}{2\sqrt{\pi}}e^{-x^2/(4gt)}. \tag{97}$$

For $1<\alpha\leqslant 2$ the function $L_{\alpha}(x)$ can be presented as the convergent expansion

$$L_{\alpha}(x)=-\frac{1}{\pi x}\sum_{n=1}^{\infty}(-x)^n\frac{\Gamma(1+n/\alpha)}{n!}\sin(n\pi/2). \tag{98}$$

The asymptotic ($x\to\infty$, $1<\alpha<2$) is given by

$$L_{\alpha}(x)\sim-\frac{1}{\pi x}\sum_{n=1}^{\infty}(-1)^n x^{-n\alpha}\frac{\Gamma(1+n\alpha)}{n!}\sin(n\pi/2), \quad x\to\infty, \tag{99}$$

with the leading term

$$L_{\alpha}(x)\sim\pi^{-1}\Gamma(1+\alpha)x^{-\alpha-1}, \quad x\to\infty. \tag{100}$$

As the result, the solution of (79) is

$$Z(x,t)=e^{i(a_2t+\theta(0))}\left(a_1^{1/2}+\varepsilon(gt)^{-1/\alpha}e^{-2a_1t}\times\int_{-\infty}^{+\infty}L_{\alpha}(x'(gt)^{-1/\alpha})\varphi(x-x')dx'+O(\varepsilon^2)\right). \tag{101}$$

This solution can be considered as a space-time synchronization in the oscillatory medium with long-range interaction decreasing as $|x|^{-(\alpha+1)}$.

For $\varphi(x)=\delta(x-x_0)$, solution (101) has the form

$$Z(x,t)=e^{i(a_2t+\theta(0))}\big(a_1^{1/2}+\varepsilon(gt)^{-1/\alpha}e^{-2a_1t}\times L_{\alpha}((x-x_0)(gt)^{-1/\alpha})+O(\varepsilon^2)\big), \tag{102}$$

and the asymptotic is

$$Z(x,t)=e^{i(a_2t+\theta(0))}(a_1^{1/2}+\varepsilon gte^{-2a_1t}\pi^{-1}\Gamma(1+\alpha)\times(x-x_0)^{-\alpha-1}+O(\varepsilon^2)), \quad x\to\infty. \tag{103}$$

This solution shows that the long-wave modes approach the limit cycle exponentially with time. For $t=1/(2a_1)$, we have the maximum of $|Z(x,t)|$ with respect to time,

$$\max_{t>0}|Z(x,t)|=a_1^{1/2}+\varepsilon g\frac{\Gamma(1+\alpha)}{2\pi e}(x-x_0)^{-\alpha-1}+O(\varepsilon^2). \tag{104}$$

As the result, we have the power law decay with respect to the coordinate for the space structures near the limit cycle $|Z|=a_1^{1/2}$.

V. NONLINEAR LONG-RANGE INTERACTION AND FRACTIONAL PHASE EQUATION

Here, we would like to show one more application of the replacement of dynamical equation by the fractional ones for a chain with long-range interaction. The model was first considered in Refs. 10, 11, and 49 with application in biology and chemistry. This model has additional interest since it can be reduced to a chain of interacting spins.

A. Nonlinear nonlocal phase coupling

Let us consider the phase equation

$$\frac{d}{dt}\theta_n(t) = \omega_n + g \sum_{m=-\infty, m\neq n}^{+\infty} J_\alpha(n-m)\sin(\theta_n - \theta_m), \tag{105}$$

where θ_n denotes the phase of the nth oscillator, ω_n its natural frequency, and

$$J_\alpha(n) = |n|^{-\alpha-1}. \tag{106}$$

For $\alpha=-1$, Eq. (105) defines the Kuramoto model[11,49–51] with sinusoidal nonlocal coupling (infinite radius of interaction). We can rewrite Eq. (105) for classical spin-like variables

$$s_n(t) = e^{i\theta_n(t)}, \quad \sin(\theta_n - \theta_m) = \frac{1}{2i}(s_n s_m^* + s_n^* s_m). \tag{107}$$

Then Eq. (105) is

$$s_n^* \frac{d}{dt} s_n = i\omega_n + \frac{g}{2} \sum_{m=-\infty, m\neq n}^{+\infty} \frac{1}{|n-m|^{\alpha+1}} [s_n s_m^* + s_n^* s_m]. \tag{108}$$

This equation describes the long-range interaction of spin variables. We also will call Eq. (108) as the phase coupling equation since $|s_n|^2=\text{const}$. Thermodynamics of the model of classical spins with long-range interactions have been studied for more than 30 years. An infinite one-dimensional Ising model with long-range interactions was considered by Dyson.[18] The d-dimensional classical Heisenberg model with long-range interaction is described in Refs. 19 and 20, and its quantum generalization with long-range interaction decreases as $|n|^{-\alpha}$ can be found in Ref. 21.

B. Phase-coupled oscillatory medium with nonlinear long-range interaction

Let us derive an equation for the continuous medium that consists of oscillators of (105) or (108) type with nonlinear long-range interaction. The medium can be defined by the field

$$S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx} \sum_{n=-\infty}^{+\infty} e^{-ikn} s_n(t). \tag{109}$$

We also will need the following momentum representations:

$$a(k,t) = \sum_{n=-\infty}^{+\infty} e^{-ikn} s_n(t). \tag{110}$$

For the left-hand side of (108), we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx} \sum_{n=-\infty}^{+\infty} e^{-ikn} s_n^* \frac{d}{dt} s_n = S^*(x,t)\frac{d}{dt}S(x,t). \tag{111}$$

For the interaction term, we similarly obtain (9)–(17):

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx} \sum_{n=-\infty}^{+\infty} e^{-ikn} \sum_{m=-\infty, m\neq n}^{+\infty} \frac{1}{|n-m|^{\alpha+1}} s_n^* s_m \\
&= S^*(x,t)\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_1 a(k_1,t)\tilde{J}_\alpha(k_1) e^{ik_1 x} \\
&= S^*(x,t)\Bigg(2\zeta(\alpha+1)S(x,t) - a_\alpha \frac{\partial^\alpha}{\partial|x|^\alpha}S(x,t) \\
&\quad + 2\sum_{n=0}^{\infty} \frac{\zeta(\alpha+1-2n)}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}S(x,t)\Bigg),
\end{aligned} \tag{112}$$

where we use (15) for $\tilde{J}_\alpha(k)$, and a_α is the same as in (18).

For the term ω_n, we use

$$\omega(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx} \sum_{n=-\infty}^{+\infty} e^{-ikn}\omega_n. \tag{113}$$

If all oscillators have the same natural frequency $\omega_n=\omega$, then $\omega(x)=\omega$.

As the result, Eq. (108) is transformed into

$$\begin{aligned}
&S^*(x,t)\frac{\partial}{\partial t}S(x,t) \\
&= i\omega(x) - f_\alpha S^*(x,t)S(x,t) \\
&\quad - g_\alpha\left(S^*(x,t)\frac{\partial^\alpha}{\partial|x|^\alpha}S(x,t) + S(x,t)\frac{\partial^\alpha}{\partial|x|^\alpha}S^*(x,t)\right) \\
&\quad + g\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!} \\
&\quad \times\left(S^*(x,t)\frac{\partial^{2n}}{\partial|x|^{2n}}S(x,t) + S(x,t)\frac{\partial^{2n}}{\partial|x|^{2n}}S^*(x,t)\right),
\end{aligned} \tag{114}$$

where

$$f_\alpha = 2g\zeta(\alpha+1), \quad g_\alpha = (1/2)a_\alpha g = g\Gamma(-\alpha)\cos(\pi\alpha/2). \tag{115}$$

Equation (114) is a fractional equation for the oscillatory medium with long-range interacting spins (108). We can call (114) the fractional phase equation.

In the infrared approximation ($k\to 0$), we can use (15)

$$\begin{aligned}
&\tilde{J}_\alpha(k) \approx 2\Gamma(-\alpha)\cos(\pi\alpha/2)|k|^\alpha + 2\zeta(\alpha+1), \\
&\qquad 0<\alpha<2, \quad \alpha\neq 1,
\end{aligned} \tag{116}$$

and Eq. (114) is reduced to

$$\begin{aligned}
S^*(x,t)\frac{\partial}{\partial t}S(x,t) &= i\omega(x) - f_\alpha - g_\alpha\Bigg(S^*(x,t)\frac{\partial^\alpha}{\partial|x|^\alpha}S(x,t) \\
&\quad + S(x,t)\frac{\partial^\alpha}{\partial|x|^\alpha}S^*(x,t)\Bigg),
\end{aligned} \tag{117}$$

where $0<\alpha<2$, $\alpha\neq 1$.

VI. CONCLUSION

A one-dimensional chain of interacting objects, say oscillators, can be considered as a benchmark for numerous applications in physics, chemistry, biology, etc. All considered models were related mainly to the oscillating objects with long-range powerwise interaction, i.e., with forces proportional to $1/|n-m|^s$ and $2<s<3$. A remarkable feature of this interaction is the possibility of replacing the set of coupled individual oscillator equations into the continuous medium equation with the fractional space derivative of the order $\alpha=s-1$, where $0<\alpha<2$, $\alpha\neq 1$. Such a transformation is an approximation and it appears in the infrared limit for the wave number $k\to 0$. This limit helps us to consider different models and related phenomena in a unified way applying different tools of fractional calculus.

A nontrivial example of the general property of the fractional linear equation is its solution with a powerwise decay along the space coordinate. From the physical point of view that means a new type of space structure or coherent structure. The scheme of the equations with fractional derivatives includes either the effect of synchronization,[8] breathers,[56–58] fractional kinetics,[1] and others.

Discrete breathers are periodic space-localized oscillations that arise in discrete and continuous nonlinear systems. Their existence was proven in Ref. 59. Discrete breathers have been widely studied in systems with short-range interactions (for a review, see Refs. 56 and 60). Energy and decay properties of discrete breathers in systems with long-range interactions have also been studied in the framework of the Klein-Gordon,[57,61] and the discrete nonlinear Schrödinger equations.[62] Therefore, it is interesting to consider breathers solution in systems with long-range interactions in the infrared approximation.

We also assume that the suggested replacement of the equations of interacting oscillators by the continuous medium equation can be used for improvement of simulations for equations with fractional derivatives.

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