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COMMENTS ON "RIEMANN–CHRISTOFFEL TENSOR IN DIFFERENTIAL GEOMETRY OF FRACTIONAL ORDER APPLICATION TO FRACTAL SPACE-TIME", [FRACTALS 21 (2013) 1350004]

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Abstract

We prove that main properties represented by Eq. (4.2) for fractional derivative of power function and the non-fractional Leibniz rule in the form (4.3) of the considered paper, cannot hold together for derivatives of non-integer order. As a result, we prove that the usual Leibniz rule (4.3) cannot hold for fractional derivatives.

Keywords: Fractional Derivative; *L*eibniz Rule; Modified Riemann–Liouville Derivatives.

In Ref. 1, the author presents as main properties of suggested fractional derivatives $\mathcal{D}^\alpha$ the equation for fractional derivative of power function (FD-PF)

$$\mathcal{D}^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \quad (\gamma > 0 \quad \alpha > 0), \tag{1}$$

and the Leibniz rule

$$\mathcal{D}^\alpha(v(x)u(x)) = (\mathcal{D}^\alpha v(x))u(x) + v(x)(\mathcal{D}^\alpha u(x)), \tag{2}$$

as Eqs. (4.2) and (4.3) of Ref. 1, where $\mathcal{D}^\alpha$ is the modified Riemann–Liouville fractional derivatives.

Note that Eq. (1) can be used for $x > 0$ since $x^{\gamma-\alpha}$ does not exist for $\gamma - \alpha < 0$ at $x = 0$.

The relations (1) and (2) cannot be performed together for fractional derivatives with orders $\alpha \neq 1$. To prove this statement, we can use the functions $u(x) = v(x) = x$ in the Leibniz rule (2). In this case, this rule is written in the form

$$\mathcal{D}^\alpha x^2 = (\mathcal{D}^\alpha x)x + x(\mathcal{D}^\alpha x). \tag{3}$$

Equation (1) gives

$$\mathcal{D}^\alpha x^2 = \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad \mathcal{D}^\alpha x = \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha}, \tag{4}$$

Substituting Eq. (4) into Eq. (3), we obtain

$$\frac{\Gamma(3)}{\Gamma(3-\alpha)} - \frac{2\Gamma(2)}{\Gamma(2-\alpha)} = 0. \tag{5}$$

If we take into account $\Gamma(3-\alpha) = (2-\alpha)\Gamma(2-\alpha)$ and $\Gamma(n) = (n-1)!$, then (5) can be represented in the form

$$\frac{1-\alpha}{\Gamma(3-\alpha)} = 0. \tag{6}$$

As a result, we demonstrate that Eq. (1) and the Leibniz rule (2) cannot be satisfied together for $\alpha \neq 1$. Analogously, we can use $u(x) = x^{\gamma_1}$ and $v(x) = x^{\gamma_2}$ with $\gamma_1, \gamma_2 \in \mathbb{R}_+$ to prove that the Leibniz rule (2) holds only for $\alpha = 1$.

In Sec. 4.2.1 of Refs. 1 and 2, there is an attempt to answer the obvious objection that the Leibniz rule (2) cannot hold for derivatives of orders $\alpha \neq 1$. The main assumption of this answer is that (2) holds only for non-differentiable functions $u(x)$ and $v(x)$. This assumption is incorrect also. Equation (2) of the Leibniz rule means that the fractional derivatives $\mathcal{D}^\alpha u(x)$, $\mathcal{D}^\alpha v(x)$ and $\mathcal{D}^\alpha(u(x)v(x))$ exist, i.e., the functions $u(x)$ and $v(x)$ should be fractionally differentiable. Therefore, arbitrary non-differentiable functions cannot be considered in the Leibniz rule (2). Using Eq. (1) (see Eq. (4.2) of Ref. 1), we can see that the author assumes that the power functions x^γ ($\gamma \in \mathbb{R}_+$) are fractionally differentiable. Using that power functions are fractionally differentiable, we can consider the Leibniz rule (2) for the power functions $u(x) = x^\beta$ and $v(x) = x^\gamma$ with $\beta, \gamma \neq \alpha$ (including integer values of β and γ), where α is the order of fractional derivative used in (2). As a result, we get by transformation similar to (3–6) that the Leibniz rule (2) holds only for $\alpha = 1$.

In addition, it is easy to see that nowhere in the "proof" of (2) given in Ref. 2, the assumption that $u(x)$, $v(x)$ are fractional differential functions but not classically differentiable is not used. Therefore, we can repeat the same "proof" for each pair $u(x)$, $v(x)$ of fractional differential functions without the useless assumption that these functions are not classically differential. This allows to use power functions x^γ in (2). As a result, Eqs. (1) and (2) lead to the statement that the Leibniz rule (2) cannot hold for $\alpha \neq 1$. In addition, this means that the "proof" of (2) suggested in Ref. 2 is incorrect.

The violation of the Leibniz rule (2) is a characteristic property of fractional-order derivatives of all types[3] and derivatives of integer orders $\alpha \neq 1$. Moreover, the fact of violation of the Leibniz rule (2) for fractional derivatives does not depend on the class of functions (in contrast to statements in Ref. 2), if the relation (1) can be used. A correct form of the Leibniz rule for fractional-order derivatives should be obtained as a generalization of the Leibniz rule for integer-order derivatives (see Sec. 2.7.2 of Refs. 4 and 5).

In addition, the chain rule (see Eqs. (4.4), (5.1), (5.2) of Ref. 1), which is used as the basis for formulation of the suggested generalization of differential geometry in Ref. 1, also cannot satisfy for fractional-order derivatives with $\alpha \neq 1$ (for example, see Ref. 6). Moreover, by using the Fourier transform it is easy to prove that the nonlinear coordinate transformation maps fractional order derivatives into pseudo-differential operator of the general form that cannot be represented as a fractional derivative. As a result, the "fractional" differential geometry of fractional differential manifold suggested in Ref. 1 and Refs. 7 and 8 is wrong.

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