Fundamenta Informaticae 151 (2017) 431–442
DOI 10.3233/FI-2017-1502
IOS Press

Interpretation of Fractional Derivatives as Reconstruction from Sequence of Integer Derivatives

Vasily E. Tarasov*
Skobeltsyn Institute of Nuclear Physics
Lomonosov Moscow State University
Moscow 119991, Russia
tarasov@theory.sinp.msu.ru

Abstract. In this paper, we propose an "informatic" interpretation of the Riemann-Liouville and Caputo derivatives of non-integer orders as reconstruction from infinite sequence of standard derivatives of integer orders. The reconstruction is considered with respect to orders of derivatives.

Keywords: Fractional derivative, Riemann-Liouville derivative, Caputo derivative, derivations of Hadamard type, sampling theorem.

1. Introduction

Derivatives of non-integer orders [1, 2] have been studied for a long time and they are associated with the names of famous mathematicians such as Riemann, Liouville, Riesz, Grünwald, Letnikov, Sonine, Marchaud, Weyl and others. Fractional-order derivatives have wide applications in physics and mechanics since it allow us to describe systems, media and fields that are characterized by power-law non-locality and memory of power-law type. The fractional-order derivatives have a set of unusual properties such us violation of the standard Leibniz and chain rules (for example, see [3, 4, 5, 6]).

*Address for correspondence: Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Leninskie gory, GSP-1, Moscow 119991, Russian Federation

There are different interpretations of the fractional derivatives and integrals such as probabilistic interpretations [13, 14, 15], geometric interpretations [16, 17, 18, 19, 20, 21], physical interpretations [19, 20, 21, 22, 23, 24, 25, 26].

In this paper, we propose an "informatic" ("computer science") interpretation of the Riemann-Liouville and Caputo derivatives of non-integer orders. It allows us to interpret these fractional-order derivatives as reconstructions from infinite sequence of derivatives of integer orders.

2. Kotel'nikov theorem on reconstruction

The theorem on the possibility of a complete reconstruction of the continuous function (signal) at a discrete reference was first proposed and proved by Vladimir Kotel'nikov [7] in 1933. This theorem is also proved by Claude Shannon [8] in 1949.

The Kotel'nikov theorem, which is also known as the sampling theorem, states that under certain restrictive conditions, function $f(t)$ can be restored from its sample, $f[n] = f(nT)$, according to the Whittaker-Shannon interpolation formula. For a given sequence of real numbers $f[n]$, the continuous function $f(t)$ is defined by the equation

$$f(t) = \sum_{n=-\infty}^{\infty} f[n] \, \mathrm{sinc} \left(\frac{t}{T} - n \right) , \tag{1}$$

where T is a period of sampling, and the sinc function is

$$\mathrm{sinc}(z) := \begin{cases} \dfrac{\sin(\pi z)}{\pi z} & z \neq 0 \\ 1 & z = 0. \end{cases} \tag{2}$$

The noted restrictive conditions for the function $f(t)$ of the interpolation formula (1) are the following: (1) $f(t)$ should be bounded. The Fourier integral transform of $f(t)$ should satisfy the property: $\mathcal{F}\{f(t)\} = \hat{f}(\omega) = 0$ for $|\omega| > \omega_0 > 0$; (2) The sampling frequency $\omega_s = 2\pi/T$ should be at least more than twice the range of frequencies, $\omega_s > 2\omega_0$.

Interpolation formula (1) reconstructs the original function $f(t)$ of continuous variable t, only if these two conditions are satisfied. The Whittaker-Shannon interpolation formula is a basis to construct a continuous-time bandlimited function from a sequence of real numbers [9, 10].

Remark 1. Let $f(x)$ be a function on $\mathbf{R}$ and let $h > 0$. The cardinal series of $f(x)$ with respect to the interval h is defined by the formal series

$$C(f, h, x) := \sum_{n=-\infty}^{\infty} f(n\,h) \ \mathrm{sinc} \left(\frac{x}{h} - n \right) . \tag{3}$$

If this series converges, then $C(f, h, x)$ is called the Whittaker cardinal function of $f(x)$ [11, 12].

3. Theorem on binomial coefficients

Let us give theorem on the binomial coefficients.

Theorem 1. *Equality*

$$\sum_{n=0}^{\infty} \binom{\beta}{n} \operatorname{sinc}(n-\alpha) = \binom{\beta}{\alpha} \quad (\beta \geq \alpha > 0) \tag{4}$$

holds for the binomial coefficients $\binom{\beta}{\alpha}$ *that are defined by the equation*

$$\binom{\beta}{\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1-\alpha)}. \tag{5}$$

Proof:
Equation (4) will be proved by using properties of hypergeometrical functions. Substitution of (5) into equation (4) gives

$$\sum_{n=0}^{\infty} \frac{\operatorname{sinc}(n-\alpha)\,\Gamma(\beta+1)}{\Gamma(n+1)\,\Gamma(\beta+1-n)} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1-\alpha)}. \tag{6}$$

Using the equation

$$\operatorname{sinc}(z) = \frac{1}{\Gamma(1+z)\,\Gamma(1-z)} \quad (z \notin \mathbf{Z}), \quad \operatorname{sinc}(z) = 1 \quad (z \in \mathbf{Z}) \tag{7}$$

and multiplying equation (6) by the factor $\Gamma(\alpha+1)$, we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha-n+1)\,\Gamma(n+1-\alpha)} \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}, \tag{8}$$

where $\beta \geq \alpha > 0$. Then we rewrite (8) in the form

$$\frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} \cdot \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \cdot \frac{\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \cdot \frac{1}{n!} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}. \tag{9}$$

Using the Pochhammer symbol which can be defined by the equation

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)}, \tag{10}$$

expression (9) can be written as

$$\frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha-n+1)_n\,(\beta-n+1)_n}{(1-\alpha)_n} \frac{1}{n!} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}. \tag{11}$$

Appling the following equality of the Pochhammer symbols

$$(-x)_n = (-1)^n \cdot (x-n+1)_n, \tag{12}$$

we rewrite equation (11) in the form

$$\frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(-\alpha)_n \, (-\beta)_n}{(1-\alpha)_n} \frac{1}{n!} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}. \tag{13}$$

Using the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$, which is defined by the equation (for example, see equation 1.6.1 of [2]) of the form

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_k \, (b)_k}{(c)_k} \frac{z^n}{n!}, \tag{14}$$

where $|z| < 1$; $a, b \in \mathbf{C}$; $c \in \mathbf{C}/Z_0^-$, we represent (13) as

$$\frac{1}{\Gamma(1-\alpha)} \, {}_2F_1(-\alpha, -\beta; 1-\alpha; 1) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}. \tag{15}$$

Using equation 1.6.9 of [2], which has the form

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (c-a-b > 0, \quad b \leq 0), \tag{16}$$

equation (15) gives the identity

$$\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\,\Gamma(1-\alpha+\alpha+\beta)}{\Gamma(1-\alpha+\alpha)\,\Gamma(1-\alpha+\beta)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}, \tag{17}$$

where $\beta \geq \alpha > 0$, which coincides with equation (4), was to be proved. □

Corollary. *For $\beta = m \in \mathbf{N}$, equality (4) of the binomial coefficients has the form*

$$\sum_{n=0}^{m} \binom{m}{n} \operatorname{sinc}(n-\alpha) = \binom{m}{\alpha}. \tag{18}$$

Proof:
For $\beta = m \in \mathbf{N}$, we can use $\binom{m}{n} = 0$ for $n > m$. Then (4) take the form (18). □

4. Riemann-Liouville fractional derivatives and derivatives of Hadamard type

Let us give a definition of the Riemann-Liouville fractional derivatives and some corresponding properties.

Definition 1. *The left-sided Riemann-Liouville fractional derivatives of order $\alpha > 0$ on finite interval $[a, b]$ is defined by*

$$^{RL}\mathcal{D}^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(n+1-\alpha)} \frac{d^{n+1}}{dx^{n+1}} \int_a^x \frac{f(z)dx}{(x-z)^{\alpha-n}} \quad (n \leq \alpha < n+1), \tag{19}$$

where we use $n = [\alpha]$ and $x > a$.

The main properties of the Riemann-Liouville derivatives of order α (see equations 2.1.16, 2.1.20, 2.1.21 of Property 2.1 of [2]) have the form

$$^{RL}\mathcal{D}^{\alpha}_{a+} (x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, \tag{20}$$

where $\alpha > 0$, $\beta > -1$, and $\beta - \alpha + 1 \neq -k$ ($k = 0, 1, 2, ...$);

$$^{RL}\mathcal{D}^{\alpha}_{a+} (x-a)^{\alpha-m} = 0 \quad (m = 1, 2, ..., n+1), \tag{21}$$

where $n = [\alpha]$. For integer values of order $\alpha \in \mathbf{N}$ (see equations 2.1.7 of Section 2.1 of [2]) the Riemann-Liouville derivative is equal to the standard integer-order derivative

$$^{RL}\mathcal{D}^{0}_{a+} f(x) = f(x), \quad ^{RL}\mathcal{D}^{n}_{a+} f(x) = \frac{d^n f(x)}{dx^n} \quad (n \in \mathbf{N}). \tag{22}$$

Let us give a definition of the fractional derivatives that are expressed by the left-sided Riemann-Liouville fractional derivatives of order $\alpha > 0$ and the factor $(x-a)^{\alpha}/\Gamma(\alpha+1)$.

Definition 2. *Fractional derivatives of the Hadamard type is the operators*

$$^{HT}\mathcal{D}^{\alpha}_{a+} := \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} \cdot {}^{RL}\mathcal{D}^{\alpha}_{a+} \quad (\alpha > 0), \tag{23}$$

where $^{RL}\mathcal{D}^{\alpha}_{a+}$ the left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$.

Using (22), the Hadamard-type fractional derivatives (23) of integer orders $n \in \mathbf{N}$ has the form

$$^{HT}\mathcal{D}^{n}_{a+} := \frac{1}{\Gamma(n+1)} (x-a)^{n} \cdot \frac{d^n}{dx^n} \quad (n \in \mathbf{N}). \tag{24}$$

Using (20) and (21), we obtain the derivatives of power functions

$$^{HT}\mathcal{D}^{\alpha}_{a+} (x-a)^{\beta} = \binom{\beta}{\alpha} (x-a)^{\beta}, \tag{25}$$

where $\alpha > 0$, $\beta > -1$, $\beta - \alpha + 1 \neq -k$ with $k = 0, 1, 2, ...$, and

$$^{HT}\mathcal{D}^{\alpha}_{a+} (x-a)^{\alpha-m} = 0 \quad (m = 1, 2, ..., n+1). \tag{26}$$

Remark 2. It is known that an action of the Hadamard operators on the power functions give the same function (see equations (2.7.21-2.7.24) of Property 2.25 in [2]). Therefore the operators (23) we call the derivations of Hadamard type.

Let us give a definition of the Caputo fractional derivative (see Section 2.4 of [2]).

Definition 3. *The Caputo fractional derivative can be defined via the Riemann-Liouville fractional derivative by the equation*

$$({}^{C}\mathcal{D}^{\alpha}_{a+}f)(x) = \left({}^{RL}\mathcal{D}^{\alpha}_{a+} \left(f(z) - \sum_{k=0}^{n} \frac{(z-a)^k}{k!} \left(\frac{d^k f(z)}{dz^k} \right)(a) \right) \right)(x), \tag{27}$$

where $n \le \alpha < n+1$, $n = [\alpha]$ *and* $x \in (a,b)$.

If the function $f(x)$ is an analytic function on the interval (a,b), then it can be represented as a convergent power series on (a,b) of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \left(\frac{d^k f(x)}{dx^k} \right)(a). \tag{28}$$

Using equation (27), we can state that the Caputo fractional derivative of order $n \le \alpha < n+1$ of function (28) can be considered as the Riemann-Liouville fractional derivative of the function

$$f_{C,\alpha}(x) = \sum_{k=[\alpha]+1}^{\infty} \frac{(x-a)^k}{k!} \left(\frac{d^k f(x)}{dx^k} \right)(a). \tag{29}$$

For the Caputo fractional derivatives, we can also define the corresponding derivatives of Hadamard type

$${}^{HT}_{C}\mathcal{D}^{\alpha}_{a} := \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} \cdot {}^{C}\mathcal{D}^{\alpha}_{a+} \quad (\alpha > 0), \tag{30}$$

where ${}^{C}\mathcal{D}^{\alpha}_{a+}$ the left-sided Caputo fractional derivative (27) of order $\alpha > 0$.

Remark 3. It should be note that the Marchaud fractional derivatives see Section 2.7 of [2]) coincide with the corresponding Riemann-Liouville fractional derivatives for a wide class of functions (for example, see Section 13 of [1]).

5. Fractional derivatives as infinite series of integer derivatives

Let us give a theorem on a sinc-representation of the fractional-order derivatives of Hadamard type as an infinite series of derivatives of integer orders.

Theorem 2. *Let $f(x)$ be an analytic function on the interval (a, b), which can be represented as a convergent power series on (a, b). Then the fractional derivatives of Hadamard type (23) of order $\alpha > 0$ can be represented in the form*

$$^{HT}\mathcal{D}^{\alpha}_{a} f(x) = \sum_{n=0}^{\infty} \mathrm{sinc}(\alpha - n)\, \mathcal{D}^{n}_{a} f(x), \tag{31}$$

where $\alpha > 0$ and $\mathrm{sinc}(\alpha - n)$ is the sinc function.

Proof:
The proof is based on an important equation that allows us to express the Riemann-Liouville derivative of non-integer order through an infinite series of derivatives of integer orders. Let $f(x)$ be an analytic function on the interval (a, b), which can be represented as a convergent power series on (a, b), then the left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$ can be represented in the form

$$^{RL}\mathcal{D}^{\alpha}_{a+} f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(x-a)^{n-\alpha}}{\Gamma(n+1-\alpha)} \frac{d^n f(x)}{dx^n} \quad x \in (a, b), \tag{32}$$

where $d^n f(x)/dx^n$ is the standard derivative of order $n \in \mathbf{N}$, and $d^0 f(x)/dx^0 := f(x)$. The proof of representation (32) is given in [1] (see Lemma 15.3 of [1]).

Substitution of the equation

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1-n)} \tag{33}$$

into (32), we obtain

$$^{RL}\mathcal{D}^{\alpha}_{a+} f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1-n)\,\Gamma(n+1-\alpha)} (x-a)^{n-\alpha} \frac{d^n f(x)}{dx^n} \quad x \in (a, b). \tag{34}$$

Let us rewrite equation (34) as

$$\frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha}\, {}^{RL}\mathcal{D}^{\alpha}_{a+} f(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha+1-n)\,\Gamma(n+1-\alpha)} \frac{(x-a)^n}{\Gamma(n+1)} \frac{d^n f(x)}{dx^n}, \tag{35}$$

where $x \in (a, b)$. Using equations (23), (24) and equation 1.5.8 of [2],

$$\Gamma(1+z)\,\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)} \quad (z \notin \mathbf{Z}_0), \quad \Gamma(1+z)\,\Gamma(1-z) = 1 \quad (z = 0), \tag{36}$$

we can write equation (35) in the form

$$^{HT}\mathcal{D}^{\alpha}_{a+} f(x) = \sum_{n=0}^{\infty} \frac{\sin(\pi(\alpha-n))}{\pi(\alpha-n)} \frac{1}{\Gamma(n+1)} (x-a)^n\, {}^{HT}\mathcal{D}^{n}_{a+} f(x) \quad (\alpha \neq m \in \mathbf{N}), \tag{37}$$

where $x \in (a, b)$ and $^{HT}\mathcal{D}^{n}_{a+}$ is the standard derivatives of order n with the factor $(x-a)^n/n!$, that is, the nth term of the Taylor's series of $f(x)$. For $\alpha = m \in \mathbf{N}$ equation (35) is an equality, since $\Gamma(\alpha+1-n)\,\Gamma(n+1-\alpha) = 1$ for $\alpha = n$, and $\Gamma(\alpha+1-n)\,\Gamma(n+1-\alpha) = 0$ for $\alpha = m \neq n$. Using the definition of the sinc function (2), we obtain equation (31). $\square$

The theorem is proved for analytic functions on the interval (a,b). Let us prove equation (31) for the power functions

$$f_{a,\beta}(x) := (x-a)^{\beta} \quad (\beta \geq \alpha > 0, \quad x > a) \tag{38}$$

by using Theorem 1.

Theorem 3. *Let $f_{a,\beta}(x)$ be a power function of type (38). Then the fractional derivatives of Hadamard type (23) of order $\alpha > 0$ can be represented in form (31), that is,*

$$^{HT}\mathcal{D}^{\alpha}_{a} f_{a,\beta}(x) = \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n)\, \mathcal{D}^{n}_{a} f_{a,\beta}(x) \quad (\beta \geq \alpha > 0) \tag{39}$$

for all $x \in (a,b)$.

Proof:
Substitution of (38) into (39) gives

$$^{HT}\mathcal{D}^{\alpha}_{a} (x-a)^{\beta} = \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n)\, \mathcal{D}^{n}_{a} (x-a)^{\beta}. \tag{40}$$

Using equation (25) in the form

$$^{HT}\mathcal{D}^{\alpha}_{a+} (x-a)^{\beta} = \binom{\beta}{\alpha} (x-a)^{\beta}, \quad {}^{HT}\mathcal{D}^{n}_{a} (x-a)^{\beta} = \binom{\beta}{n} (x-a)^{\beta}, \tag{41}$$

and substituting the r.h.s. of (41) into equation (40), we get

$$\sum_{n=0}^{\infty} \binom{\beta}{n} \operatorname{sinc}(n-\alpha) = \binom{\beta}{\alpha} \quad (\beta \geq \alpha > 0), \tag{42}$$

which is proved in Theorem 1 on the binomial coefficients. □

Remark 4. For $\alpha = m \in \mathbf{N}$, we can use

$$\operatorname{sinc}(m-n) = \delta_{m,n} \quad (n, m \in \mathbf{N}), \tag{43}$$

and equation (31) gives

$$^{HT}\mathcal{D}^{\alpha}_{a+} f(x) = \sum_{n=0}^{\infty} \delta_{m,n} \frac{(x-a)^n}{\Gamma(n+1)} \cdot \frac{d^n f(x)}{dx^n} = \frac{(x-a)^m}{\Gamma(m+1)} \cdot \frac{d^m f(x)}{dx^m}. \tag{44}$$

This expression means that equation (31) holds for the Hadamard-type derivatives (23) of integer orders $\alpha = m \in \mathbf{N}$.

Remark 5. Equation (31) of the fractional-order derivative can be represented in the form

$$ {}^{HT}\mathcal{D}^{\alpha}_{a} f(x) = \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n) \, \frac{(x-a)^n}{n!} \, \frac{d^n f(x)}{dx^n}, \tag{45} $$

where $\alpha > 0$ and $x > a$.

Remark 6. Representation (45) for the functions $f_{C,\alpha}$, which are defined by (29), allows us to use equation (31) of Theorem 2 for the Caputo fractional derivatives of the Hadamard type (30). In this case, expression (31) is represented in the form

$$ {}^{HT}_{C}\mathcal{D}^{\alpha}_{a} f(x) = \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n) \, \mathcal{D}^{n}_{a} f_{C,\alpha} \quad (\alpha > 0), \tag{46} $$

where $f_{C,\alpha}$ is defined by (29).

Remark 7. Using that the Marchaud fractional derivatives coincide with the corresponding Riemann-Liouville fractional derivatives for a wide class of functions (see Section 13 of [1]), the sinc-represetation (31) can also be applied for the Marchaud fractional-order derivatives of these functions.

6. Reconstruction of fractional derivative

Equation (31) can be considered as an analog of formula (1). This equation allows us to construct continuous-order derivatives from a sequence of integer-order derivatives.

If we consider the sequence

$$ f_x[n] := \frac{(x-a)^n}{\Gamma(n+1)} \, \frac{d^n f(x)}{dx^n} \quad (n \in \mathbf{N}), \tag{47} $$

which is sequence of derivatives of integer order $n \in \mathbf{N}$ at fixed points $x \in (a,b)$, then interpolation formula (1) with $T = 1$ defines the continuous analog $f_x(\alpha)$ of $f_x[n]$. This analog can be considered as a derivative of non-integer order α, that is defined by the equation

$$ f_x(\alpha) = \sum_{n=0}^{\infty} f_x[n] \operatorname{sinc}(\alpha - n) \, . \tag{48} $$

This equation can be used to define a "continuous-order" derivative (${}^{CO}\mathcal{D}^{\alpha}_{x}$) via integer-order derivatives (${}^{IO}\mathcal{D}^{n}_{x}$) by applying the functions

$$ f_x(\alpha) := \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \, {}^{CO}\mathcal{D}^{\alpha}_{x} f(x), \quad f_x[n] := \frac{(x-a)^n}{\Gamma(n+1)} \, {}^{IO}\mathcal{D}^{n}_{x} f(x), \tag{49} $$

where $f(x)$ is the function for which the derivatives of equation (49) exist. As a result, we have a new approach to definition of derivatives of non-integer orders that is based on the equation

$$ \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \, {}^{CO}\mathcal{D}^{\alpha}_{x} f(x) := \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n) \, \frac{(x-a)^n}{\Gamma(n+1)} \, {}^{IO}\mathcal{D}^{n}_{x} f(x). \tag{50} $$

We can also consider some "continuous-order" and integer-order derivatives of Hadamard type that are connected by the expression

$$ {}^{HTCO}\mathcal{D}^{\alpha}_{x} f(x) = \sum_{n=0}^{\infty} \operatorname{sinc}(\alpha - n)\ {}^{HTIO}\mathcal{D}^{n}_{x} f(x). \tag{51} $$

It should be emphasized that possibility to use definitions (50) and (51) has been proved (see the proof of Theorem 2) in the case of an analytic function $f(x)$ with $x \in (a, b)$ and the standard definition of integer-order derivatives ${}^{IO}\mathcal{D}^{n}_{x} f(x) := d^n f(x)/dx^n$. In this case, the "continuous-order" derivative is the Riemann-Liouville fractional derivative (${}^{CO}\mathcal{D}^{\alpha}_{x} = {}^{RL}\mathcal{D}^{\alpha}_{a+}$). Equations (31), (50) and (51) can be considered as analogs of the interpolation formula (1), which constructs continuous-order derivatives from a sequence of integer-order derivatives. Equations (50) and (51) are analogs of the interpolation formula (1), which constructs continuous-order derivatives from a sequence of integer-order derivatives. As a result, these equations allows us to get continuous-order derivatives from a sequence of derivatives of integer orders.

As a continuous-order derivative, we can consider the Riemann-Liouville fractional derivative ${}^{RL}\mathcal{D}^{\alpha}_{a+}$, the Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha}_{a+}$, the Marchaud fractional derivative ${}^{M}\mathcal{D}^{\alpha}_{a+}$. In this cases, the integer-order derivatives are the standard integer derivatives ${}^{IO}\mathcal{D}^{n}_{x} = d^n/dx^n$. We assume that there are other examples of "continuous-order" and integer-order derivatives, which are connected by interpolation formula (50).

7. Conclusion

We propose an interpretation of the Riemann-Liouville and Caputo derivatives of non-integer orders. Equations (31), (50) and (51) can be considered as analogs of the interpolation formula (1), which constructs continuous-order derivatives from a sequence of integer-order derivatives. Equations (31), (45), (46), (50) allow us to interpret the fractional-order derivatives as a reconstruction from infinite sequences of standard derivatives of integer orders. These formulas reconstruct the fractional derivatives $\alpha > 0$ with respect to order $\alpha \in \mathbf{R}_+$ from the integer derivatives of orders $n \in \mathbf{Z}_+$. Using the suggested "informatic" ("computer science") interpretation, we can say that the fractional derivatives (continuous-order derivatives) can be restored from integer-order (discrete-order) derivatives by formulas (31), (45), (46), (50). It should be noted that infinity of sequences of integer derivatives plays a fundamental role in representation of fractional derivatives that describe nonlocality and memory in physics and mechanics. We assume that the interpolation formulas, which reconstruct the fractional derivatives of order $\alpha > 0$ from the integer-order derivatives, can be generalized for the lattice fractional calculus [27, 28] and the exact fractional differences [29, 30, 31]. The main advantage of the suggested interpretation is that it allows us to obtain geometric and physical interpretations by reconstruct of with respect to order from interpretation of integer order derivatives.

References

[1] Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives Theory and Applications Gordon and Breach, New York, 1993. ISBN-10:2881248640, 13:978-2881248641

[2] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam, 2006. ISBN:0444518320

[3] Tarasov VE. No violation of the Leibniz rule. No fractional derivative Communications in Nonlinear Science and Numerical Simulation. 2013:18(11):2945–2948. doi:10.1016/j.cnsns.2013.04.001

[4] Ortigueira MD, Tenreiro Machado JA. What is a fractional derivative? Journal of Computational Physics. 2015;293:4–13. doi:10.1016/j.jcp.2014.07.019

[5] Tarasov VE. On chain rule for fractional derivatives Communications in Nonlinear Science and Numerical Simulation. 2016;30(1-3):1–4. doi:10.1016/j.cnsns.2015.06.007

[6] Tarasov VE. Leibniz rule and fractional derivatives of power functions Journal of Computational and Nonlinear Dynamics. 2016:11(3):031014. doi:10.1115/1.4031364

[7] Kotel'nikov VA. On the transmission capacity of 'ether' and wire in electric communications Physics-Uspekhi. 2006;46(7):736-744. doi:10.1070/PU2006v049n07ABEH006160

[8] Shannon CE. Communication in the presence of noise Proceedings of the IEEE. 1998;86(2):447–457. doi:10.1109/JPROC.1998.659497

[9] Stenger F. Numerical Methods Based on Sinc and Analytic Functions Springer-Verlag, New York, 1993. doi:10.1007/978-1-4612-2706-9

[10] Stenger F. Handbook of Sinc Numerical Methods CRC Press, Taylor and Francis, Boca Raton, London, New York, 2011. ISBN:1439821585, 9781439821589

[11] Whittaker ET. On the functions which are represented by the expansions of the interpolation-theory Proceedings of the Royal Society of Edinburgh. 1915;35:181–194. doi:10.1017/S0370164600017806

[12] McNamee J, Stenger F, Whitney EL. Whittaker's cardinal function in retrospect Mathematics of Computation. 1971;25:141–154. doi:10.2307/2005140

[13] Stanislavsky AA. Probabilistic interpretation of the integral of fractional-order Theoretical and Mathematical Physics. 2004;138(3):418-431. doi:10.1023/B:TAMP.0000018457.70786.36

[14] Tenreiro Machado JA. A probabilistic interpretation of the fractional-order differentiation Fractional Calculus and Applied Analysis. 2009;6(1):73–80.

[15] Tenreiro Machado JA. Fractional derivatives: Probability interpretation and frequency response of rational approximations Communications in Nonlinear Science and Numerical Simulation. 2009;14(9-10):3492–3497. doi:10.1016/j.cnsns.2009.02.004

[16] Ben Adda F. Geometric interpretation of the fractional derivative Journal of Fractional Calculus. 1997; 11:21–51

[17] Ben Adda F. Geometric interpretation of the differentiability and gradient of real order Comptes Rendus de l'Academie des Sciences - Series I - Mathematics. 1997;326(8):931–934. doi:10.1016/S0764-4442(98)80116-X

[18] Podlubny I, Despotovic V, Skovranek T, McNaughton BH. Shadows on the walls: Geometric interpretation of fractional integration The Journal of Online Mathematics and Its Applications. 2007;7:1664

[19] Podlubny I. Geometrical and physical interpretation of fractional integration and fractional differentiation Fractional Calculus and Applied Analysis. 2002;5(4):367–386

[20] Moshrefi-Torbati M, Hammond JK. Physical and geometrical interpretation of fractional operators Journal of the Franklin Institute. 1998;335(6):1077–1086. doi:10.1016/S0016-0032(97)00048-3

[21] Cioc R. Physical and geometrical interpretation of Grunwald-Letnikov differintegrals: Measurement of path and acceleration Fractional Calculus and Applied Analysis. 2016;19(1):161–172. doi:10.1515/fca-2016-0009

[22] Nigmatullin RR. A fractional integral and its physical interpretation Theoretical and Mathematical Physics. 1992;90(3):242–251. doi:10.1007/BF01036529

[23] Rutman RS. On physical interpretations of fractional integration and differentiation Theoretical and Mathematical Physics. 1995;105(3):1509–1519. doi:10.1007/BF02070871

[24] Heymans N, Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives Rheologica Acta. 2006;45(5):765–772. doi:10.1007/s00397-005-0043-5

[25] Gomez-Aguilara JF, Razo-Hernandez R, Granados-Lieberman D. A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response Revista Mexicana de Fisica. 2014;60:32–38

[26] Bandyopadhyay B, Kamal S. Essence of fractional order calculus, physical interpretation and applications in Stabilization and Control of Fractional Order Systems: A Sliding Mode Approach Springer International Publishing, Heidelberg, 2015: 1–54. doi:10.1007/978-3-319-08621-7

[27] Tarasov VE. Toward lattice fractional vector calculus Journal of Physics A. 2014;47(35):355204. doi:10.1088/1751-8113/47/35/355204

[28] Tarasov VE. Lattice fractional calculus Applied Mathematics and Computation. 2015;257:12–33. doi:10.1016/j.amc.2014.11.033

[29] Tarasov VE. Exact discrete analogs of derivatives of integer orders: Differences as infinite series Journal of Mathematics. 2015;2015:134842. doi:10.1155/2015/134842

[30] Tarasov VE. Exact discretization by Fourier transforms Communications in Nonlinear Science and Numerical Simulation. 2016;37:31-61. doi:10.1016/j.cnsns.2016.01.006

[31] Tarasov VE. United lattice fractional integro-differentiation Fractional Calculus and Applied Analysis. 2016;19(3):625-664. doi:10.1515/fca-2016-0034