International Journal of Applied Mathematics

Volume 27 No. 5 2014, 491-518

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v27i5.7

FRACTIONAL-ORDER VARIATIONAL DERIVATIVE

Vasily E. Tarasov

Skobeltsyn Institute of Nuclear Physics
Lomonosov Moscow State University
Moscow, 119991, RUSSIA

Abstract: We consider some possible approaches to the fractional-order generalization of definition of variation (functional) derivative. Some problems of formulation of a fractional-order variational derivative are discussed. To give a consistent definition of the fractional-order variations, we use a fractional generalization of the Gateaux differential.

AMS Subject Classification: 26A33, 49S05
Key Words: fractional derivative, variational derivative, Gateaux differential

1. Introduction

The derivatives of non-integer order are well-known in mathematics, see for example, [1]-[5]. The fractional calculus has a long history since 1695, when the derivative of order $\alpha = 0.5$ has been discussed by Leibniz [6, 7, 8, 9, 10]. Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. The interest to fractional equations has been growing continually during the last few years because of numerous applications, [11]-[17].

In mathematics and theoretical physics, the variational (functional) derivative is a generalization of the usual derivative that arises in the calculus of variations. In a variational (functional) derivative, instead of differentiating a function with respect to a variable, one differentiates a functional with respect to a function. A fractional generalization of variational (functional) derivative is suggested in this paper.

Received: September 14, 2014

In this paper some problems of formulation of a fractional-order variational derivative are discussed. To give a definition of fractional variation, we suggest to use a fractional generalization of the Gateaux differential.

In Section 2, some properties of the Riemann-Liouville or Caputo fractional derivatives are noted. In Section 3, we give definitions of variational (functional) derivatives of integer orders. In Section 4, we discuss problems of different possible ways to define a fractional generalization of variational (functional) derivatives. In Section 5, we suggest a definition of fractional-order variational (functional) derivatives by using the proposed fractional generalization of Gateaux differential. In Section 6, a fractional variation of fields that is defined by fractional exterior derivatives are considered. A conclusion is given in Section 7.

2. Fractional Derivative

The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grünwald, and Letnikov, see e.g. [1]-[5]. The authors of many papers use the fractional derivative D^α_x in the Riemann-Liouville or Caputo forms. Let us give definitions of these derivatives and some properties.

Definition 1. ([1]) The Riemann-Liouville fractional derivative of the function $f(x)$ belonging to the space $AC^n[a,b]$ of absolutely continuous functions is defined on $[a,b]$ by the equation

$$D^\alpha_x f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(y)\,dy}{(x-y)^{\alpha-m+1}}, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, m is the first integer number greater than or equal to α.

In equation (1), the initial point of the fractional derivative can be set to zero. Then the derivative of powers k of x is

$$D^\alpha_x (x)^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha} \quad (x > a), \tag{2}$$

where $k \geq 1$, and $\alpha \geq 0$. Note that the derivative (1) of a constant C needs not be zero:

$$D^\alpha_{x_i} C = \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)} C.$$

Therefore we see that constants C in the equation $V(x) = C$ cannot define a stationary state for the equation $D^\alpha_x V(x) = 0$. In order to define stationary values, we should consider solutions of the equations $D^\alpha_{x_i} V(x) = 0$.

The Riemann-Liouville fractional derivative has some notable disadvantages in the physical applications such as the hyper-singular improper integral, where the order of singularity is higher than the dimension, and nonzero of the fractional derivative of constants, which would entail that dissipation does not vanish for a system in equilibrium. The desire to formulate initial value problems for physical systems leads to the use of the so-called Caputo fractional derivatives, see e.g. [22, 23, 24] (see also [3, 4]) rather than Riemann-Liouville fractional derivative.

Definition 2. The Caputo fractional derivative of the function $f(x)$ belonging to the space $AC^n[a, b]$ of absolutely continuous functions is defined on $[a, b]$ by the equation

$$D^\alpha_x f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{dy}{(x-y)^{\alpha-m+1}} \frac{d^m f(y)}{dy^m} \quad (x > a), \tag{3}$$

where $f^{(m)}(y) = d^m f(y)/dy^m$, and m is the first integer number greater than or equal to α.

This definition is of course more restrictive than (1), in that requires the absolute integrability of the derivative of order m. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desire order of fractional derivative. The Riemann-Liouville fractional derivative is computed in the reverse order. Integration by part of (3) leads us to the relation

$$D^\alpha_{*\,x} f(x) = D^\alpha_x f(x) - \sum_{k=0}^{m-1} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0+). \tag{4}$$

It is observed that the second term in Eq. (4) regularizes the Caputo fractional derivative to avoid the potentially divergence from singular integration at $x = 0+$. In addition, the Caputo fractional differentiation of a constant results in zero. If the Caputo fractional derivative is used instead of the Riemann-Liouville fractional derivative then the stationary values are the same as those for the usual case $(V(x)-C = 0)$. The Caputo formulation of fractional calculus can be more applicable to definition of fractional variation than the Riemann-Liouville formulation.

3. Variational Derivatives of Integer Order

In mathematics and theoretical physics, the variational (functional) derivative is a generalization of the usual derivative that arises in the calculus of variations. In a variational (functional) derivative, instead of differentiating a function with respect to a variable, one differentiates a functional with respect to a function.

3.1. Definition by Increment and Taylor Series

The variational derivative can be defined in the following way. Let us consider the increment of the functional $F[u]$ that is defined by the equation

$$\Delta F[u] = F[u+h] - F[u], \tag{5}$$

and consider an integer-order variational derivative.

Definition 3. If this increment of the functional $F[u]$ exists, and can be represented in the form

$$\Delta F[u] = \delta F(u,h) + \omega(h,u), \tag{6}$$

where

$$\lim_{||h||\to 0} \frac{||\omega(h,u)||}{||h||} = 0,$$

then δF is called the first variation or Frechet derivative [29, 30, 26] of functional F. The function $h = h(x)$ is called the variation, and it is denoted by δu.

Example 1. Let us define the functional

$$F[u] = \int_{x_1}^{x_2} f(u)dx \tag{7}$$

in some Banach space E. The increment of the functional $F[u]$ is defined by the equation

$$\Delta F[u] = F[u+h] - F[u] = \int_{x_1}^{x_2} f(u+h)dx - \int_{x_1}^{x_2} f(u)dx. \tag{8}$$

Here we suppose that the variation $h(x) = \delta u(x)$ is equal to zero in boundary points x_1 and x_2. Let us expand the integrand $f(x, u+h)$ in the power series up to first order

$$f(u+h) = f(u) + \frac{\partial f(u)}{\partial u} h + R(h,u),$$

where

$$\lim_{h\to 0} h^{-1} R(h, u) = 0. \tag{9}$$

Then we have

$$\delta F[u] = \int_{x_1}^{x_2} \frac{\partial f(u)}{\partial u} h\, dx. \tag{10}$$

The variational derivative for functional (7) has the form

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f(u)}{\partial u}. \tag{11}$$

Example 2. Let us consider the functional

$$F[u] = \int_{x_1}^{x_2} f(u, u_x) dx. \tag{12}$$

We can derive the first variation of the functional by the equation

$$\delta F[u] = F[u + \delta u] - F[u] = \int_{x_1}^{x_2} f(u + h, [u + h]_x) dx - \int_{x_1}^{x_2} f(u, u_x) dx. \tag{13}$$

Here we suppose that the variation $h(x) = \delta u(x)$ is equal to zero in boundary points x_1 and x_2. Let us expand the integrand $f(u + \delta u, (u + \delta u)_x)$ in power series up to first order

$$f(u + h, (u + h)_x) = f(u, u_x) + \frac{\partial f}{\partial u} h + \frac{\partial f}{\partial u_x} h_x,$$

where $h_x = dh/dx$. Then we get the variation of the functional

$$\delta F[u] = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial u} h + \frac{\partial f}{\partial u_x} h_x \right] dx. \tag{14}$$

Integrating the second term by part, and supposing

$$h(x_1) = h(x_2) = 0,$$

we get the result

$$\delta F[u] = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial u} - \frac{d}{dx} \left(\frac{\partial f}{\partial u_x} \right) \right] h(x)\, dx. \tag{15}$$

Using this relation, we get the variational derivative of the functional (12) in the form

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx} \left(\frac{\partial f}{\partial u_x} \right). \tag{16}$$

Example 3. If we consider the functional

$$F[u] = \int_{x_1}^{x_2} f(x, u, u_x, ...u^{(n)})dx, \tag{17}$$

then the variation of the functional is defined by the equation

$$\delta F[u] = \int_{x_1}^{x_2} \sum_{m=0}^{n} \frac{\partial f}{\partial u^{(m)}} (\delta u)^{(m)} dx, \tag{18}$$

and the variational derivative has the form

$$\frac{\delta F[u]}{\delta u} = \sum_{m=0}^{n} (-1)^m \frac{d^m}{dx^m} \left(\frac{\partial f}{\partial u^{(m)}} \right), \tag{19}$$

where $u^{(m)} = d^m u(x)/dx^m$.

3.2. Definition by Composite Function of Parameter

There is another approach to definition of variation and variational derivative. Let us consider the functional (12), where u is a function of coordinates x and parameter a, i.e., $u = u(x, a)$, and

$$F[u] = \int_{x_1}^{x_2} f(u(x, a), u_x(x, a))dx. \tag{20}$$

The derivative of F with respect a can be written as

$$\frac{dF}{da} = \int_{x_1}^{x_2} dx \frac{df}{da} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial u} \frac{\partial u}{\partial a} + \frac{\partial f}{\partial u_x} \frac{\partial u_x}{\partial a} \right] dx. \tag{21}$$

Using

$$\frac{\partial u_x}{\partial a} = \frac{\partial}{\partial a} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\partial u}{\partial a},$$

the conditions $\delta u(x_1, a) = \delta u(x_2, a) = 0$, and integrating by part, we get

$$\frac{dF}{da} = \int_{x_1}^{x_2} \frac{\partial u}{\partial a} \left[\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u_x} \right) \right] dx. \tag{22}$$

As a result, we have

$$\frac{dF}{da} = \int_{x_1}^{x_2} \frac{\partial u(x)}{\partial a} \frac{\delta F[u]}{\delta u(x)} dx. \tag{23}$$

A fractional-order generalization of this approach is difficult to realize. The reason of this difficulty is related to the difficulty to generalize the rule of differentiating a composite functions

$$\frac{df(u(a))}{da} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial a} \tag{24}$$

for the fractional order case. It is known that coordinate transformations are connected with the derivative of a composite function $D_t^1 f(u(x)) = (D_u^1 f)(u = u(x))\,(D_x^1 u)(x)$. The formula of fractional derivative of a composite function (see Equation 2.209, Section 2.7.3, page 98, [3]) is the following:

$$D_x^\alpha f(u(x)) = \frac{(x-a)^\alpha}{\Gamma(1-\alpha)} f(u(x)) + \sum_{k=1}^{\infty} C_k^\alpha \frac{k!(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} (D^m f)(u(x)) \sum \prod_{r=1}^{k} \frac{1}{a_r!}\Big(\frac{(D_x^r u)(x))}{r!}\Big)^{a_r}, \tag{25}$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \ldots, a_k$ such that

$$\sum_{r=1}^{k} r a_r = k, \quad \sum_{r} a_r = m. \tag{26}$$

4. Problems to Formulate Fractional-Order Variational Derivative

Let us consider some problems in the formulation of the fractional-order generalization of variational derivatives.

Problem 1. To define the fractional variation, we can use the fractional Taylor series (see Section 2.6 in [1]). Expanding the integrand $f(u+h)$ in fractional Taylor power series

$$f(u+h) = \frac{1}{\Gamma(\alpha+1)}(h)^\alpha D_u^\alpha f(u) + \frac{1}{\Gamma(\alpha+2)}(h)^{\alpha+1} D_u^\alpha f(u) + \ldots, \tag{27}$$

we can see that this series cannot have the term $f(u)$. Therefore, we cannot consider the increment

$$\Delta f(u) = f(u+h) - f(u). \tag{28}$$

In order to use the Taylor series for the definition, we can consider the increment in the form

$$\Delta_q f = f(u + h) - f(u + qh), \tag{29}$$

where $0 < q < 1$. In this case, we have

$$\Delta f(u) = \frac{1 - q^{\alpha}}{\Gamma(\alpha + 1)} h^{\alpha} D^{\alpha}_u f(u) + \frac{1 - q^{\alpha+1}}{\Gamma(\alpha + 2)} h^{\alpha+1} D^{\alpha}_u f(u) + \ldots . \tag{30}$$

This approach can be connected with the q-analysis [31], and fractional q-derivatives [32, 33]. We will consider the fractional variational q-derivative in the next paper.

We can consider the increment of the functional $F[u]$ that is defined by the equation

$$\Delta_q F[u] = F[u + h] - F[u + qh]. \tag{31}$$

If this increment of the functional $F[u]$ exists, and can be represented in the form

$$\Delta_q F[u] = \delta^{\alpha} F(u, h^{\alpha}) + \omega(h^{\alpha}, u), \tag{32}$$

where

$$\lim_{||h|| \to 0} \frac{||\omega(h^{\alpha}, u)||}{||h^{\alpha}||} = 0,$$

then $\delta^{\alpha} F$ can be considered as a fractional variation of functional F.

Problem 2. To define the fractional generalization of variation and fractional exterior variational calculus [21], we can use an analogy with the definition of fractional exterior derivative. If the partial derivatives in the definition of the exterior derivative

$$d = dx_i \partial / \partial x_i$$

are allowed to assume fractional order, a fractional exterior derivative can be defined [20] by the equation

$$d^{\alpha} = (dx_i)^{\alpha} D^{\alpha}_{x_i}, \tag{33}$$

where D^{α}_x is the fractional derivative with respect to x. Using this analogy, we can define the fractional variation in the following way. For the point u of functional space, we can define the fractional variation $\delta F[u]$ of the functional

$$F[u] = \int_{x_1}^{x_2} f(u, u_x) dx,$$

where $u_x = du/dx$, by the equation

$$\delta^\alpha F[u] = \int_{x_2}^{x_2} dx \left[(\delta u)^\alpha D^\alpha_u f(u, u_x) + (\delta u_x)^\alpha D^\alpha_{u_x} f(u, u_x) \right]. \tag{34}$$

Unfortunately, this approach leads to difficulties with the realization of the integration by part in the second term. It is easy to see that the variation $(\delta u_x)^\alpha$ cannot be represented as some operator $\hat{A}(d/dx)$ acts on the variation $(\delta u)^\alpha$, i.e., we have

$$(\delta u_x(x))^\alpha = \left(\frac{d}{dx} \delta u(x) \right)^\alpha \neq \hat{A} \left(\frac{d}{dx} \right) (\delta u)^\alpha. \tag{35}$$

In the particular case, $(\delta u_x(x))^\alpha = (d(\delta u)/dx)^\alpha \neq d(\delta u)^\alpha/dx$ if $\alpha \neq 1$.

Problem 3. To define the fractional variational derivative, we can use fractional derivative with respect to function (see Section 18.2 in [1]). The fractional Riemann-Liouville defivative of the function $f(x)$ with respect the function $u(x)$ of order α $(0 < \alpha < 1)$ is:

$$D^\alpha_{0+,u(x)} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{du(x)}{dx} \right)^{-1} \frac{d}{dx} \int_0^x dy \frac{f(y)}{[u(x) - u(y)]^\alpha} \frac{du(y)}{dy}. \tag{36}$$

Using the function $u = u(x, a)$, we can define fractional derivative with respect to function of parameter

$$D^\alpha_{0+,u(x,a)} f(x, a) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{du(x,a)}{da} \right)^{-1} \frac{d}{da} \int_0^a db \frac{f(x,b)}{[u(x,a) - u(x,b)]^\alpha} \frac{du(x,b)}{db}. \tag{37}$$

As a result, the fractional variation can be defined by the equation

$$\delta^\alpha F[u(x)] = \int_{x_1}^{x_2} dx D^\alpha_{u(x,a)} f(u(x,a)) (\delta u(x,q))^\alpha, \quad \frac{\delta^\alpha F[u]}{\delta u^\alpha} = D^\alpha_{u(x,a)} f(u(x)). \tag{38}$$

This approach leads to difficulties with the realization of integration by part in the second term in (35). In this paper, this approach to definition is not considered.

To avoid all these difficulties and problems, we suggest to define a fractional variational derivative by using a fractional-order generalization of the Gateaux differential.

5. Fractional Generalization of Gateaux Differential

In this section, we consider 3 steps to define the fractional generalization of variation by using some generalization of the Gateaux differential. We consider these steps in order to explain the final definition of fractional Gateaux variation.

5.1. Variations of Integer Order

Suppose the functional $F[u]$ is continuous (smooth) map (with certain boundary conditions) from everywhere dense subset $D(F)$ of Banach space to space R. Let us define the Gateaux differential [28, 29, 30] of a functional $F[u]$ at the point $u(x)$ of subset $D(F)$ of the functional Banach space.

Definition 4. The Gateaux differential (or first variation) of a fucntional $F[u]$ is defined by the equation

$$\delta F[u,h] = \left(\frac{d}{d\epsilon} F[u+\epsilon h] \right)_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{F[u+\epsilon\ h] - F[u]}{\epsilon} \tag{39}$$

if the limit exists for all $h(x) \in D(F)$. The function $h(x)$ is called a variation of function $u(x)$ and denoted by $\delta u(x) = h(x)$.

A first variation of the functional $F[u]$ at the point $u = u(x)$ is defined as a first derivative of functional $F[u+\epsilon h]$ with respect to parameter ϵ for $\epsilon = 0$.

If the Gateaux differential of the functional

$$F[u] = \int dx f(x, u, u_x, ...) \tag{40}$$

is linear with respect to $h(x)$, then we can write

$$\delta F[u,h] = \int dx\, E(x, u, u_x, ...) h(x), \tag{41}$$

and $E(x, u, u_x, ...)$ is called the variational derivative and is denoted by $\delta F/\delta u$.

Example. For the example, we can consider the functional

$$F[u] = \int_{x_1}^{x_2} [u(x)]^2 dx. \tag{42}$$

The functional $F[u+\epsilon h]$ has the form

$$F[u+\epsilon h] = \int_{x_1}^{x_2} [u(x)]^2 dx + 2\epsilon \int_{x_1}^{x_2} u(x)h(x)dx + \epsilon^2 \int_{x_1}^{x_2} [h(x)]^2 dx. \quad (43)$$

The derivative of first order with respect to parameter ϵ is

$$\frac{d}{d\epsilon} F[u+\epsilon h] = 2\int_{x_1}^{x_2} u(x)h(x)dx + 2\epsilon \int_{x_1}^{x_2} [h(x)]^2 dx. \quad (44)$$

As a result, the variational derivative is equal

$$\left(\frac{d}{d\epsilon} F[u+\epsilon h]\right)_{\epsilon=0} = 2\int_{x_1}^{x_2} u(x)h(x)dx. \quad (45)$$

5.2. Step 1.

It seems that we can define a fractional variation by the equation

$$\delta^\alpha F[u,h] = \left(\left[\frac{d}{d\epsilon}\right]^\alpha F[u+\epsilon h]\right)_{\epsilon=0}. \quad (46)$$

Let us consider the fractional Riemann-Liouville derivative of functional (43) with respect to parameter ϵ:

$$D^\alpha_\epsilon F[u+\epsilon h] = (D^\alpha_\epsilon 1) \int_{x_1}^{x_2} [u(x)]^2 dx + 2(D^\alpha_\epsilon \epsilon) \int_{x_1}^{x_2} u(x)h(x)dx$$
$$+ (D^\alpha_\epsilon \epsilon^2) \int_{x_1}^{x_2} [h(x)]^2 dx. \quad (47)$$

Using (2),

$$D^\alpha_\epsilon \epsilon^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} \epsilon^{k-\alpha}, \quad (48)$$

we have

$$D^\alpha_\epsilon F[u+\epsilon h] = \frac{\epsilon^{-\alpha}}{\Gamma(1-\alpha)} \int_{x_1}^{x_2} [u(x)]^2 dx + \frac{2\epsilon^{1-\alpha}}{\Gamma(2-\alpha)} \int_{x_1}^{x_2} u(x)h(x)dx +$$
$$+ \frac{2}{\Gamma(3-\alpha)} \epsilon^{k-\alpha} \int_{x_1}^{x_2} [h(x)]^2 dx. \quad (49)$$

If the derivatives with respect to parameter ϵ in the definition of variational (39) are allowed to assume fractional order, a fractional variational can be defined. Unfortunately, if we define the fractional variation of the functional by the equation

$$\delta^\alpha F[u] = (D^\alpha_\epsilon F[u+\epsilon h])_{\epsilon=0}\,, \tag{50}$$

then we have some problems about incorrectness of definition (50). These problems are following:

1) The first term of right hand side of equation (49) leads us to infinity. If ϵ tends to zero, then we get $\epsilon^{-\alpha} \to \infty$, and $D^\alpha_\epsilon F[u+\epsilon h] \to \infty$. Therefore the first term that is follows from the relation $D^\alpha_\epsilon C = 0$ must be removed. For this aim we can use the Caputo fractional derivative [3, 22, 23, 24]. Using (4) for $0 < \alpha < 1$, we have

$$D^\alpha_{*\ \epsilon} f(\epsilon) = D^\alpha_\epsilon f(\epsilon) - \frac{\epsilon^{-\alpha}}{\Gamma(2-\alpha)} f(0+). \tag{51}$$

2) The second term of the right hand side of equation (49) is proportional to $\epsilon^{1-\alpha}$. This proportionality leads us to zero in the limit $\epsilon \to 0$, and we cannot derive some nonzero relation. Therefore we must consider the functional $F[u+(\epsilon h)^\alpha]$ in the definition.

As the result, we cannot use the definitions (46) and (50).

5.3. Step 2.

Let us consider the following definition of fractional variational functional $F[u]$:

$$\delta^\alpha F[u] = (D^\alpha_\epsilon F[u+(\epsilon h)^\alpha])_{\epsilon=0}\,, \tag{52}$$

where $D^\alpha_\epsilon = D^\alpha_*$ is a Caputo fractional derivative of order α with respect to ϵ.

If we consider the functional (42), then

$$F[u+(\epsilon h)^\alpha] = \int_{x_1}^{x_2} [u(x)]^2 dx + 2\epsilon^\alpha \int_{x_1}^{x_2} u(x) h^\alpha(x) dx + \epsilon^{2\alpha} \int_{x_1}^{x_2} [h(x)]^{2\alpha} dx. \tag{53}$$

The fractional derivative (52) of this functional is equal to

$$D^\alpha_\epsilon F[u+(\epsilon h)^\alpha] = 2\Gamma(\alpha+1) \int_{x_1}^{x_2} u(x) h^\alpha(x) dx + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \epsilon^\alpha \int_{x_1}^{x_2} [h(x)]^{2\alpha} dx. \tag{54}$$

Therefore, we get the following equation

$$(D_\epsilon^\alpha F[u+(\epsilon h)^\alpha])_{\epsilon=0} = 2\Gamma(\alpha+1)\int_{x_1}^{x_2} u(x)h^\alpha(x)dx. \tag{55}$$

Note that for $\alpha = 1$, we get the usual relation

$$\delta^{\alpha=1}F[u] = 2\int_{x_1}^{x_2} u(x)h(x)dx. \tag{56}$$

Unfortunately, if we put $\alpha = 0$, we get

$$F[u] = \delta^{\alpha=0}F[u] = 2\int_{x_1}^{x_2} u(x)(h(x))^0 dx = 2\int_{x_1}^{x_2} u(x)dx \neq F[u]. \tag{57}$$

Therefore we cannot use the definition (52).

5.4. Step 3.

The usual definition (39) can be rewritten in the form

$$\delta F[u] = \left(\frac{d}{d\epsilon}F\left[u\left(1+\frac{\epsilon h}{u}\right)\right]\right)_{\epsilon=0}. \tag{58}$$

The fractional variation can be defined by the equation

$$\delta^\alpha F[u] = \left(D_\epsilon^\alpha F\left[u\left(1+\left[\frac{\epsilon h}{u}\right]^\alpha\right)\right]\right)_{\epsilon=0}. \tag{59}$$

This definition is more consistent in order to realize the physical dimensions. In this definition the parameter ϵ is dimensionless.

If we consider the functional (42), then

$$F\left[u\left(1+\left[\frac{\epsilon h}{u}\right]^\alpha\right)\right] = F[u+(\epsilon h)^\alpha u^{1-\alpha}] =$$

$$\int_{x_1}^{x_2}[u(x)]^2dx + 2\epsilon^\alpha\int_{x_1}^{x_2} u^{2-\alpha}(x)h^\alpha(x)dx + \epsilon^{2\alpha}\int_{x_1}^{x_2} u^{2-2\alpha}(x)[h(x)]^{2\alpha}dx. \tag{60}$$

The fractional derivative of this functional is equal to

$$D_\epsilon^\alpha F[u+(\epsilon h)^\alpha u^{1-\alpha}] = 2\Gamma(\alpha+1)\int_{x_1}^{x_2} u^{2-\alpha}(x)h^\alpha(x)dx$$
$$+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\epsilon^\alpha\int_{x_1}^{x_2} u^{2-2\alpha}(x)[h(x)]^{2\alpha}dx. \tag{61}$$

As a result, we get

$$\left(D^\alpha_\epsilon F[u+(\epsilon h)^\alpha u^{1-\alpha}]\right)_{\epsilon=0} = 2\Gamma(\alpha+1)\int_{x_1}^{x_2} u^{2-\alpha}(x)h^\alpha(x)dx. \tag{62}$$

For $\alpha=1$, we get the usual relation. For $\alpha=0$, we have

$$F[u] = \delta^{\alpha=0}F[u] = 2\int_{x_1}^{x_2} u^2(x)dx \neq F[u]. \tag{63}$$

If we consider the functional

$$F[u] = \int_{x_1}^{x_2} u^n(x)dx, \tag{64}$$

then definition (59) leads to the relation

$$\left(D^\alpha_\epsilon F[u+(\epsilon h)^\alpha u^{1-\alpha}]\right)_{\epsilon=0} = n\Gamma(\alpha+1)\int_{x_1}^{x_2} u^{n-\alpha}(x)h^\alpha(x)dx \tag{65}$$

and, for $\alpha=0$

$$F[u] = \delta^{\alpha=0}F[u] = n\int_{x_1}^{x_2} u^n(x)dx \neq F[u]. \tag{66}$$

Therefore we have to implement some changes in the definition (59). This modification is suggested in the next subsection.

5.5. Definition of Variation of Fractional Order

Taking into account the remarks in the form of Step 1-3 and using Eq. (58) for variation of integer order, we can define the fractional variation in the following form.

Definition 5. The fractional-order variation $\delta^\alpha F[u]$ of the functional $F[u]$ is defined by the equation

$$\delta^\alpha F[u] = \left(D^\alpha_\epsilon F\left[u\left(1+\frac{\epsilon h}{u}\right)^\alpha\right]\right)_{\epsilon=0}, \tag{67}$$

or, in an equivalent form

$$\delta^\alpha F[u] = \left(D^\alpha_\epsilon F\left[u^{1-\alpha}(u+\epsilon h)^\alpha\right]\right)_{\epsilon=0}, \tag{68}$$

where $\epsilon \geq 0$.

Example. If we consider the functional (42), we have

$$F\left[u^{1-\alpha}(u+\epsilon h)^{\alpha}\right] = \int_{x_1}^{x_2} u^{2-2\alpha} h^{2\alpha} (\epsilon + u/h)^{2\alpha} dx. \tag{69}$$

Using the relation

$$D_{\epsilon}^{\alpha} (\epsilon + u/h)^{2\alpha} = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} (\epsilon + u/h)^{\alpha}, \tag{70}$$

where we consider $\epsilon + u(x)/h(x) = \epsilon - \epsilon_0$ with $\epsilon_0 = \epsilon_0(x) = -u(x)/h(x)$, we get the fractional derivative of functional (69) in the form

$$D_{\epsilon}^{\alpha} F\left[u^{1-\alpha}(u+\epsilon h)^{\alpha}\right] = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \int_{x_1}^{x_2} u^{2-2\alpha} h^{2\alpha} (\epsilon + u/h)^{\alpha} dx. \tag{71}$$

As the result, we get

$$\left(D_{\epsilon}^{\alpha} F\left[u^{1-\alpha}(u+\epsilon h)^{\alpha}\right]\right)_{\epsilon=0} = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \int_{x_1}^{x_2} dx u^{2-\alpha}(x) h^{\alpha}(x). \tag{72}$$

For $\alpha = 1$, and $\alpha = 0$, we get the usual relations. Therefore definition (67) satisfies the correspondent requirements for $\alpha = 0$ and $\alpha = 1$. As the result, we get the fractional variational derivative of order $0 \leq \alpha \leq 1$ in the form (68).

Proposition 1. *The fractional-order variation derivative (67) of the functional*

$$F[u] = \int_{x_1}^{x_2} u^n(u) dx \tag{73}$$

has the form

$$\delta^{\alpha} F[u] = \lambda(\alpha, n) \int_{x_1}^{x_2} dx \, (D_u^{\alpha} u^n)(x) h^{\alpha}(x), \tag{74}$$

where

$$\lambda(\alpha, n) = \frac{\Gamma(n\alpha+1)\Gamma(n+1-\alpha)}{\Gamma((n-1)\alpha+1)\Gamma(n+1)}. \tag{75}$$

Proof. Using (73), we have

$$F\left[u^{1-\alpha}(u+\epsilon h)^{\alpha}\right] = \int_{x_1}^{x_2} u^{n-n\alpha} h^{n\alpha} (\epsilon + u/h)^{n\alpha} dx. \tag{76}$$

From the relation

$$D^{\alpha}_{\epsilon}(\epsilon + u/h)^{n\alpha} = \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}(\epsilon + u/h)^{(n-1)\alpha}, \tag{77}$$

we get the fractional derivative of functional (73) in the form

$$\delta^{\alpha} F[u] = \left(D^{\alpha}_{\epsilon} F\left[u^{1-\alpha}(u+\epsilon h)^{\alpha}\right]\right)_{\epsilon=0} = \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \int_{x_1}^{x_2} dx u^{n-\alpha}(x) h^{\alpha}(x). \tag{78}$$

This equation can be represented in the form

$$\delta^{\alpha} F = \int_{x_1}^{x_2} dx (D^{\alpha}_{u} u^{n}) h^{\alpha}(x), \tag{79}$$

up to numerical factor. For equation (79), we have

$$(D^{\alpha}_{u} u^{n}) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} u^{n-\alpha}. \tag{80}$$

This ends the proof. □

The fractional variational derivative of order $\alpha \geq 1$ can be defined by the usual relation

$$\delta^{\alpha} = \delta^{[\alpha]} \delta^{\{\alpha\}} \tag{81}$$

by analogy with fractional derivative

$$D^{\alpha}_{x} = \frac{d^{[\alpha]}}{dx^{[\alpha]}} D^{\{\alpha\}}_{x}, \tag{82}$$

where $[\alpha]$ is the integer part of α, and $\{\alpha\}$ is the fractional part of number α, i.e., $\{\alpha\} = \alpha - [\alpha]$.

5.6. Functional with Derivative of Field

Let us consider the functional $F[u]$ of the form

$$F[u] = \int_{x_1}^{x_2} f(u, u_x) dx, \tag{83}$$

where $u_x = \frac{du(x)}{dx}$. The first variation of this functional is defined by

$$\delta F[u] = \left(\frac{d}{d\epsilon} F[u+\epsilon h]\right)_{\epsilon=0} = \lim_{\epsilon\to 0} \left(\frac{d}{d\epsilon} \int_{x_1}^{x_2} f(u+\epsilon h, [u+\epsilon h]_x)\right) dx, \tag{84}$$

where we use

$$[u + \epsilon h]_x = u_x + \epsilon h_x. \tag{85}$$

In order to use the equation

$$D^\alpha_\epsilon (\epsilon - \epsilon_0)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (\epsilon - \epsilon_0)^{\beta-\alpha}, \tag{86}$$

for the relation $[\epsilon + u(x)/h(x)]$, we consider $\epsilon_0 = \epsilon_0(x) = -u(x)/h(x)$. The definition of the fractional variation of functional (83) can be realized in the following form.

Definition 6. The fractional-order variation derivative of the functional (83) is defined by the equation

$$\delta^\alpha F[u] = \lim_{\epsilon \to 0} \int_{x_1}^{x_2} \left(D^\alpha_\epsilon f \left(u^{1-\alpha}(u+\epsilon h)^\alpha, [u^{1-\alpha}(u+\epsilon h)^\alpha]_x \right) \right). \tag{87}$$

Using the definition (68), we have the fractional variation of functional (83) in the form

$$\delta^\alpha F[u] = \int_{x_1}^{x_2} \left(D^\alpha_\epsilon f \left(u^{1-\alpha}(u+\epsilon h)^\alpha, [u^{1-\alpha}(u+\epsilon h)^\alpha]_x \right) \right)_{\epsilon=0}. \tag{88}$$

It is easy to see that the expression $[u^{1-\alpha}(u+\epsilon h)^\alpha]_x$ cannot be represented in the form of similar Eq. (85). This expression has the form

$$\begin{aligned} [u^{1-\alpha}(u+\epsilon h)^\alpha]_x = [(1-\alpha)u^{-\alpha}(u+\epsilon h)^\alpha + \alpha u^{1-\alpha}(u+\epsilon h)^{\alpha-1}]u_x \\ + \alpha u^{1-\alpha}(u+\epsilon h)^{\alpha-1}\epsilon h_x. \end{aligned} \tag{89}$$

For $\alpha = 0$, we have

$$[u^{1-\alpha}(u+\epsilon h)^\alpha]_x = u_x, \tag{90}$$

and for $\alpha = 1$, we have

$$[u^{1-\alpha}(u+\epsilon h)^\alpha]_x = u_x + \epsilon h_x. \tag{91}$$

Let us give the proposition for a special form of the functional.

Proposition 2. *The fractional-order variation derivative (87) of the functional*

$$F[u] = \int_{x_1}^{x_2} u(x) u_x(x) dx. \tag{92}$$

has the form

$$\delta^{\alpha} F[u] = \int_{x_1}^{x_2} dx \, \left(A_1(\alpha)\, u_x(x) h(x) + A_2(\alpha) u^{2-\alpha}(x) \alpha h^{\alpha-1}(x) h_x(x) \right), \quad (93)$$

where

$$A_1(\alpha) = \left[\frac{(1-\alpha)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} + \frac{\alpha\Gamma(2\alpha)}{\Gamma(\alpha)} \right], \quad A_2(\alpha) = \frac{\alpha^2 \Gamma(2\alpha)}{\Gamma(\alpha+1)}. \quad (94)$$

Proof. Let us use the functional

$$F[u^{1-\alpha}(u+\epsilon h)^{\alpha}] = \int_{x_1}^{x_2} dx = u^{1-\alpha}(u+\epsilon h)^{\alpha} [u^{1-\alpha}(u+\epsilon h)^{\alpha}]_x$$
$$= \int_{x_1}^{x_2} dx \Big((1-\alpha) u^{1-2\alpha} h^{2\alpha} (\epsilon + u/h)^{2\alpha} u_x + \alpha u^{2-2\alpha} h^{2\alpha-1} (\epsilon + u/h)^{2\alpha-1} u_x$$
$$+ \alpha u^{2-2\alpha} h^{2\alpha-1} (\epsilon + u/h)^{2\alpha-1} \epsilon h_x \Big). \quad (95)$$

Consider $\epsilon_0 = \epsilon_0(x) = -u(x)/h(x)$. We can use the following relations

$$D^{\alpha}_{\epsilon} (\epsilon - \epsilon_0)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (\epsilon - \epsilon_0)^{\beta-\alpha}, \quad (96)$$

and

$$D^{\alpha}_{\epsilon} (\epsilon - \epsilon_0)^{\beta} (\epsilon - 0)^{\gamma} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}$$
$$F_{2;1}\left(-\gamma, \beta+1; \beta+1-\alpha; -\frac{\epsilon - \epsilon_0}{\epsilon_0} \right) (\epsilon_0)^{\gamma} (\epsilon - \epsilon_0)^{\beta-\alpha}. \quad (97)$$

Note that Eq. (96) is satisfied for $\beta > -1$, and Eq. (97) is satisfied for $\beta > -1$, $\epsilon > \epsilon_0 > 0$. Here $F_{2;1}(a, b, c, z)$ is the Gauss hypergeometric function [1]:

$$F_{2;1}(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b_k)}{(c)_k} \frac{z^k}{k!},$$

and

$$(z)_k = z(z+1)...(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

For the function $F_{2;1}(a, b, c, z)$ there exists the Euler integral representation

$$F_{2;1}(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad (98)$$

where $Re(c) > Re(b) > 0$, $|arg(1-z)| < \pi$, and the relation

$$F_{2;1}[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{99}$$

where $Re(c-a-b) > 0$. For the case $\alpha > 1$, we can use

$$\frac{d^k}{dz^k} F_{2;1}[a,b;c;z] = \frac{(a)_k (b_k)}{(c)_k} F_{2;1}[a+k, b+k; c+k; z], \tag{100}$$

where $k = [\alpha]$ is the integer part of α. As a result, we get

$$\delta^\alpha F[u] = \int_{x_1}^{x_2} dx \left[\frac{(1-\alpha)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} + \frac{\alpha\Gamma(2\alpha)}{\Gamma(\alpha)} \right] u_x(x) h(x) + \int_{x_1}^{x_2} dx \frac{\alpha\Gamma(2\alpha)}{\Gamma(\alpha+1)} u^{2-\alpha} \alpha h^{\alpha-1} h_x. \tag{101}$$

This ends the proof. □

For $\alpha \to 1$, we have the usual relation

$$\delta F[u] = \int_{x_1}^{x_2} dx [u_x h + u h_x].$$

Note that we can use

$$\alpha h^{\alpha-1} h_x = (h^\alpha)_x,$$

and then

$$\int_{x_1}^{x_2} dx \frac{\Gamma(2\alpha)\Gamma(1-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} u^{2-\alpha} \alpha h^{\alpha-1} h_x = - \int_{x_1}^{x_2} dx \frac{\Gamma(2\alpha)\Gamma(1-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} (u^{2-\alpha})_x h^\alpha. \tag{102}$$

This allows us to realize integration by part in the second term of Eq. (101).

6. Fractional Variation of Fields

In this section we consider the definition of fractional variational derivative without using the increment

$$\Delta F[u] = F[u+h] - F[u]$$

of the functional $F[u]$, and without using the derivative with respect to parameter ϵ as in the Gateaux derivative. We suppose that the functional $F[u]$ has some densities $f(u, u_x, ...)$.

If $\mathbf{u} = (u_1, ..., u_m)(x,t)$ is a smooth vector-function defined in the region $W \subset R^n$, then the variation of the functional

$$F[\mathbf{u}] = \int_W f(\mathbf{u}, \mathbf{u}_x) dx$$

can be defined by the relation

$$\delta F[u] = \int_W \delta f(\mathbf{u}, \mathbf{u}_x) dx = \int_W \left[\frac{\partial f}{\partial u^\mu} \delta u^\mu + \frac{\partial f}{\partial u^\mu_x} \delta u^\mu_x \right] dx. \tag{103}$$

To define the fractional generalization of variation and fractional exterior variational calculus [21], we can use an analogy with the definition of fractional exterior derivative. If the partial derivatives in the definition of the exterior derivative

$$d = dx_i \partial / \partial x_i$$

are allowed to assume fractional order, a fractional exterior derivative can be defined [20] by the equation

$$d^\alpha = (dx_i)^\alpha D^\alpha_{x_i}, \tag{104}$$

where D^α_x are the fractional derivative with respect to x. Using this analogy, we can define the fractional variation in the following way. For the point u of functional space, we can define the fractional variation $\delta F[u]$ of the functional

$$F[u] = \int_{x_1}^{x_2} f(u, u_x) dx, \tag{105}$$

where $u_x = du/dx$, by the equation

$$\delta^\alpha F[u] = \int_{x_1}^{x_2} \delta f(u, u_x) dx = \int_{x_2}^{x_2} dx \left[(\delta u)^\alpha D^\alpha_u f(u, u_x) + (\delta u_x)^\alpha D^\alpha_{u_x} f(u, u_x) \right]. \tag{106}$$

This approach has a difficulty with the realization of integration by part in the second term of (106). It is easy to see that the variation $(\delta u_x)^\alpha$ cannot be represented as some operator acts on the variation $(\delta u)^\alpha$, i.e., we have

$$(\delta u_x(x))^\alpha = \left(\frac{d}{dx} \delta u(x) \right)^\alpha \neq \frac{d}{dx} (\delta u)^\alpha. \tag{107}$$

In order to resolve this difficulty we can use the following.

Let us define the fractional variation of the functional

$$F[u] = \int f(u_1, u_2, ..., u_m)dx, \tag{108}$$

by the equation

$$\delta^\alpha F[u] = \int \left[\sum_{k=1}^{m} (D^\alpha_{u_k} f)(\delta u_k)^\alpha\right] dx \tag{109}$$

by analogy with

$$\delta F[u] = \int \sum_{k=1}^{m} (D^1_{u_k} f)\delta u_k dx. \tag{110}$$

In this case, we have

$$\delta^\alpha u_l = \int \left[\sum_{k=1}^{m} (D^\alpha_{u_k} u_l)(\delta u_k)^\alpha\right]. \tag{111}$$

It is known, that

$$\delta u_l = \int \left[\sum_{k=1}^{m} (D^1_{u_k} u_l)\delta u_k\right] dx = \int \left[\delta_{kl}\delta(y-x)\delta u_k\right] dx. \tag{112}$$

Using the Caputo fractional derivative, we have

$$D^\alpha_{u_k(y)} u_l(x) = \frac{u^{1-\alpha}}{\Gamma(2-\alpha)}\delta_{kl}\delta(y-x). \tag{113}$$

Here the Caputo fractional derivative leads to the δ_{kl}, i.e.,

$$D^\alpha_{u_k(y)} u_l(x) = 0 \quad k \neq l.$$

As the result, we get

$$\delta^\alpha u_l = \frac{u^{1-\alpha}}{\Gamma(2-\alpha)}\delta_{kl}(\delta u_k)^\alpha. \tag{114}$$

Therefore, we have

$$(\delta u_k)^\alpha = \Gamma(2-\alpha)u_k^{\alpha-1}\delta^\alpha u_k. \tag{115}$$

For the Riemann-Liouville fractional derivatives, we can derive some analogous relation:

$$(\delta u_k)^\alpha = A_{kl}\left(\Gamma(2-\alpha)u_k^{\alpha-1}; \Gamma(1-\alpha)u_k^{-1}u_l^\alpha\right)\delta^\alpha u_k. \tag{116}$$

Substituting Eq. (115) in Eq. (109), we have the following definition.

Definition 7. The fractional order variation $\delta^{\alpha}F[u]$ of the functional (108) is defined by the equation

$$\delta^{\alpha}F[u] = \Gamma(2-\alpha)\int\left[\sum_{k=1}^{m}(D^{\alpha}_{u_k}f)u_k^{\alpha-1}\delta^{\alpha}u_k\right]dx. \tag{117}$$

We can define the fractional variation for derivative $du_k(x)/dx$ by the equation

$$\delta^{\alpha}\frac{d}{dx}u_k(x) = \frac{d}{dx}\delta^{\alpha}u_k(x), \tag{118}$$

where we suppose that the fields u_k do not connected by some constraint. Analogously, we have

$$\delta^{\alpha}D^{\beta}_{x}u_k(x) = D^{\beta}_{x}\delta^{\alpha}u_k(x). \tag{119}$$

Let us consider $u_1 = u$, and $u_2 = u_x$ and the functional

$$F[u] = \int f(u_1, u_2)dx = \int f(u, u_x)dx. \tag{120}$$

As the result, we have

$$\delta^{\alpha}F[u] = \Gamma(2-\alpha)\int\left[(D^{\alpha}_{u_k}f)u_k^{\alpha-1}\delta^{\alpha}u - \frac{d}{dx}\left(D^{\alpha}_{u_{kx}}f)u_{kx}^{\alpha-1}\right)\right]\delta^{\alpha}u_k dx. \tag{121}$$

We can use this relation only if u_x and u can be considered as independent values, i.e., for the case

$$D^{\alpha}_{u_x}u = 0, \quad D^{\alpha}_{u}u_x = 0. \tag{122}$$

Note that Eq. (122) cannot be satisfied in the general case. Therefore, we have to consider the functional

$$F[u] = \int\left[f(u_1, u_2) + \lambda\left(u_2 - \frac{d}{dx}u_1\right)\right]dx, \tag{123}$$

instead of the functional (120).

Proposition 3. *The fractional-order variation equation*

$$\delta^{\alpha}F[u] = 0$$

of the functional (123) gives

$$(D^{\alpha}_{u_1}f)u_1^{2\alpha-2} - \frac{d}{dx}\left(u_1^{\alpha-1}u_2^{\alpha-1}D^{\alpha}_{u_2}f\right) = 0. \tag{124}$$

where $u_2 = du_1/dx$.

Proof. The fractional-order variation of the functional (123) gives

$$\delta^\alpha F[u] = \Gamma(2-\alpha) \int dx \left[(D^\alpha_{u_1} f) u_1^{\alpha-1} - \lambda \left(D^\alpha_{u_1} \frac{d}{dx} u_1 \right) u_1^{\alpha-1} \right] \delta^\alpha u_1$$
$$+\Gamma(2-\alpha) \int dx \left[(D^\alpha_{u_2} f) u_2^{\alpha-1} + \lambda (D^\alpha_{u_2} u_2) u_2^{\alpha-1} \right] \delta^\alpha u_2$$
$$+\Gamma(2-\alpha) \int dx \left[u_2 - \frac{d}{dx} u_1 \right] (D^\alpha_\lambda \lambda) \lambda_k^{\alpha-1} \delta^\alpha \lambda. \quad (125)$$

Using

$$\lambda \left(D^\alpha_{u_1} \frac{d}{dx} u_1 \right) u_1^{\alpha-1} = \lambda u_1^{\alpha-1} \frac{d}{dx} \left(D^\alpha_{u_1} u_1 \right)$$
$$= \frac{d}{dx} \left(\lambda u_1^{\alpha-1} D^\alpha_{u_1} u_1 \right) - D^\alpha_{u_1} u_1 \frac{d}{dx} \left(\lambda u_1^{\alpha-1} \right), \quad (126)$$

we get the following fractional variation of the functional

$$\delta^\alpha F[u] = \Gamma(2-\alpha) \int dx \left[(D^\alpha_{u_1} f) u_1^{\alpha-1} + D^\alpha_{u_1} u_1 \frac{d}{dx} \left(\lambda u_1^{\alpha-1} \right) \right] \delta^\alpha u_1$$
$$+\Gamma(2-\alpha) \int dx \left[(D^\alpha_{u_2} f) u_2^{\alpha-1} + \lambda (D^\alpha_{u_2} u_2) u_2^{\alpha-1} \right] \delta^\alpha u_2$$
$$+\Gamma(2-\alpha) \int dx \left[u_2 - \frac{d}{dx} u_1 \right] (D^\alpha_\lambda \lambda) \lambda_k^{\alpha-1} \delta^\alpha \lambda. \quad (127)$$

As the result, we get the field equations

$$(D^\alpha_{u_1} f) u_1^{\alpha-1} + D^\alpha_{u_1} u_1 \frac{d}{dx} \left(\lambda u_1^{\alpha-1} \right) = 0, \quad (128)$$

$$(D^\alpha_{u_2} f) + \lambda (D^\alpha_{u_2} u_2) = 0, \quad (129)$$

$$u_2 - \frac{d}{dx} u_1 = 0. \quad (130)$$

From the Eq. (129), we derive the Lagrange multiplier

$$\lambda = -\frac{D^\alpha_{u_2} f}{D^\alpha_{u_2} u_2}. \quad (131)$$

Substituting this equation in Eq. (128), we have

$$(D^\alpha_{u_1} f) u_1^{\alpha-1} - D^\alpha_{u_1} u_1 \frac{d}{dx} \left(\frac{u_1^{\alpha-1}}{D^\alpha_{u_2} u_2} D^\alpha_{u_2} f \right) = 0. \quad (132)$$

Using $D^\alpha_u u = u^{1-\alpha}/\Gamma(2-\alpha)$, we get (124).

Equation (124) is the fractional Euler-Lagrange equation.

7. Conclusion

In the general case, the equation of motion cannot be derived from the stationary action principle. The class of equations that can be derived from stationary action principle by using fractional variation is a wider class than the usual class equations that can be derived by usual (integer, first order) variation. The usual (integer order) equations of motion can be considered as special case of equations that can be derived by fractional variation such that $\alpha = 1$.

It can seems that the fractional variations are abstract and formal constructions. For this reason, we would like to pay attention that the suggested fractional variations can have wide application in study of fractional gradient type equations and fractional generalization of Lyapunov direct (second) method in the theory of stability.

The possible importance of fractional variations are connected with the following ideas. The class of gradient dynamical systems is a restricted class of all dynamical systems. However these systems have important property. The gradient system can be described by one function - potential, and the study of the system can be reduced to research of potential. For example, the way of chemical reactions is defined from the analysis of potential energy surfaces [35, 36, 37]. The fractional gradient systems has been suggested in Refs. [18, 19]. The fractional gradient systems are non-gradient dynamical systems that can be described by one function - some potential. For example, the Lorenz equations and Rössler equations are fractional gradient systems [18, 19]. Therefore the study the some non-gradient system can be reduced to research of potential. For example, the way of some chemical reactions with dissipation, dynamical chaos and self-organizing can be considered by the analysis of some potential energy surfaces. The suggested fractional variations allow us to define the fractional generalization of gradient type equations that can have wide applications for the description of dissipative structures [38, 39]. The suggested approach can also be generalized for lattice systems by using the lattice fractional calculus [34].

Appendix

The following rules for variational defivatives are known:

- The variation derivative of the field $u(x)$ is defined by the equation

$$\frac{\delta u(x)}{\delta u(y)} = \delta(y - x), \tag{133}$$

where we use

$$\delta u(x) = \int \delta(y-x)\delta u(y). \tag{134}$$

The variational derivatives of linear functional

$$F[u] = \int g(x)u(x)dx \tag{135}$$

can be calculated by the simple formula

$$\frac{\delta F[u]}{\delta u(y)} = \int g(x)\frac{\delta u(x)}{\delta u(y)}dx = \int g(x)\delta(y-x)dx = h(y).$$

- If the u function in the functional is affected by differential operators, then, in order to make use of the rule (133), one should at first "throw them over" to the left, fulfilling integration by parts. For example,

$$\frac{\delta}{\delta u(y)}\int g(x)[\nabla u(x)]dx = -\frac{\delta}{\delta u(y)}\int [\nabla g(x)]u(x)dx = -\nabla g(y).$$

We assumed here that on the boundary of integration domain the product $u(x)h(x)$ becomes zero.

- The variational derivative of nonlinear functionals is calculated according to the rule of differentiating a complex function similarly to partial derivatives:

$$\frac{\delta}{\delta u(y)}\int f(u(x))dx = \int \frac{\delta f(u)}{\delta u(x)}\frac{\delta u(x)}{\delta u(y)}dx$$
$$= \int \frac{\delta f(u)}{\delta u(x)}\delta(y-x)dx = \frac{\delta f(u(y))}{\delta u(y)}.$$

For example,

$$\frac{\delta}{\delta u(y)}\int [u(x)]^n dx = n[u(y)]^{n-1}.$$

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Integrals and Derivatives of Fractional Order and Applications*, Nauka i Tehnika, Minsk (1987); and *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach, New York (1993).

[2] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, Harlow and Wiley, New York (1994).

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1998).

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006).

[5] D. Valerio, J.J. Trujillo, M. Rivero, J.A. Tenreiro Machado, D. Baleanu, Fractional calculus: A survey of useful formulas, *The European Physical Journal. Special Topics*, **222**, No 8 (2013), 1827-1846.

[6] B. Ross, A brief history and exposition of the fundamental theory of fractional calculus. In: *Fractional Calculus and Its Applications*, Lecture Notes in Mathematics. **457** (Ed. B. Ross), Springer-Verlag, Berlin (1975), 1-36.

[7] L. Debnath, A brief historical introduction to fractional calculus, *International Journal of Mathematical Education in Science and Technology*, **35**, No 4 (2004), 487-501.

[8] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Communications in Nonlinear Science and Numerical Simulation*, **16**, No 3 (2011), 1140-1153.

[9] J.A. Tenreiro Machado, A.M. Galhano, J.J. Trujillo, Science metrics on fractional calculus development since 1966, *Fractional Calculus and Applied Analysis*, **16**, No 2 (2013), 479-500.

[10] J.A. Tenreiro Machado, A.M.S.F. Galhano, J.J. Trujillo, On development of fractional calculus during the last fifty years, *Scientometrics*, **98**, No 1 (2014), 577-582.

[11] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, Singapore (2010).

[12] V.E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, New York (2011).

[13] J. Klafter, S.C. Lim, R. Metzler (Eds.), *Fractional Dynamics. Recent Advances*, World Scientific, Singapore (2011).

[14] M.M. Meerschaert, A. Sikorskii, *Stochastic Models for Fractional Calculus*, Walter de Gruyter, Berlin/Boston (2012).

[15] V.E. Tarasov, Review of some promising fractional physical models, *International Journal of Modern Physics B*, **27**, No 9 (2013), 1330005.

[16] V. Uchaikin, R. Sibatov, *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors, Dielectrics and Nanosystems*, World Scientific, Singapore (2013).

[17] T. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, Wiley-ISTE, London, Hoboken (2014).

[18] V.E. Tarasov, Fractional generalization of gradient and Hamiltonian systems, *Journal of Physics A*, **38**, No 26 (2005), 5929-5943.

[19] V.E. Tarasov, Fractional generalization of gradient systems, *Letters in Mathematical Physics*, **73**, No 1 (2005), 49-58.

[20] K. Cottrill-Shepherd, M. Naber, Fractional differential forms, *Journal of Mathematical Physics* **42**, No 5 (2001), 2203-2212.

[21] R. Aldrovandi, R.A. Kraenkel, On exterior variational calculus, *Journal of Physics A*, **21**, No 6 (1988), 1329-1339.

[22] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, *Geophysical Journal of the Royal Astronomical Society*, **13**, No 5 (1967), 529-539.

[23] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism, *Pure and Applied Geophysics*, **91**, No 1 (1971), 134-147.

[24] R. Gorenflo, F. Mainardi, Fractional calculus, integral and differential equations of fractional order, In: *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi (Eds.), Springer Verlag, Wien and New York (1997), 223-276.

[25] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag, Berlin, Heidelberg (2010).

[26] M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Moscow, GosTehIzdat (1956) in Russsian; Holden-Day, San Francisco (1964).

[27] O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, *Journal of Mathematical Analysis and Applications*, **272**, No 1 (2002), 368-379.

[28] R. Gateaux, Fonctions iVune infiniti des variables indipendantes, *Bulletin de la Societe Msathematique de France*, **47**, (1919), 70-96.

[29] M. Frechet, La notion de differentielle dans l'analyse generale, *Annales Scientifiques de l'Ecole Normale Ssuperieure*, **42**, (1925), 293-323.

[30] M. Frechet, Sur la notion de differentielle, *Journal de Mathematiques*, **16**, (1937), 233-250.

[31] V. Kac, P. Cheung, *Quantum Calculus*, Springer, New York (2002).

[32] W.A. Al-Salam, Some Fractional q-Integrals and q-Derivatives, *Proceedings of the Edinburgh Mathematical Society*, **15**, No 2 (1966), 135-140.

[33] R.P. Agarwal, Certain fractional q-integrals and q-derivatives, *Mathematical Proceedings of the Cambridge Philosophical Society*, **66**, No 2 (1969), 365-370.

[34] V.E. Tarasov, Toward lattice fractional vector calculus, *Journal of Physics A*, **47**, No 35 (2014), 355204.

[35] R.D. Levine, J. Bernstein, *Molecular Reaction Dynamics*, Oxford University Press, New York (1974).

[36] K. Fukui, Formulation of the reaction coordinate, *The Journal of Physical Chemistry*, **74** (1970), 4161-4163.

[37] K. Fukui, The path of chemical reactions the IRC approach, *Accounts of Chemical Research*, **14**, No 12 (1981), 363-368.

[38] G. Nicolis, I. Prigogine, *Self-Organization in Nonequilibrium Systems: From Dissipative Structures to Order through Fluctuations*, John Wiley, New York (1977).

[39] R.Z. Sagdeev, D.A. Usikov, G.M. Zaslavsky, *Nonlinear Physics*, Harwood Academic, New York (1988).