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FRACTIONAL DERIVATIVE AS FRACTIONAL POWER OF DERIVATIVE

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Definitions of fractional derivatives as fractional powers of derivative operators are suggested. The Taylor series and Fourier series are used to define fractional power of self-adjoint derivative operator. The Fourier integrals and Weyl quantization procedure are applied to derive the definition of fractional derivative operator. Fractional generalization of concept of stability is considered.

Keywords: Fractional derivative; fractional power of derivative; fractional stability.

Mathematics Subject Classfication: 26A33

1. Introduction

The theory of integrals and derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov [1, 2]. Fractional analysis has found many applications in recent studies in mechanics and physics. The interest in fractional equations has been growing continually during the last few years because of numerous applications. In a short period of time the list applications becomes long. For example, it includes the chaotic dynamics [3, 4], material sciences [5–9], mechanics of fractal and complex media [10–18], quantum mechanics [19, 23], physical kinetics [3, 24–26], plasma physics [27, 30], electromagnetic theory [28–30], astrophysics [33], long-range dissipation [34, 42], non-Hamiltonian mechanics [35–41], long-range interaction [44–47].

It is known that we can define a fractional power of operator [48–54]. The integer power of operator can be easily realized. Therefore, we can realize the fractional power as the integer power series. In this paper, we use equations that represent the fractional power as a series of integer powers series. This representation allows us to define the fractional power of operator as a series of integer powers of operator. As the result, we obtain the definition of fractional derivatives as a fractional power of derivative operator.

Note that the well-known Riemann–Liouville fractional derivative can be represented as a power series of derivatives of integer order [1]:

$$D^{\alpha}_{a+} = \sum_{n=0}^{\infty} A_n(x,a,\alpha)\frac{d^n}{dx^n}, \tag{1.1}$$

where

$$A_n(x,a,\alpha) = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}\frac{(x-a)^{n-\alpha}}{\Gamma(n+1-\alpha)} \tag{1.2}$$

for the functions that are analytical in the interval (a, b).

In Sec. 2, we point out some well-known definitions of functions of bounded and unbounded operators. In Sec. 3, the fractional derivatives are defined as fractional powers of coordinates that considered as Taylor series. In Sec. 4, the fractional derivatives are considered as fractional powers of coordinates that considered as Fourier series for the interval. In Sec. 5, the fractional derivatives are defined as fractional powers of coordinates by using the Fourier integrals. In Sec. 6, the fractional derivatives are defined as fractional powers of coordinates by using the Weyl quantization. In Sec. 7, using fractional derivatives, we define the stability with respect to fractional variations.

2. Function of Bounded and Unbounded Operators

Let us point out some well-known definitions of functions of bounded and unbounded operators [48–52].

2.1. *Power series*

Let us consider a bounded linear operator A that is defined on the linear space E, and $A \in L(E,E)$, where $L(E,E)$ is a space of linear maps of E. Suppose the function $f(x)$ is an analytical function of the variable x such that it can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n.$$

Then, we can define

$$f(A) = \sum_{n=0}^{\infty} f_n A^n. \tag{2.1}$$

The operator $f(A)$ is a linear bounded operator A on space E. For example, the exponential function of operator is defined by

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

2.2. *Cauchy's integral formula*

The definition of operator function by power series can be generalized for wider class of functions. To realize this generalization, we use Cauchy's integral formula instead of power series. Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)dz}{z - z_0}, \tag{2.2}$$

where the integral is a contour integral along the contour Γ enclosing the point z_0.

We can define the algebraic isomorphism between an operator algebra and some functions [51]. The function $f(z) = z$ corresponds to the operator A. The function $f(z - z_0) = (z - z_0)^{-1}$ corresponds to the resolvent operator $R(z, A) = (A - zI)^{-1}$. If $|z| > r_A$, where r_A is a spectral radius:

$$r_A = \lim_{n\to\infty} \sqrt{\|A^n\|},$$

then the resolvent operator exists. The function of linear bounded operator is defined by

$$f(A) = -\frac{1}{2\pi i} \oint_\Gamma f(z)R(z, A)dz, \tag{2.3}$$

where

$$R(z, A) = (A - zI)^{-1}, \quad z \in \rho(A).$$

Here $\Gamma = \partial G \in \sigma(A)$, where $\sigma(A)$ is a spectrum of operator A, and $\rho(A) \subset G$. For example, we can define the operator

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = -\frac{1}{2\pi i} \oint_{\partial G} e^{zt} R(z, A)dz \tag{2.4}$$

that is corresponded to the function $\exp(zt)$, and $\sigma(A) \subset G$.

As the second example, the operator E_z that corresponds to the Heaviside function $\theta(z - z_0)$, where $\theta(z - z_0) = 0$ for $z_0 \geq z$, and $\theta(z - z_0) = 1$ for $z_0 < z$ is defined by

$$E_z = E(z, A) = \frac{1}{2\pi i} \oint_\Gamma \theta(z - z_0)R(z_0, A)dz_0, \tag{2.5}$$

and is called the spectral operator. The operator E_z can be denoted by $\theta(zI - A)$.

2.3. *Spectral representation of selfadjoint unbounded operator*

It is known that spectral function E_z exists for all selfadjoint operators A, and

$$Ax = \int_{-\infty}^{+\infty} z dE_z x, \tag{2.6}$$

where

$$\|Ax\|^2 = \int_{-\infty}^{+\infty} |z|^2 d(E_z x, x) < \infty. \tag{2.7}$$

Then, the function $f(A)$ of selfadjoint operator A can be defined by the equation

$$f(A)x = \int_{-\infty}^{+\infty} f(z)dE_z x. \tag{2.8}$$

As operator A, we can consider the selfadjoint derivative $-i\partial/\partial x$. For the father information about this approach, we can use Refs. [48, 49, 52, 67, 68].

3. Fractional Derivatives by Taylor Series

3.1. *Definition of Taylor series*

A one-dimensional Taylor series, which is an expansion of a real function $f(x)$ about a point $x = a$, is given by

$$f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n, \tag{3.1}$$

where

$$f_n = \frac{1}{n!} f^{(n)}(a), \tag{3.2}$$

and $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at the point $x = a$.

Suppose the function $f(x)$ has all derivatives in the interval $|x-a| < a_0$, and the condition

$$\lim_{n\to\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 0 \tag{3.3}$$

is satisfied, then the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3.4}$$

converges to the function $f(x)$ for all intervals $|x-a| < a'$, where $a' < a_0$. This representation of the functions can be used to define fractional power of operator.

3.2. *Taylor series for fractional power of coordinate*

The Taylor series for fractional power of coordinate $f(x) = x^{\alpha}$ about a point $x = a > 0$ is

$$x^{\alpha} = \sum_{n=0}^{\infty} f_n (x-a)^n, \tag{3.5}$$

where

$$f_n = \frac{1}{n!}(x^{\alpha})^{(n)}(a) = \frac{B(\alpha,n)}{n! a^{n-\alpha}}, \tag{3.6}$$

$$B(\alpha,n) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1). \tag{3.7}$$

If $m-1 < \alpha < m$, then

$$B(\alpha,n) = (-1)^{n-m}\frac{(\alpha+1-m)_m}{(m-\alpha)_{n-m}} = (-1)^{n-m}\frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(m-\alpha)\Gamma(\alpha+1-m)}, \tag{3.8}$$

where

$$(z)_m = z(z+1)\cdots(z+m-1).$$

For $0 < \alpha < 1$,

$$B(\alpha, n) = (-1)^{n-1}\alpha(1-\alpha)_{n-1} = (-1)^{n-1}\frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)}. \tag{3.9}$$

Using

$$(x-a)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k a^{n-k}, \quad \binom{n}{k} = \frac{n!}{(n-k)!k!}, \tag{3.10}$$

we rewrite Eq. (3.5) in the form

$$x^\alpha = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C(n,k,\alpha,a)x^k, \tag{3.11}$$

where

$$C(n,\alpha,a) = \frac{(-1)^{k+1}a^{\alpha-k}}{(n-k)!k!}\frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)}. \tag{3.12}$$

Equation (3.11) represents the fractional power of coordinate as a series of integer powers. This representation allows us to define the fractional power of operator as a series of integer powers of operator.

It is known that selfadjoint operators have the real eigenvalues. Using Eq. (3.11) with $a > 0$, we can define the fractional power of the selfadjoint operator A by

$$A^\alpha = \sum_{n=0}^{\infty}\frac{B(\alpha,n)}{n!a^{n-\alpha}}A^n. \tag{3.13}$$

For the operator $A = -i\partial/\partial x$, we have

$$\left(-i\frac{d}{dx}\right)^\alpha = \sum_{n=0}^{\infty}\frac{B(\alpha,n)}{n!a^{n-\alpha}}\left(-i\frac{d}{dx}-a\right)^n, \tag{3.14}$$

or, in the equivalent form

$$\left(-i\frac{d}{dx}\right)^\alpha = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-i)^k C(n,k,\alpha,a)\frac{d^k}{dx^k}. \tag{3.15}$$

As the result, we get that fractional derivative is defined as a series of integer powers of selfadjoint derivative operator.

3.3. *Examples of computation of fractional derivatives*

Let us consider the fractional derivative (3.14) of constant c:

$$\left(-i\frac{d}{dx}\right)^\alpha c = \sum_{n=0}^{\infty}\frac{A(\alpha,n)}{n!a^{n-\alpha}}\left(-i\frac{d}{dx}-a\right)^n c. \tag{3.16}$$

Using

$$\left(-i\frac{d}{dx}-a\right)^n c = (-a)^n c, \tag{3.17}$$

we get

$$\left(-i\frac{d}{dx}\right)^{\alpha} c = a^{\alpha} c \sum_{n=0}^{\infty} (-1)^n \frac{B(\alpha, n)}{n!}. \tag{3.18}$$

If $0 < \alpha < 1$, then

$$\left(-i\frac{d}{dx}\right)^{\alpha} c = -a^{\alpha} c \sum_{n=0}^{\infty} \frac{\Gamma(n-\alpha)}{\Gamma(n+1)\Gamma(1-\alpha)}. \tag{3.19}$$

Let us consider the fractional derivative of a power x^m. From (3.15),

$$\left(-i\frac{d}{dx}\right)^{\alpha} x^m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-i)^k C(n,k,\alpha,a)(x^m)^{(k)}. \tag{3.20}$$

Using

$$(x^m)^{(k)} = m(m-1)\cdots(m-k+1)x^{m-k} = \frac{m!}{(m-k)!} x^{m-k} \tag{3.21}$$

for $k \leq m$, and $(x^m)^{(k)} = 0$ for $k > m$, we get

$$\left(-i\frac{d}{dx}\right)^{\alpha} x^m = \sum_{n=0}^{m} \sum_{k=0}^{n} (-i)^k C(n,k,\alpha,a) \frac{m!}{(m-k)!} x^{m-k}. \tag{3.22}$$

4. Fractional Derivatives by Fourier Series

4.1. *Fourier series*

Fourier series of a function $f(x) \in L_2[-l,l]$ is an expansion in terms of an infinite sum of sines and cosines. Since sines and cosines form a complete orthogonal system over $[-l,l]$, the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right], \tag{4.1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{l}\int_{-l}^{+l} f(x)dx, \\ a_n &= \frac{1}{l}\int_{-l}^{+l} f(x)\cos\left(\frac{\pi n x}{l}\right) dx, \\ b_n &= \frac{1}{l}\int_{-l}^{+l} f(x)\sin\left(\frac{\pi n x}{l}\right) dx, \end{aligned} \tag{4.2}$$

and n is a positive integer number.

Let us consider the Fourier series for $f(x) = |x|^\alpha \in L^2[-l;l]$, where α is a positive fractional power. The Fourier series of this function for $x \in [-l, +l]$ is

$$|x|^\alpha = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k x}{l}\right), \tag{4.3}$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} |y|^\alpha \cos\left(\frac{\pi k y}{l}\right) dy = \frac{2}{l}\int_0^l y^\alpha \cos\left(\frac{\pi k y}{l}\right) dy, \quad k = 0, 1, 2, \ldots. \tag{4.4}$$

The cosine can be represented as a power series. Therefore, Eq. (4.3) allows us to present the fractional power as a series of integer powers series. Then, we can define fractional derivative on the interval $[-l, l]$ as a fractional power of derivative.

4.2. *Fractional derivative*

The fractional power of derivative operator for the interval $[-l; l]$, we can define by

$$\left(-i\frac{d}{dx}\right)^\alpha = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(-i\frac{\pi k}{l}\frac{d}{dx}\right). \tag{4.5}$$

It is known that we can define $\exp(A)$ for the selfadjoint operator $A = -id/dx$ by

$$e^{-id/dx} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\left(\frac{d}{dx}\right)^n. \tag{4.6}$$

Using

$$\cos(A) = \frac{1}{2}\left(e^{iA} + e^{-iA}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(iA)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2(n!)}\left[(A)^n + (-A)^n\right] = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2(n!)} A^n = \sum_{m=0}^{\infty} \frac{1}{(2m)!} A^{2m},$$

we have

$$\cos\left(-i\frac{d}{dx}\right) = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{d}{dx}\right)^{2m}. \tag{4.7}$$

Equations (6.15) and (4.7) allows us to define the fractional power of operator as a series of integer powers of operator. Using (4.7), we rewrite (4.5) as

$$\left(-i\frac{d}{dx}\right)^\alpha = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{\pi k}{l}\right)^{2m}\left(\frac{d}{dx}\right)^{2m}, \tag{4.8}$$

or by the equivalent equation

$$\left(-i\frac{d}{dx}\right)^\alpha = \frac{a_0}{2} + \sum_{m=0}^{\infty} S_m \left(\frac{d}{dx}\right)^{2m}, \tag{4.9}$$

where

$$S_m = \frac{1}{(2m)!} \sum_{k=1}^{\infty} \left(\frac{\pi k}{l} \right)^{2m} a_k. \tag{4.10}$$

Let us compute the coefficients a_k by

$$a_k = \frac{1}{l^\alpha} \int_0^l y^\alpha \cos(\pi k y) dy.$$

As the result, we obtain

$$a_k = 2^\alpha \pi^{-1/2-\alpha} l^{-\alpha} k^{-1-\alpha} \left(\frac{2^{-\alpha} \pi^{-1/2+\alpha} k^\alpha (\alpha+3) \sin(\pi k)}{(\alpha+1)(\alpha+3)} \right.$$
$$+ \frac{2^{-\alpha} \pi^{-1/2+\alpha} k^\alpha [\pi k \cos(\pi k) - \sin(\pi k)]}{\alpha+1} + \frac{2^{-\alpha} \sqrt{k} \alpha L(\alpha+1/2, 3/2, \pi k) \sin(\pi k)}{\alpha+1}$$
$$\left. - \frac{2^{-\alpha} [\pi k \cos(\pi k) - \sin(\pi k)] L(\alpha+3/2, 1/2, \pi k)}{\sqrt{k} \pi (\alpha+1)} \right),$$

where $L(\mu, \nu, z)$ is the Lommel function [55].

4.3. *Complex Fourier series*

The real-valued function $f(x)$, which is defined on [−L/2, L/2], can be presented by

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i(2\pi n/L)x}, \tag{4.11}$$

where

$$f_n = \frac{1}{L} \int_{-L/2}^{+L/2} f(x) e^{-i(2\pi n x/L)} dx. \tag{4.12}$$

The operator function $f(A)$ is defined by

$$f(A) = \sum_{n=-\infty}^{\infty} f_n e^{i(2\pi n/L)A}, \tag{4.13}$$

where f_n are defined in (4.12), and

$$e^{i(2\pi n/L)A} = \sum_{k=0}^{\infty} i^k \frac{(2\pi n/L)^k}{k!} A^k. \tag{4.14}$$

Substitution of (4.14) into (4.13) gives

$$f(A) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} f_n \frac{(i2\pi n/L)^k}{k!} A^k. \tag{4.15}$$

Then the fractional power of selfadjoint derivative on $[-L/2, +L/2]$ can be presented by

$$\left(-i\frac{d}{dx}\right)^{\alpha} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} f(n,k) \left(\frac{d}{dx}\right)^{k}, \tag{4.16}$$

where

$$f(n,k) = \frac{(2\pi n)^k}{k! L^{k+1}} \int_{+L/2}^{-L/2} |x|^{\alpha} \cos(nx) dx. \tag{4.17}$$

Equation (4.16) represents the fractional power as a series of integer powers. This representation defines the fractional power of derivative as a series with integer powers of derivatives.

5. Fourier Transform and Fourier Integral

5.1. *Fractional derivative by Fourier integral*

Let us consider a function $f(x)$ with n variables x. Suppose A_1, $A_2, \dots, A_n$ are n elements of commutative operator algebra $\mathcal{A}$. For example, $\mathcal{A}$ is an algebra of operators in linear space. We denote by $\tilde{f}$ the Fourier transform for $f(x)$:

$$\tilde{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx, \quad xy = x_1 y_1 + \cdots + x_n y_n, \tag{5.1}$$

where

$$f(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \tilde{f}(y) e^{i(y_1 x_1 + \cdots + y_n x_n)} dy = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} dy \tilde{f}(y) e^{iyA}. \tag{5.2}$$

The operator function $f(A)$ of elements $A_1, \dots, A_n$ is defined by

$$f(A) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \tilde{f}(y) e^{i(y_1 A_1 + \cdots + y_n A_n)} dy = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} dy \tilde{f}(y) e^{iyA}. \tag{5.3}$$

Substitution of Eq. (5.1) into Eq. (5.3) get

$$f(A) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \tilde{f}(y) e^{iyA} dy = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx f(x) e^{iy(A-x)}. \tag{5.4}$$

In general, we must give the exact definition of these integrals and description of possible class of symbols and algebras of elements $A_1, \dots, A_n$.

To define the fractional power of the operator A, we use (5.3) in the form

$$A^{\alpha} = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy |x|^{\alpha} e^{iy(A-x)}. \tag{5.5}$$

This equation can be considered as a definition of fractional power of operator A.

For $f(x) = |x|^{\alpha}$, where $\alpha \neq -1, -3, \dots$, we have [56]:

$$\tilde{f}(y) = \int_{-\infty}^{+\infty} dx |x|^{\alpha} e^{-ixy} = -2\sin(\pi\alpha/2)\Gamma(\alpha+1)|y|^{-\alpha-1}.$$

Then Eq. (5.5) gives

$$A^{\alpha} = -\frac{2\sin(\pi\alpha/2)\Gamma(\alpha+1)}{(2\pi)^n} \int_{-\infty}^{+\infty} dy|y|^{-\alpha-1} e^{iyA}. \tag{5.6}$$

For the selfadjoint derivative operators

$$A_1 = -i\frac{\partial}{\partial x_1}, \dots, A_n = -i\frac{\partial}{\partial x_n},$$

Eq. (5.6) is

$$\left(-i\frac{\partial}{\partial x}\right)^{\alpha} = -\frac{2\sin(\pi\alpha/2)\Gamma(\alpha+1)}{(2\pi)^n} \int_{-\infty}^{+\infty} dy|y|^{-\alpha-1} \exp\left(y\frac{\partial}{\partial x}\right). \tag{5.7}$$

As the result, we obtain the fractional derivative operator as a fractional power of selfadjoint derivative operator.

5.2. *Fractional power of selfadjoint derivative operator*

Let us consider selfadjoint derivative operators

$$D_x = -i\frac{\partial}{\partial x} = \left(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}\right). \tag{5.8}$$

It is easy to prove that $f(p) = p^{\alpha} \in S^{\infty}(R^n_p)$. Here, S^{∞} is the space of symbols that are slowly growth on the infinity. This space is defined as

$$S^{\infty}(R^n) = \cup_l \cap_k S^k_l(R^n),$$

where $S^k_l(R^n)$ is a space of functions from the class $C^k(R^n)$ with the norm

$$\|f\|_{S^k_l(R^n)} = \sup_{R^n}(1+|x|^2)^{l/2}\left(\sum_{|a|}\|f^{(a)}(x)\|\right). \tag{5.9}$$

The space $S^{\infty}(R^n)$ is defined by

$$\exists r, \quad \forall s \quad \|f\|_{r,s} = \sup_x\left\{(1+|x|)^r \left|\left(\frac{\partial}{\partial x}\right)^s f(x)\right|\right\} < \infty. \tag{5.10}$$

The fractional powers of the operator (5.8) are elements of algebra $\mathcal{L}(S^{\infty}, S^{\infty})$ of all continuous linear maps of the space $S^{\infty}(R^n)$. Then, the fractional derivative operator

$$D^{\alpha}_x = \left(-i\frac{\partial}{\partial x}\right)^{\alpha} \tag{5.11}$$

is a fractional power of selfadjoint derivative operator. The operator (5.11) acts on the arbitrary function $u(x) \in C^{\infty}(R^n)$ by

$$\left(-i\frac{\partial}{\partial x}\right)^{\alpha} u(x) = \tilde{F}_{p\to x} p^{\alpha} F_{y\to p} u(y), \tag{5.12}$$

where

$$F_{y\to p}u(y) = \left(\frac{1}{2\pi i}\right)^{n/2}\int_{-\infty}^{+\infty} e^{ipy}u(y)dy, \tag{5.13}$$

is the direct Fourier transform, and

$$\tilde{F}_{p\to x}\Psi(p) = \left(\frac{1}{2\pi i}\right)^{n/2}\int_{-\infty}^{+\infty} e^{-ipx}\Psi(p)dp \tag{5.14}$$

is the Fourier transform.

Proposition 5.1. *The operator*

$$D_x^\alpha = \left(-i\frac{\partial}{\partial x}\right)^\alpha \tag{5.15}$$

has the symbol

$$\mathrm{symb}\{D_x^\alpha\}(p) = p^\alpha. \tag{5.16}$$

Proof. Using Eq. (5.12), we get

$$\begin{aligned}
D_x^\alpha e_p(x) &= \left(-i\frac{\partial}{\partial x}\right)^\alpha e_p(x) = \left(-i\frac{\partial}{\partial x}\right)^\alpha e^{ipx} = \tilde{F}_{z\to x} z^\alpha F_{y\to z} e^{ipy} \\
&= \tilde{F}_{z\to x}\big(z^\alpha (2\pi i)^{n/2}\delta(z-p)\big) \\
&= \tilde{F}_{z\to x}\big(p^\alpha (2\pi i)^{n/2}\delta(z-p)\big) = p^\alpha e^{ipx} = p^\alpha e_p(x).
\end{aligned}$$

As the result, we obtain

$$D_x^\alpha e_p(x) = p^\alpha e_p(x). \tag{5.17}$$

Multiplying both sides of (5.17) on $e_{-p}(x)$, we get $e_{-p}(x)D_x^\alpha e_p(x) = p^\alpha$ that proves (5.16). □

6. Fractional Derivative by Quantization Map

6.1. *Quantization procedure for coordinate representation*

Let us consider the quantum mechanics in coordinate representation. It is known that quantization $\hat{Q}$ is a linear map of coordinate q and momentum p into selfadjoint operators

$$\hat{Q}(q) = \hat{q} = q, \quad \hat{Q}(p) = \hat{p} = -i\hbar\frac{\partial}{\partial q}, \quad \hat{Q}(1) = \hat{I}. \tag{6.1}$$

Using linearity of quantization map [57, 58], we get

$$\hat{Q}(aq + bp) = a\hat{q} + b\hat{p}. \tag{6.2}$$

Obviously, we have

$$\hat{Q}([aq + bp]^n) = [a\hat{q} + b\hat{p}]^n. \tag{6.3}$$

Using the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \tag{6.4}$$

we get

$$\hat{Q}(\exp[i(aq + bp)]) = \exp[(i/\hbar)(a\hat{q} + b\hat{p})]. \tag{6.5}$$

This allows us to define the function of operators $\hat{q}$ and $\hat{p}$ by using the Fourier transforms [57–68].

6.2. *Weyl quantization*

Canonical quantization defines a map of real functions into selfadjoint operators [57–68]. A classical observable is described by some real function $A(q,p)$ from a function space $\mathcal{M}$. Quantization of this function leads to selfadjoint operator $\hat{A}(\hat{q},\hat{p})$ from some operator space $\hat{\mathcal{M}}$.

Let us consider main points of the usual method of canonical quantization [57, 60, 67, 68]. Let q_k be canonical coordinates and p_k are canonical momenta, where $k = 1, \dots, n$. The basis of the space $\mathcal{M}$ of functions $A(q,p)$ is defined by functions

$$W(a,b,q,p) = e^{(i/\hbar)(aq+bp)}, \quad aq = \sum_{k=1}^{n} a_k q_k. \tag{6.6}$$

Quantization transforms coordinates q_k and momenta p_k to operators $\hat{q}_k$ and $\hat{p}_k$. Weyl quantization of the basis functions (6.6) leads to the Weyl operators

$$\hat{Q}(W(a,b,q,p)) = \hat{W}(a,b,\hat{q},\hat{p}) = e^{(i/\hbar)(a\hat{q}+b\hat{p})}, \quad a\hat{q} = \sum_{k=1}^{n} a_k \hat{q}_k. \tag{6.7}$$

Operators (6.7) form a basis of the operator space $\hat{\mathcal{M}}$. Classical observable, characterized by the function $A(q,p)$, can be represented in the form

$$A(q,p) = \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} \tilde{A}(a,b) W(a,b,q,p) d^n a d^n b, \tag{6.8}$$

where

$$\tilde{A}(a,b) = \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} \tilde{A}(a,b) W(a,b,q,p) d^n a d^n b, \tag{6.9}$$

i.e. $\tilde{A}(a,b)$ is the Fourier image of the function $A(q,p)$. Quantum observable $\hat{A}(\hat{q},\hat{p})$, which coresponds to $A(q,p)$, is

$$\hat{Q}(A(q,p)) = \hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} \tilde{A}(a,b) \hat{W}(a,b,\hat{q},\hat{p}) d^n a d^n b. \tag{6.10}$$

This formula can be considered as an operator expansion for $\hat{A}(\hat{q},\hat{p})$ in the operator basis (6.7). Substitution of (6.9) into (6.10) gives

$$\hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^{2n}} \int_{-\infty}^{+\infty} d^n a \, d^n b \int_{-\infty}^{+\infty} d^n q \, d^n p \, A(q,p) \hat{W}(a,b,\hat{q} - q\hat{I}, \hat{p} - p\hat{I}). \tag{6.11}$$

The function $A(q, p)$ is called the Weyl symbol of the operator $\hat{A}(\hat{q}, \hat{p})$. Canonical quantization defined by (6.11) is called the Weyl quantization. The Weyl operator (6.7) in formula (6.11) leads to Weyl quantization. Another basis operator leads to different quantization scheme [60].

6.3. *Fractional derivative by Weyl quantization map*

Let us consider a quantization map of real function $f(p) = |p|^{\alpha}$ into selfadjoint operator. Quantization of this function leads to some selfadjoint operator $\hat{f}(\hat{p})$ from the operator space $\hat{\mathcal{M}}_p$, where $\hat{p}_k = -i\partial/\partial x_k$ and $k = 1, \dots, n$. Then

$$D_x^{\alpha} = f(\hat{p}) = \hat{p}^{\alpha} = (-i\partial/\partial x_k)^{\alpha}.$$

The basis of the space $\mathcal{M}_p$ is defined by functions

$$W(a, p) = e^{ibp}, \quad ap = \sum_{k=1}^{n} a_k p_k. \tag{6.12}$$

Quantization maps p_k into $\hat{p}_k = -i\partial/\partial x_k$. Weyl quantization of the functions (6.12) leads to

$$\hat{Q}(W(a, b)) = \hat{W}(a, \hat{p}) = e^{ia\hat{p}}, \quad a\hat{p} = \sum_{k=1}^{n} a_k \hat{p}_k. \tag{6.13}$$

The operators (6.13) form a basis of the operator space $\hat{\mathcal{M}}_p$. Using Fourier transform, the function $f(p) = |p|^{\alpha}$ can be presented by

$$f(p) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \tilde{f}(a) W(a, p) d^n a, \tag{6.14}$$

where

$$\tilde{f}(a) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} f(p) W(a, p) d^n p \tag{6.15}$$

i.e. $\tilde{f}(a)$ is the Fourier image of the function $f(p)$. Quantum observable $\hat{f}(\hat{p})$, which corresponds to $f(p)$, is defined by formula

$$\hat{Q}(f(p)) = \hat{f}(\hat{p}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \tilde{f}(a) \hat{W}(a, \hat{p}) d^n a. \tag{6.16}$$

This formula can be considered as an operator expansion for $\hat{f}(\hat{p})$ in the operator basis (6.7). From (6.15) and (6.16), we obtain

$$\hat{f}(\hat{p}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} d^n a \int_{-\infty}^{+\infty} d^n p \; f(p) \hat{W}(a, \hat{p} - p\hat{I}). \tag{6.17}$$

The function $f(p)$ is the Weyl symbol of the operator $\hat{f}(\hat{p})$.

For $f(p) = |p|^\alpha$, where α is a positive real number, we obtain

$$D_x^\alpha = \left(-i\frac{d}{dx}\right)^\alpha = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} d^n a \int_{-\infty}^{+\infty} d^n p \, |p|^\alpha \hat{W}(a, \hat{p} - p\hat{I}). \tag{6.18}$$

As the result, we have the definition of fractional derivatives on R^n as a fractional power of selfadjoint derivative.

7. Fractional Stability

In this section, we use the fractional generalization of variations of variables. Fractional integrals and derivatives are used for stability problems [69–73]. In this paper, we consider the properties of dynamical systems with respect to fractional variations [41]. We formulate stability with respect to motion changes at fractional changes of variables. Some systems can be unstable "in sense of Lyapunov", and be stable with respect to fractional variation.

7.1. *Fractional variation derivative*

Let us consider dynamical system that is defined by the ordinary differential equations. Suppose that the motion of dynamical system is described by the equations

$$\frac{d}{dt} y_k = F_k(y), \quad k = 1, \ldots, n. \tag{7.1}$$

Here $y_1, \ldots, y_n$ be real variables that define the state of dynamical system.

Let us consider the variation δy_k of variables y_k. The unperturbed motion is satisfied to zero value of variation $\delta y_k = 0$. The variation describes that as function $f(y)$ changes at changes of argument y. The first variation describes changes of function with respect to the first power of changes of y:

$$\delta f(y) = D_y^1 f(y) dy, \tag{7.2}$$

where

$$D_y^1 f(y) = \frac{\partial f(y)}{\partial y}.$$

The second variation describes changes of function with respect to the second power of changes of y:

$$\delta^2 f(y) = D_y^2 f(y)(dy)^2. \tag{7.3}$$

The variation δ^n of integer order n is defined by the derivative of integer order $D_y^n f(y) = \partial^n f / \partial y^n$.

Let us define the variation of fractional order as a fractional exterior derivative of the function (zero-form) by the equation

$$\delta^\alpha f = D_y^\alpha f \, (\delta y)^\alpha, \tag{7.4}$$

where D_y^α is a fractional derivative with respect to y.

The fractional variation of order α describes the function $f(y)$ changes with respect to fractional power of variable y changes. The variation of fractional order is defined by the derivative of fractional order.

7.2. *Equations for fractional variations*

Let us derive the equations for fractional variations $\delta^\alpha y_k$. We consider the fractional variation of Eq. (7.1) in the form:

$$\delta^\alpha \frac{d}{dt} y_k = \delta^\alpha F_k(y), \quad k = 1, \ldots, n. \tag{7.5}$$

Using the definition of fractional variation (7.4), we have

$$\delta^\alpha F_k(y) = [D^\alpha_{y_l} F_k](\delta y_l)^\alpha, \quad k = 1, \ldots, n. \tag{7.6}$$

From Eq. (7.6), and the property of variation

$$\delta^\alpha \frac{d}{dt} y_k = \frac{d}{dt} \delta^\alpha y_k, \tag{7.7}$$

where $y_k = y_k(t, a)$, we obtain

$$\frac{d}{dt} \delta^\alpha y_k = [D^\alpha_{y_l} F_k]\, (\delta y_l)^\alpha, \quad k = 1, \ldots, n. \tag{7.8}$$

Note that in the left-hand side of Eq. (7.8), we have fractional variation of $\delta^\alpha y_k$, and in the right-hand side — fractional power of variation $(\delta y_k)^\alpha$.

Let us consider the fractional variation of the variable y_k. Using Eq. (7.4), we get

$$\delta^\alpha y_k = [D^\alpha_{y_l} y_k]\, (\delta y_l)^\alpha, \quad k = 1, \ldots, n. \tag{7.9}$$

For the Riemann–Liouville fractional derivative,

$$D^\alpha_{y_l} y_k \neq 0$$

if $k \neq l$. Therefore for simplification of our transformations, we use the fractional derivative as a fractional power of derivative

$$D^\alpha_{y_l} y_k = \delta_{kl} D^\alpha_{y_l} y_l, \tag{7.10}$$

where δ_{kl} is a Kronecker symbol. Substituting Eq. (7.10) into Eq. (7.9), we can express the fractional power of variations $(\delta y_k)^\alpha$ through the fractional variation $\delta^\alpha y_k$:

$$(\delta y_k)^\alpha = D^\alpha_{y_k} y_k \delta^\alpha y_k. \tag{7.11}$$

Substitution of Eq. (7.11) into Eq. (7.8) get

$$\frac{d}{dt} \delta^\alpha y_k = \left[D^\alpha_{y_l} y_l\right] \left[D^\alpha_{y_l} F_k\right] \delta^\alpha y_l. \tag{7.12}$$

Here we mean the sum on the repeated index l from 1 to n. Equation (7.12) is equations for fractional variations. Let us denote x_k the fractional variations $\delta^\alpha y_k$:

$$x_k = \delta^\alpha y_k = \left[D^\alpha_{y_k} y_k\right] (\delta y_k)^\alpha. \tag{7.13}$$

As the result, we obtain the differential equation for fractional variations

$$\frac{d}{dt} x_k = a_{kl}(\alpha) x_l, \tag{7.14}$$

where

$$a_{kl}(\alpha) = \left[D^\alpha_{y_l} y_l\right] D^\alpha_{y_l} F_k. \tag{7.15}$$

Using the matrix $X^t = (x_1, \ldots, x_n)$, and $A_\alpha = \|a_{kl}(\alpha)\|$, we can rewrite Eq. (7.14) in the matrix form

$$\frac{d}{dt} X = A_\alpha X. \tag{7.16}$$

Equation (7.16) is a linear differential equation. To define the stability with respect to fractional variations, we consider the characteristic equation

$$\mathrm{Det}[A_\alpha - \lambda E] = 0 \tag{7.17}$$

with respect to λ. If the real part $\mathrm{Re}[\lambda_k]$ of all eigenvalues λ for the matrix A_α are negative, then the unperturbed motion is asymptotically stable with respect to fractional variations. If the real part $\mathrm{Re}[\lambda_k]$ of one of the eigenvalues λ of the matrix A_α is positive, then the unperturbed motion is unstable with respect to fractional variations.

A system is said to be stable with respect to fractional variations if for every ϵ, there is a δ_0 such that:

$$\|\delta^\alpha y(t_0)\| < \delta_0 => \|\delta^\alpha y(\alpha, t)\| < \epsilon \quad \forall t \in R_+. \tag{7.18}$$

The dynamical system is said to be asymptotically stable with respect to fractional variations $\delta^\alpha y(t, \alpha)$ if as

$$t \to \infty, \quad \|\delta^\alpha y(t, \alpha)\| \to 0. \tag{7.19}$$

The concept of stability with respect to fractional variations is wider than the usual Lyapunov or asymptotic stability. Fractional stability includes concept of "integer" stability as a special case $\alpha = 1$. Some systems can be unstable with respect to first variation of states, and be stable with respect to fractional variation. Therefore fractional derivatives expand our possibility to research the properties of dynamical systems.

References

[1] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications* (Gordon and Breach, New York, 1993).
[2] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).
[3] G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.* **371** (2002) 461-580.
[4] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).
[5] R. Hilfer (ed.), *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
[6] M. Caputo, *Elasticita e Dissipazione* (Zanichelli, Bologna, 1969) (in Italian).
[7] R. R. Nigmatullin, The realization of the generalized transfer equation in a medium with fractal geometry, *Phys. Status Solidi. B* **133** (1986) 425–430.
[8] R. R. Nigmatullin, Fractional integral and its physical interpretation, *Theor. Math. Phys.* **90** (1992) 242–251.
[9] A. Le Mehaute, R. R. Nigmatullin and L. Nivanen, *Fleches du temps et Geometric Fractale* (Hermes, Paris, 1998), Chap. 5 (in French).
[10] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics* (Springer, New York, 1997).
[11] V. E. Tarasov, Continuous medium model for fractal media, *Phys. Lett. A* **336** (2005) 167–174.
[12] V. E. Tarasov, Fractional Fokker-Planck equation for fractal media, *Chaos* **15** (2005) 023102.
[13] V. E. Tarasov, Possible experimental test of continuous medium model for fractal media, *Phys. Lett. A* **341** (2005) 467–472.
[14] V. E. Tarasov, Fractional hydrodynamic equations for fractal media, *Ann. Phys.* **318** (2005) 286–307.
[15] V. E. Tarasov, Wave equation for fractal solid string, *Mod. Phys. Lett. B* **19** (2005) 721–728.
[16] V. E. Tarasov, Dynamics of fractal solid, *Int. J. Mod. Phys. B* **19** (2005) 4103–4114.
[17] V. E. Tarasov, Fractional Chapman-Kolmogorov equation, *Mod. Phys. Lett. B* **21** (2007) 163–174.
[18] V. E. Tarasov and G. M. Zaslavsky, Fractional Ginzburg-Landau equation for fractal media, *Phys. A* **354** (2005) 249–261.
[19] N. Laskin, Fractional quantum mechanics, *Phys. Rev. E* **62** (2000) 3135–3145.
[20] N. Laskin, Fractional quantum mechanics and Levy path integrals, *Phys. Lett. A* **268** (2000) 298–305.
[21] N. Laskin, Fractals and quantum mechanics, *Chaos* **10** (2000) 780–790.
[22] N. Laskin, Fractional Schrodinger equation, *Phys. Rev. E* **66** (2002) 056108.
[23] M. Naber, Time fractional Schrodinger equation, *J. Math. Phys.* **45** (2004) 3339–3352.
[24] G. M. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos, *Phys. D* **76** (1994) 110–122.
[25] A. I. Saichev and G. M. Zaslavsky, Fractional kinetic equations: Solutions and applications, *Chaos* **7** (1997) 753–764.
[26] G. M. Zaslavsky and M. A. Edelman, Fractional kinetics: From pseudochaotic dynamics to Maxwell's demon, *Phys. D* **193** (2004) 128–147.
[27] B. A. Carreras, V. E. Lynch and G. M. Zaslavsky, Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model, *Phys. Plasmas* **8** (2001) 5096–5103.

[28] J. Lutzen, Liouville's differential calculus of arbitrary order and its electrodynamical origin, in *Proc. 19th Nordic Congress Mathenzaticians* (Icelandic Mathematical Society, Reykjavik, 1985), pp. 149–160.

[29] L. M. Zelenyi and A. V. Milovanov, Fractal topology and strange kinetics: From percolation theory to problems in cosmic electrodynamics, *Phys. Uspekhi* **47** (2004) 749–788.

[30] V. E. Tarasov, Electromagnetic field of fractal distribution of charged particles, *Phys. Plasmas* **12** (2005) 082106.

[31] V. E. Tarasov, Multipole moments of fractal distribution of charges, *Mod. Phys. Lett. B* **19** (2005) 1107–1118.

[32] V. E. Tarasov, Magnetohydrodynamics of fractal media, *Phys. Plasmas* **13** (2006) 052107.

[33] V. E. Tarasov, Gravitational field of fractal distribution of particles, *Celes. Mech. Dynam. Astron.* **19** (2006) 1–15.

[34] F. Mainardi and R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes, *J. Comput. Appl. Math.* **118** (2000) 283–299.

[35] V. E. Tarasov, Fractional generalization of Liouville equation, *Chaos* **14** (2004) 123–127.

[36] V. E. Tarasov, Fractional systems and fractional Bogoliubov hierarchy equations, *Phys. Rev. E* **71** (2005) 011102.

[37] V. E. Tarasov, Fractional Liouville and BBGKI equations, *J. Phys. Conf. Ser.* **7** (2005) 17–33.

[38] V. E. Tarasov, Transport equations from Liouville equations for fractional systems, *Int. J. Mod. Phys. B* **20** (2006) 341–354.

[39] V. E. Tarasov, Fractional generalization of gradient and Hamiltonian systems, *J. Phys. A* **38** (2005) 5929–5943.

[40] V. E. Tarasov, Fractional generalization of gradient systems, *Lett. Math. Phys.* **73** (2005) 49–55.

[41] V. E. Tarasov, Fractional variations for dynamical systems: Hamilton and Lagrange approaches, *J. Phys. A* **39** (2006) 8049–8425.

[42] V. E. Tarasov and G. M. Zaslavsky, Dynamics with low-level fractionality, *Phys. A* **368** (2006) 399–415.

[43] V. E. Tarasov, Psi-series solution of fractional Ginzburg-Landau equation, *J. Phys. A* **39** (2006) 8395–8407.

[44] N. Laskin and G. M. Zaslavsky, Nonlinear fractional dynamics of lattice with long-range interaction, *Phys. A* **368** (2006) 38–54.

[45] V. E. Tarasov and G. M. Zaslavsky, Fractional dynamics of coupled oscillators with long-range interaction, *Chaos* **16** (2006) 023110.

[46] V. E. Tarasov and G. M. Zaslavsky, Fractional dynamics of systems with long-range interaction, *Commun. Nonlin. Sci. Numer. Simul.* **11** (2006) 885–898.

[47] N. Korabel, G. M. Zaslavsky and V. E. Tarasov, Coupled oscillators with power-law interaction and their fractional dynamics analogues, *Commun. Nonlin. Sci. Numer. Simul.* **12** (2007) accepted; math-ph/0603074.

[48] M. A. Krasnolelsky, P. P. Zabreiko, E. I. Pustilnik and P. E. Sobolevsky, *Integral Operators in Spaces of Integrable Functions* (Nauka, Moscow, 1966), Sec. 3.4 (in Russian).

[49] M. A. Krasnolelsky and P. E. Sobolevsky, *Dokladi Acad. Nauk SSSR* **129** (1959) 499–502.

[50] S. G. Krein, *Linear Differential Equations in Banach Space* (Nauka, Moscow, 1967), Sec. 1.5 (in Russian); and Translations of Mathematical Mongraphs, Vol. 29 (American Mathematical Society, Providence, 1971)

[51] Yu. L. Dalechky and S. G. Krein, *Stability of Solutions of Differential Equations in Banach Space* (Nauka, Moscow, 1970), Sec. 1.2 (in Russian) and (American Mathematical Society, Providence, 1974).

[52] S. G. Krein (ed.), *Function Analysis*, 2nd edn. (Nauka, Moscow, 1970) (in Russian).

[53] V. P. Maslov *Operational Methods*, Sec. 3.4. (Mir, Moscow, 1976).

[54] M. V. Karasev and V. P. Maslov, *Nonlinear Poisson Brackets. Geometry and Quantization* (American Mathematical Society, New York, 1993), Appendix 1.

[55] Y. Luke, *The Special Functions and Their Approximations* (Academic Press, New York, 1969) Vol. 1, Chap. 6.

[56] Yu. A. Brichkov and A. P. Prudnikov, *Integral Transformations of Generalized Functions* (Nauka, Moscow, 1977) (in Russian).

[57] F. A. Berezin and M. A. Shubin, *The Schroedinger Equation*, Sec. 5. (Kluwer, Dordrecht, 1991).

[58] F. A. Berezin, Non-Wiener functional integrals, *Theor. Math. Phys.* **6**(2) (1971) 194–212.

[59] F. A. Berezin, About one representation of operators by functionals, *Trudi Moskovskogo Matematicheskogo Obschestva* **17** (1967) 117–196.

[60] N. L. Balazs and B. K. Jennings, Wigner's function and other distribution functions in mock phase spaces, *Phys. Rep.* **104** (1984) 347–391.

[61] R. F. V. Anderson, The Weyl functional calculus, *J. Funct. Anal.* **4** (1969) 240–267.

[62] R. F. V. Anderson, On the Weyl functional calculus, *J. Funct. Anal.* **6** (1970) 110–115.

[63] V. E. Tarasov, Quantization of non-Hamiltonian and dissipative systems, *Phys. Lett. A* **288** (2001) 173–183.

[64] V. E. Tarasov, Weyl quantization of dynamical systems with flat phase space, *Moscow Univ. Phys. Bull.* **56**(6) (2001) 5–10.

[65] V. E. Tarasov, Quantization of non-Hamiltonian systems, *Theor. Phys.* **2** (2001) 150–160.

[66] V. E. Tarasov, Path integral for quantum operations, *J. Phys. A* **37** (2004) 3241–3257.

[67] V. E. Tarasov, *Mathematical Introduction to Quantum Mechanics* (MAI, Moscow, 2000) (in Russian).

[68] V. E. Tarasov, *Quantum Mechanics. Lectures on Theory Foundation*, 2nd edn. (Vuzovskaya kniga, Moscow, 2005) (in Russian).

[69] S. Momani and S. B. Hadid, Lyapunov stability solutions of fractional integrodifferential equations, *Int. J. Math. Math. Sci.* **47** (2004) 2503–2507.

[70] S. B. Hadid and J. G. Alshamani, Liapunov stability of differential equations of noninteger order, *Arab J. Math* **7** (1986) 5–17.

[71] Y. Q. Chen and K. L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, *Nonlin. Dynam.* **29** (2002) 191–200.

[72] T. D. Khusainov, Stability analysis of a linear-fractional delay system, *Differential Equations* **37** (2001) 1184–1188.

[73] D. Matignon, Stability result on fractional differential equations with applications to control processing, in *IMACS — SMC Proceeding* (Lille, France, 1996), pp. 963–968.