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DYNAMICS OF FRACTAL SOLIDS

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We describe the fractal solid by a special continuous medium model. We propose to describe the fractal solid by a fractional continuous medium model, where all characteristics and fields are defined everywhere in the volume but they follow some generalized equations which are derived by using fractional integrals of fractional order. The order of fractional integral can be equal to the fractal mass dimension of the solid. Fractional integrals are considered as an approximation of integrals on fractals. We suggest the approach to compute the moments of inertia for fractal solids. The dynamics of fractal solids are described by the usual Euler's equations. The possible experimental test of continuous medium model for fractal solids is considered.

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1. Introduction

Derivatives and integrals of fractional order[1–3] have many applications in recent studies in condensed matter physics. The interest to fractional analysis has grown continually in the last years. Fractional analysis has numerous applications: kinetic theories,[4–9] statistical mechanics of fractal systems,[10–12] dynamics in a complex or porous media,[13–18] electrodynamics,[19–24] and many others.

In order to use fractional derivatives and fractional integrals for fractal media, we must use some continuous medium model.[14] We propose to describe the fractal medium by a fractional continuous medium,[14,15] where all characteristics and fields are defined everywhere in the volume but they follow some generalized equations which are derived by using fractional integrals. In many problems the real fractal structure of matter can be disregarded and the medium can be replaced by some fractional continuous mathematical model. Smoothing of the microscopic characteristics over the physically infinitesimal volume transforms the initial fractal medium into fractional continuous model[14] that uses the fractional integrals. The order of fractional integral is equal to the fractal mass dimension of the medium.

The fractional integrals allow us to take into account the fractality of the media. In order to describe the fractal medium by continuous medium model we must use the fractional integrals which are considered as an approximation of integrals on fractals. In Ref. 25, authors prove that integrals on net of fractals can be approximated by fractional integrals. In Refs. 10–12, we proved that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor. To prove this we use the formulas of dimensional regularizations.[26]

In this paper, we use the fractional integrals to describe fractal solids. We consider the fractal solid by using the fractional continuous medium model. To describe a fractal solid, we use integrals of fractional order. We prove that equations of motion for fractal solid have the same form as the equations for usual solids. We suggest the approach to compute the moments of inertia for fractal solids and consider the possible experimental testing of the continuous medium model for fractal solids.

2. Mass of Fractal Solids

The fractal solid is characterized by the fractal dimensions. It is known that fractal dimension can best be calculated by box-counting method which means drawing a box of size R and counting the mass inside. The mass fractal dimension[27,28] can be easily measured for fractal solids. The properties of the fractal solid like mass obeys a power law relation $M \sim R^D$, where M is the mass of fractal solid, R is a box size (or a sphere radius), and D is a mass fractal dimension. The power law relation $M \sim R^D$ can be naturally derived by using the fractional integral. In Ref. 14, we prove that the mass fractal dimension is connected to the order of fractional integrals.

Let us consider the region W of solid in three-dimensional Euclidean space E^3. The volume of the region W is denoted by $V(W)$. The mass of the region W in the fractal solid is denoted by $M(W)$. The fractality of solid means that the mass of this solid in any region W of Euclidean space E^3 increases more slowly than the volume of this region. For the ball region of the fractal solid, this property can be described by the power law $M \sim R^D$, where R is the radius of the ball W.

Fractal solid is called a homogeneous if the following property is satisfied: for all regions W and W' of the homogeneous fractal solid such that the volumes are equal $V(W) = V(W')$, we have the masses of these regions equal too i.e. $M(W) = M(W')$. Note that the wide class of the fractal media satisfies the homogeneous property. In Refs. 14 and 15, the fractional continuous medium model for the fractal media has been suggested. Note that the fractality and homogeneity properties in the fractional continuous model are realized in the following forms:

(1) Homogeneity: The local density of the homogeneous fractal solid in the continuous model has the form $\rho(\mathbf{r}) = \rho_0 = \text{const}$.

(2) Fractality: The mass of the ball region W of fractal solid obeys a power law relation $M \sim R^D$, where $D < 3$, R is the radius of the ball.

The mass of the region W in the solid with integer mass dimension is derived by the equation realized by the fractional generalization of the equation

$$M_3(W) = \int_W \rho(\mathbf{r}) dV_3 \,. \tag{1}$$

We can consider the fractional generalization of this equation. Let us define the fractional integral in Euclidean space E^3 in the Riesz form.[1] The fractional generalization of Eq. (1) can be realized in the following form

$$M_D(W) = \int_W \rho(\mathbf{r}) dV_D \,, \tag{2}$$

where $dV_D = c_3(D, \mathbf{r}) dV_3$, and

$$c_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}|^{D-3} \,. \tag{3}$$

Here, we use the initial points in the fractional integrals which are set to zero. The numerical factor in Eqs. (2) and (3) has this form in order to derive the usual integral in the limit $D \to (3-0)$. Note that the usual numerical factor $\gamma_3^{-1}(D) = \Gamma(1/2)/2^D \pi^{3/2} \Gamma(D/2)$, which is used in Ref. 1 leads to $\gamma_3^{-1}(3-0) = \Gamma(1/2)/2^3 \pi^{3/2} \Gamma(3/2) = 1/(4\pi^{3/2})$ in the limit $D \to (3-0)$.

In order to have the usual dimensions of the physical values, we can use vector $\mathbf{r}$ and coordinates x, y, z as dimensionless values.

We can rewrite Eq. (2) in the form

$$M_D(W) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W \rho(\mathbf{r}) |\mathbf{r}|^{D-3} dV_3 \,. \tag{4}$$

If we consider the homogeneous fractal solid ($\rho(\mathbf{r}) = \rho_0 = \text{const.}$) and the ball region $W = \{\mathbf{r} : |\mathbf{r}| \leq R\}$, then we have

$$M_D(W) = \rho_0 \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W |\mathbf{r}|^{D-3} dV_3 \,. \tag{5}$$

Using the spherical coordinates, we get

$$M_D(W) = \frac{\pi 2^{5-D}\Gamma(3/2)}{\Gamma(D/2)} \rho_0 \int_W |\mathbf{r}|^{D-1} d|\mathbf{r}| = \frac{2^{5-D}\pi\Gamma(3/2)}{D\Gamma(D/2)} \rho_0 R^D \,. \tag{6}$$

As a result, we have $M(W) \sim R^D$, i.e., we derive equation $M \sim R^D$ up to the numerical factor. Therefore, the fractal solid with non-integer mass dimension D can be described by fractional integral of order D. Note that the interpretation of the fractional integration is connected with fractional dimension.[10,11] This interpretation follows from the well-known formulas for dimensional regularizations.[26] The fractional integral can be considered as a integral in the fractional dimension space up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$.

3. Moment of Inertia of Fractal Solids

3.1. *Fractional equation for moment of inertia*

The moment of inertia of a solid body with density $\rho(\mathbf{r})$ with respect to a given axis is defined by the volume integral

$$I = \int_W \rho(\mathbf{r})\mathbf{r}_\perp^2 \, dV_3 \,, \tag{7}$$

where $\mathbf{r}_\perp^2$ is the perpendicular distance from the axis of rotation. This can be broken into components as

$$I_{kl} = \int_W \rho(\mathbf{r})(\mathbf{r}^2\delta_{kl} - x_k x_l) dV_3 \,, \tag{8}$$

for a continuous mass distribution. Here, $\mathbf{r} = x_k \mathbf{e}_k$ is the distance to a point (not the perpendicular distance) and δ_{kl} is the Kronecker delta. Depending on the context, I_{kl} may be viewed either as a tensor or a matrix.

The fractional generalization of Eq. (8) has the form

$$I_{kl}^{(D)} = \int_W \rho(\mathbf{r})(\mathbf{r}^2\delta_{kl} - x_k x_l) dV_D \,, \tag{9}$$

where $dV_D = c_3(D, \mathbf{r})dV_3$. The moment of inertia tensor is symmetric ($I_{kl}^{(D)} = I_{lk}^{(D)}$).

The principal moments are given by the entries in the diagonalized moment of inertia matrix. The principal axes of a rotating body are defined by finding values of λ such that

$$(I_{kl}^{(D)} - \lambda\delta_{kl})\omega_l = 0 \,, \tag{10}$$

which is an eigenvalue problem. Here, $\boldsymbol{\omega} = \omega_k \mathbf{e}_k$ is the angular velocity vector. The tensor $I_{kl}^{(D)}$ may be diagonalized by transforming to appropriate coordinate system. The moments of inertia in the coordinate system, corresponding to the eigenvalues of the tensor, are known as principal moments of inertia.

3.2. *Moment of inertia of fractal solid sphere*

For a fractal solid sphere with radius R, and mass M, the moment of inertia can be derived by Eq. (9). The moment of inertia can be computed directly by noting that the component of the radius perpendicular to the z-axis in spherical coordinates is

$$\mathbf{r}_\perp^2 = (r \, \sin \, \phi)^2 \,, \tag{11}$$

where ϕ is the angle from the z-axis. Using the fractional generalization of Eq. (7), we have

$$I_z^{(D)} = \int_W \rho(\mathbf{r})\mathbf{r}_\perp^2 \, dV_D$$

$$= \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(\mathbf{r})(r \, \sin \, \phi)^2 r^{D-1} \, \sin \, \phi d\phi d\theta dr$$

$$= \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(\mathbf{r}) r^{D+1} \sin^3 \phi d\phi d\theta dr$$

$$= \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(\mathbf{r}) r^{D+1} (1 - \cos^2 \phi) \sin \phi d\phi d\theta dr \, .$$

Making the change of variables

$$u = \cos \phi \, , \qquad du = - \sin \phi d\phi \, , \tag{12}$$

we then allow the integral to be written simply and solved by quadrature. For homogeneous fractal solid sphere ($\rho(\mathbf{r}) = \rho_0$), we have

$$I_z^{(D)} = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_{-1}^{1} \rho_0 r^{D+1} (1 - u^2) du d\theta dr$$

$$= \frac{2^{5-D}\Gamma(3/2)}{3\Gamma(D/2)} \int_0^R \int_0^{2\pi} \rho_0 r^{D+1} d\theta dr$$

$$= \frac{\pi 2^{6-D}\Gamma(3/2)}{3\Gamma(D/2)} \rho_0 \int_0^R r^{D+1} dr \, .$$

As a result, we get

$$I_z^{(D)} = \frac{\pi 2^{6-D}\Gamma(3/2)}{3(D+2)\Gamma(D/2)} \rho_0 R^{D+2} \, . \tag{13}$$

The mass of the fractal solid sphere is defined by Eq. (2). Therefore, we have

$$M_D = \int_W \rho(\mathbf{r}) dV_D = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(\mathbf{r}) r^{D-1} \sin \phi d\phi d\theta dr$$

$$= \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(\mathbf{r}) r^{D-1} \sin \phi d\phi d\theta dr \, .$$

Making the change of variables (12) then allows the integral to be written simply and solved by quadrature. For homogeneous solid sphere ($\rho(\mathbf{r}) = \rho_0$), we get

$$M_D = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \int_{-1}^{1} \rho_0 r^{D+1} du d\theta dr$$

$$= \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \rho_0 r^{D+1} (u)_{-1}^{+1} d\theta dr$$

$$= \frac{2^{4-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \int_0^{2\pi} \rho_0 r^{D-1} d\theta dr$$

$$= \frac{\pi 2^{5-D}\Gamma(3/2)}{\Gamma(D/2)} \rho_0 \int_0^R r^{D-1} dr \, .$$

As a result, we have

$$M_D = \frac{\pi 2^{5-D}\Gamma(3/2)}{D\Gamma(D/2)} \rho_0 R^D \, . \tag{14}$$

Substituting ρ_0 from Eq. (14) in Eq. (13), we get the moment of inertia for fractal solid sphere in the form

$$I_z^{(D)} = \frac{2D}{3(D+2)} M_D R^2 \,. \tag{15}$$

If $D = 3$, then we have the usual relation $I_z^{(3)} = (2/5)MR^2$. If $D = (2+0)$, then we have $I_z^{(2+0)} = (1/3)MR^2$. Note that fractal solid sphere with dimension $D = (2+0)$ cannot be considered as a spherical shell that has $I_z = (2/3)MR^2$. In fractal solid sphere, we have the homogeneous distribution of fractal matter in the volume.

Because of the symmetry of the sphere, each principal moment is the same, so the moment of inertia of the sphere taken about any diameter is Eq. (15).

The moments of inertia $I_z^{(D)}$ and $I_z^{(3)}$ are connected by the relation

$$I_z^{(D)}/I_z^{(3)} = 1 + \frac{2(D-3)}{3(D+2)} \,. \tag{16}$$

Using $2 < D \leq 3$, we get $(5/6) < I_z^{(D)}/I_z^{(3)} \leq 1$.

3.3. *Moment of inertia for fractal solid cylinders*

The equation for the moment of inertia of homogeneous cylinder with integer mass dimension has the well-known form

$$I_z^{(2)} = \rho_0 \int_S (x^2+y^2) dS_2 \int_L dz \,. \tag{17}$$

Here, z is the cylinder axis, and $dS_2 = dxdy$. The fractional generalization of Eq. (17) can be defined by the equation

$$I_z^{(\alpha)} = \rho_0 \int_S (x^2+y^2) dS_\alpha \int_L dl_\beta \,, \tag{18}$$

where we use the following notations

$$dS_\alpha = c(\alpha)(\sqrt{x^2+y^2})^{\alpha-2} dS_2 \,, \qquad dS_2 = dxdy \,,$$
$$c(\alpha) = \frac{2^{2-\alpha}}{\Gamma(\alpha/2)} \,, \qquad dl_\beta = \frac{|z|^{\beta-1}}{\Gamma(\beta)} dz \,. \tag{19}$$

The numerical factor in Eq. (18) has this form in order to derive usual integral in the limit $\alpha \to (2-0)$ and $\beta \to (1-0)$. The parameters α and β are

$$1 < \alpha \leq 2 \,, \qquad 0 < \beta \leq 1 \,.$$

If $\alpha = 2$ and $\beta = 1$, then Eq. (18) has form (17). The parameter α is a fractal mass dimension of the cross-section of the cylinder. This parameter can be easily calculated from the experimental data. It can be calculated by box-counting method for the cross-section of the cylinder.

Substituting Eq. (19) in Eq. (18), we get

$$I_z^{(\alpha)} = \frac{\rho_0 c(\alpha)}{\Gamma(\beta)} \int_S (x^2+y^2)^{\alpha/2} dS_2 \int_0^H z^{\beta-1} dz \,. \tag{20}$$

Here, we consider the cylindrical region W that is defined by the relations

$$L = \{z : 0 \leq z \leq H\}\,, \qquad S = \{(x, y) : 0 \leq x^2 + y^2 \leq R^2\}\,. \tag{21}$$

Using the cylindrical coordinates (ϕ, r, z), we have

$$dS_2 = dxdy = rdrd\phi\,, \qquad (x^2 + y^2)^{\alpha/2} = r^\alpha\,. \tag{22}$$

Therefore the moment of inertia is defined by

$$I_z^{(\alpha)} = \frac{2\pi\rho_0 c(\alpha)}{\Gamma(\beta)} \int_0^R r^{\alpha+1} dr \int_0^H z^{\beta-1} dz = \frac{2\pi\rho_0 c(\alpha)}{(\alpha+2)\beta\Gamma(\beta)} R^{\alpha+2} H^\beta\,. \tag{23}$$

As a result, we have the moment of inertia of the fractal solid cylinder in the form

$$I_z^{(\alpha)} = \frac{2\pi\rho_0 c(\alpha)}{(\alpha+2)\beta\Gamma(\beta)} R^{\alpha+2} H^\beta\,. \tag{24}$$

If $\alpha = 2$ and $\beta = 1$, we get $I_z^{(2)} = (1/2)\pi\rho_0 R^4 H$.

The mass of the usual homogeneous cylinder (21) is defined by the equation

$$M = \rho_0 \int_S dS_2 \int_L dz = 2\pi\rho_0 \int_0^R rdr \int_0^H dz = \pi\rho_0 R^2 H\,. \tag{25}$$

We can consider the fractional generalization of this equation. The mass of the fractal solid cylinder (21) can be defined by the equation

$$M_\alpha = \rho_0 \int_S dS_\alpha \int_L dl_\beta\,, \tag{26}$$

where dS_α and dl_β are defined by Eq. (19). Using the cylindrical coordinates, we get the mass of fractal solid cylinder in the form

$$M_\alpha = \frac{2\pi\rho_0 c(\alpha)}{\Gamma(\beta)} \int_0^R r^{\alpha-1} dr \int_0^H z^{\beta-1} dz = \frac{2\pi\rho_0 c(\alpha)}{\alpha\beta\Gamma(\beta)} R^\alpha H^\beta\,. \tag{27}$$

As a result, we have

$$M_\alpha = \frac{2\pi\rho_0 c(\alpha)}{\alpha\beta\Gamma(\beta)} R^\alpha H^\beta\,. \tag{28}$$

Substituting mass (28) into the moment of inertia (24), we get the relation

$$I_z^{(\alpha)} = \frac{\alpha}{\alpha+2} M_\alpha R^2\,. \tag{29}$$

Note that Eq. (29) has not the parameter β. If $\alpha = 2$, we have the well-known relation $I_z^{(2)} = (1/2)MR^2$ for the homogeneous cylinder that has the integer mass dimension $D = 3$ and $\alpha = 2$.

Let us consider the fractal solid cylinder with the mass and radius that are equal to mass and radius of the homogeneous solid cylinder with integer mass dimension. In this case, the moments of inertia of these cylinders are connected by the equation

$$I_z^{(\alpha)} = \frac{2\alpha}{\alpha+2} I_z^{(2)}\,. \tag{30}$$

Here, $I_z^{(2)}$ is the moment of inertia for the cylinder with integer mass dimension $D = 3$ and $\alpha = 2$. For example, the parameter $\alpha = 1.5$ leads us to the relation $I^{(3/2)} = (6/7)I_z^{(2)}$. Using $1 \leq \alpha \leq 2$, we have the relation

$$(2/3) \leq I_z^{(\alpha)}/I_z^{(2)} \leq 1\,. \tag{31}$$

As a result, the fractal solid cylinder with the mass M, and radius R, has the moment of inertia $I_z^{(\alpha)}$ such that

$$I_z^{(\alpha)}/I_z^{(2)} = 1 + \frac{\alpha - 2}{\alpha + 2}\,, \tag{32}$$

where α is a fractal mass dimension of cross-section of the cylinder ($1 < \alpha \leq 2$). The parameter α can be calculated by box-counting method for the cross-section of the cylinder. Here, $I^{(2)}$ is the moment of inertia of usual cylinder with the mass M, and radius R.

4. Equations of Motion for Fractal Solids

4.1. *Euler's equations for fractal solids*

The moment of momentum $\mathbf{L} = L_k \mathbf{e}_k$ is defined by the equation

$$\mathbf{L} = \int_W [\mathbf{r}, \mathbf{v}]\rho(\mathbf{r})dV_3\,, \tag{33}$$

where $[\,,\,]$ is a vector product. The vector $\mathbf{r} = x_k \mathbf{e}_k$ is a radius vector, and $\mathbf{v} = v_k \mathbf{e}_k$ is a velocity of points with masses $dM_3 = \rho(\mathbf{r})dV_3$. The fractional generalization of Eq. (33) has the form

$$\mathbf{L}^{(D)} = \int_W [\mathbf{r}, \mathbf{v}]\rho(\mathbf{r})dV_D\,, \tag{34}$$

where we use $dM_D = \rho(\mathbf{r})dV_D$. Using $\mathbf{v} = [\boldsymbol{\omega}, \mathbf{r}]$, we get moment of momentum in the form

$$L_k^{(D)} = I_{kl}^{(D)} \omega_l\,. \tag{35}$$

The moment of inertia tensor $I_{kl}^{(D)}$ is related to the angular momentum vector $\mathbf{L}^{(D)}$ by Eq. (35), where $\boldsymbol{\omega} = \omega_k \mathbf{e}_k$ is the angular velocity vector.

For a fractal solid with one point fixed, if the angular momentum $\mathbf{L}^{(D)}$ is measured in the frame of the rotating body, we have the equation

$$\frac{d\mathbf{L}^{(\mathbf{D})}}{dt} + [\omega, \mathbf{L}^{(D)}] = \mathbf{N}\,, \tag{36}$$

where $\boldsymbol{\omega}$ is the angular velocity vector and $\mathbf{N} = N_k \mathbf{e}_k$ is the torque (moment of force). For components, we have

$$\frac{dL_k^{(D)}}{dt} + \varepsilon_{klm} \omega_l L_m^{(D)} = N_k\,, \tag{37}$$

where L_k are defined by the relations

$$L_k^{(D)} = \int_W \rho(\mathbf{r}) \varepsilon_{klm} x_l v_m dV_D \,. \tag{38}$$

Here, ε_{klm} is the permutation symbol, $\boldsymbol{\omega} = \omega_k \mathbf{e}_k$ is the angular frequency, and $\mathbf{N} = N_k \mathbf{e}_k$ is the external torque.

If the principle body axes are chosen, $L_k^{(D)} = I_k^{(D)} \omega_k$, then

$$\frac{d(I_k^{(D)} \omega_k)}{dt} + \varepsilon_{klm} \omega_l \omega_m I_m^{(D)} = N_k \,. \tag{39}$$

These are Euler's equations of motion. Taking the principal axes frame, we get the Euler's equations of motion for fractal solid in the form

$$I_x^{(D)} \frac{d\omega_x}{dt} + (I_z^{(D)} - I_y^{(D)}) \omega_y \omega_z = N_x \,,$$

$$I_y^{(D)} \frac{d\omega_y}{dt} + (I_x^{(D)} - I_z^{(D)}) \omega_x \omega_z = N_y \,,$$

$$I_z^{(D)} \frac{d\omega_z}{dt} + (I_y^{(D)} - I_x^{(D)}) \omega_x \omega_y = N_z \,,$$

where $I_x^{(D)}$, $I_y^{(D)}$, and $I_z^{(D)}$ are the principal moments of inertia. As a result, we proved that equations of motion for fractal solid have the same form as the equations for usual solids.

For general non-rigid motion, the equation of motion is Liouville's equation[10,11] which can be considered as the generalization of Euler's equations of motion to systems that are not rigid. In Eulerian form, the rotating axes are chosen to coincide with the instantaneous principle axes of the continuous system. For general non-rigid motion, Euler's equations are then replaced by Eq. (36) or, in component form (37). The extension of the Liouville equation to include collisions is known as the Bogoliubov equations.[11,12]

4.2. *Pendulum with fractal solids*

In this section, we consider the possible experimental testing of the continuous medium model for fractal solids. In this test we suggest measuring the period of pendulum with a fractal solid which is unequal to the period of the usual solid with the same mass and form.

Let us consider the Maxwell pendulum with the fractal solid cylinder. Usually, the Maxwell pendulum is used to demonstrate transformations between gravitational potential energy and rotational kinetic energy. The device has some initial gravitational potential energy when the string is winding on the small axis. When released, this gravitational potential energy is converted into rotational kinetic energy, with a lesser amount of translational kinetic energy. We consider the Maxwell pendulum as a cylinder that is suspended by string. The string is wound on the cylinder.

The equations of motion for Maxwell pendulum have the form

$$M_\alpha \frac{dv_y}{dt} = M_\alpha g - T \,, \qquad I_z^{(\alpha)} \frac{d\omega_z}{dt} = RT \,, \tag{40}$$

where g is the acceleration such that $g \simeq 9.81$ (m/s^2); the axis z is a cylinder axis, T is a string tension, M_α is a mass of the cylinder. Using $v_y = \omega_z R$, we have

$$M_\alpha \frac{dv_y}{dt} = M_\alpha g - \frac{I_z^{(\alpha)}}{R^2} \frac{dv_y}{dt} \,.$$

As a result, we get the acceleration of the cylinder

$$a_y^{(\alpha)} = \frac{dv_y}{dt} = \frac{M_\alpha g}{M_\alpha + I_z^{(\alpha)}/R^2} \,. \tag{41}$$

Substituting Eq. (29) in Eq. (41), we get

$$a_y^{(\alpha)} = \left(1 - \frac{\alpha}{2\alpha + 2}\right) g \,. \tag{42}$$

For the fractal mass dimension of the cross-section of the cylinder $\alpha = 1.5$, we get $a_y^{(\alpha)} = (3/5)g \simeq 6.87$ (m/s^2). For the cylinder with integer mass dimension of the cross-section ($\alpha = 2$), we have $a_y^{(2)} = (2/3)g \simeq 6.54$ (m/s^2). The period $T_0^{(\alpha)}$ of oscillation for this Maxwell pendulum is defined by the equation

$$T_0^{(\alpha)} = 4t_0 = 4\sqrt{2L/a_y^{(\alpha)}} \,,$$

where L is a string length, and the time t_0 satisfies the equation $a_y^{(\alpha)} t_0^2/2 = L$. Therefore, we get the relation for the periods

$$(T_0^{(\alpha)}/T_0^{(2)})^2 = 1 + \frac{1}{3} \frac{\alpha - 2}{\alpha + 2} \,. \tag{43}$$

If we consider $2 < D < 3$ such that $1 < \alpha < 2$, we can see that

$$(8/9) < \left(T_0^{(\alpha)}/T_0^{(2)}\right)^2 < 1 \,. \tag{44}$$

Note the parameter α can be calculated by box-counting method for the cross-section of the cylinder. For $\alpha = 1.5$, we have $(T_0^{(\alpha)}/T_0^{(2)})^2 = 0.952$.

A simple experiment to test the fractional continuous model[14,15] for fractal media is proposed. This experiment allows us to prove that the fractional integrals can be used to describe fractal media. For example, the experiment can be realized by using the sandstone. Note that Katz and Thompson[29] presented experimental evidence indicating that the pore spaces of a set of sandstone samples are fractals and are self-similar over three to four orders of magnitude in length extending from 10 angstrom to 100 μm. The deviation $T_0^{(\alpha)}$ from $T_0^{(2)}$ is no more that 6 per cent. Therefore, the precision of the experiments must be high.

5. Conclusion

In this paper we consider mechanics of fractal solids which are described by a fractional continuous medium model.[14,15] In the general case, the fractal solid cannot be considered as a continuous solid. There are points and domains that are not filled of particles. In Refs. 14 and 15, we suggest considering the fractal media as special (fractional) continuous media. We use the procedure of replacement of the medium with fractal mass dimension by some continuous model that uses the fractional integrals. This procedure is a fractional generalization of Christensen approach.[30] Suggested procedure leads to the fractional integration and differentiation to describe fractal media. The fractional integrals are considered as approximation of integrals on fractals.[25] Note that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor.[10–12] The fractional integrals are used to take into account the fractality of the media.

In this paper we suggest computing the moments of inertia for fractal solids. The simple experiments[17] to test the fractional continuous model[14,15] for fractal media can be performed. This experiment allows us to prove that the fractional integrals can be used to describe fractal solids.

Note that the fractional continuous models of fractal media can have a wide application. This is due in part to the relatively small numbers of parameters that define a random fractal medium of great complexity and rich structure. In order to describe the media with non-integer mass dimension, we must use the fractional calculus. Smoothing of the microscopic characteristics over the physically infinitesimal volume transform the initial fractal medium into fractional continuous model that uses the fractional integrals. The order of fractional integral is equal to the fractal mass dimension of medium. The fractional continuous model allows us to describe dynamics for wide class fractal media.[8,9,15,16,23]

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