

TRANSPORT EQUATIONS FROM LIOUVILLE EQUATIONS FOR FRACTIONAL SYSTEMS

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We consider dynamical systems that are described by fractional power of coordinates and momenta. The fractional powers can be considered as a convenient way to describe systems in the fractional dimension space. For the usual space the fractional systems are non-Hamiltonian. Generalized transport equation is derived from Liouville and Bogoliubov equations for fractional systems. Fractional generalization of average values and reduced distribution functions are defined. Gasdynamic equations for fractional systems are derived from the generalized transport equation.

Keywords: Transport equation; Liouville equation; Bogoliubov equations; fractional integrals; gasdynamic equations.

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1. Introduction

Integrals and derivatives of fractional order^{1,2} have found many applications in recent studies in physics.^{3–5} The interest in fractional equations has been growing continually during the last few years because of numerous applications: kinetic theories of systems with chaotic and pseudochaotic dynamics,^{6–15} electrodynamics,^{19–20} dynamical systems^{21–26} and many others.

The new type of problems has rapidly increased interest in areas in which the fractal features of a process or the medium impose a necessity of applying the tools that are non-traditional in “regular” smooth physical equations. In many problems the real fractal structure of matter can be disregarded and the medium can be replaced by some smoothed continuous mathematical model where fractional integrals appear.^{11,12,14,20} The order of fractional integral is equal to the fractal mass dimension of medium and in this way one can take into account the fractality of media. It was proved that integrals on net of fractals can be approximated by fractional integrals¹⁶ and that fractional integrals can be considered as integrals over the space with fractional dimension up to a numerical factor.^{17,18}

It is known that Bogoliubov equations can be derived from the Liouville equation and the definition of average value.^{29–32} In Ref. 17, the Liouville equation for fractional systems is derived from the fractional normalization condition. In Ref. 18, the first Bogoliubov equation for fractional systems is derived from the Liouville equation. It is known that gasdynamic equations can be derived from generalized transport equation²⁸ which is directly derived from first Bogoliubov equation.^{29,31,33} In this paper, the gasdynamic equations for fractional systems are derived from the generalized Enskog transport equations.

In Sec. 2, the fractional average values and some notations are considered. In Sec. 3, we define the reduced one-particle and two-particle distribution functions. In Sec. 4, the Liouville and first fractional Bogoliubov equations for fractional systems are considered. In Sec. 5, we derive the fractional analog of the Enskog transport equation. In Sec. 6, the gasdynamic equations for fractional systems are derived from the generalized transport equations. Finally, a short conclusion is given in Sec. 7.

2. Fractional Systems

Let us consider a dynamical system that is described by the coordinates \bar{q}_k and momenta \bar{p}_k that satisfy the equations of motion:

$$\frac{d\bar{q}_k}{d\bar{t}} = \frac{\bar{p}_k}{M}, \quad \frac{d\bar{p}_k}{d\bar{t}} = f_k(\bar{q}, \bar{p}, \bar{t}) \quad k = 1, \dots, n. \quad (1)$$

Let us introduce the *dimensionless variables*

$$q_k = \frac{\bar{q}_k}{q_0}, \quad p_k = \frac{\bar{p}_k}{p_0}, \quad t = \frac{\bar{t}}{t_0}, \quad F_k = \frac{f_k}{F_0}.$$

where q_0 is a characteristic scale in the configuration space; p_0 is a characteristic momentum, F_0 is a characteristic value of force, and t_0 is a typical time. Then Eq. (1) has the form

$$\frac{dq_k}{dt} = \frac{p_k}{m}, \quad \frac{dp_k}{dt} = Z F_k(q, p, t), \quad (2)$$

where

$$m = M q_0 / t_0 p_0, \quad Z = \frac{t_0 F_0}{p_0}, \quad (3)$$

are dimensionless parameters. Using the dimensionless variables (q, p, t) , we can generalize Eq. (2) for arbitrary powers of q_k and p_k :

$$\frac{dq_k^\alpha}{dt} = \frac{p_k^\alpha}{m}, \quad \frac{dp_k^\alpha}{dt} = Z F_k^\alpha(q^\alpha, p^\alpha, t), \quad (4)$$

where

$$q_k^\alpha = \beta(q)(q_k)^\alpha = \text{sgn}(q_k)|q_k|^\alpha, \quad (5)$$

$$p_k^\alpha = \beta(p)(p_k)^\alpha = \text{sgn}(p_k)|p_k|^\alpha. \quad (6)$$

Here $k = 1, \dots, n$, and $\beta(x) = [\text{sgn}(x)]^{\alpha-1}$. The function $\text{sgn}(x)$ is equal to $+1$ for $x \geq 0$, and -1 for $x < 0$.

A system is called a *fractional system* if the phase space is described by the fractional powers of coordinates (5) and momenta (6). We can describe the fractional systems in the usual phase space (q, p) and in the fractional phase space (q^α, p^α) . In the second case, the equations of motion for the fractional systems are more simple. Therefore we use the fractional phase space. The fractional space is considered as a space with the fractional measure that is used in the fractional integrals.

The generalization of Hamiltonian system is described by

$$\frac{dq_k^\alpha}{dt} = \frac{\partial H}{\partial p_k^\alpha}, \quad \frac{dp_k^\alpha}{dt} = -\frac{\partial H}{\partial q_k^\alpha}, \tag{7}$$

where H is an analog of the Hamiltonian. Using the fractional power Poisson brackets

$$\{A, B\}_\alpha = \sum_{k=1}^n \left(\frac{\partial A}{\partial q_k^\alpha} \frac{\partial B}{\partial p_k^\alpha} - \frac{\partial A}{\partial p_k^\alpha} \frac{\partial B}{\partial q_k^\alpha} \right), \tag{8}$$

we rewrite Eq. (7) in the form

$$\frac{dq_k^\alpha}{dt} = \{q_k^\alpha, H\}_\alpha, \quad \frac{dp_k^\alpha}{dt} = \{p_k^\alpha, H\}_\alpha. \tag{9}$$

These equations describe the system in the fractional phase space (q^α, p^α) . For the usual phase space (q, p) , Eq. (7) has the form

$$\frac{dq_k}{dt} = \frac{(q_k p_k)^{1-\alpha}}{\alpha^2} \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{(q_k p_k)^{1-\alpha}}{\alpha^2} \frac{\partial H}{\partial q_k}. \tag{10}$$

and these systems are non-Hamiltonian systems. A classical system is called Hamiltonian if the right-hand sides of the equations

$$\frac{dq_k}{dt} = g_k(q, p), \quad \frac{dp_k}{dt} = f_k(q, p) \tag{11}$$

satisfy the Helmholtz conditions:²⁷

$$\frac{\partial g_k}{\partial p_l} - \frac{\partial g_l}{\partial p_k} = 0, \quad \frac{\partial g_k}{\partial q_l} - \frac{\partial f_l}{\partial p_k} = 0, \quad \frac{\partial f_k}{\partial q_l} - \frac{\partial f_l}{\partial q_k} = 0. \tag{12}$$

It is easy to prove these conditions are not satisfied for Eq. (10). Therefore the dynamical system (10) is a non-Hamiltonian system. The fractional phase space allows us to write Eq. (9) in the simple form (7) and describe some non-Hamiltonian systems as a Hamiltonian systems in generalized space.

If $dq_k^\alpha/dt = p_k^\beta/m$, then an analog of Hamiltonian

$$H_{\alpha,\beta} = \sum_{k,l=1}^n \frac{\alpha p_k^{\alpha+\beta}}{m(\alpha+\beta)} + U(q). \tag{13}$$

The omega function for system (11) is defined by

$$\Omega = \sum_{k=1}^n \left(\frac{\partial g_k}{\partial q_k} + \frac{\partial f_k}{\partial p_k} \right), \tag{14}$$

and describes the velocity of phase volume change. If $\Omega < 0$, then the system is called a dissipative system. If $\Omega \neq 0$, then the system is a generalized dissipative system. For system (10), the omega function (14) is not equal to zero, and the systems is the general dissipative system.

It is not hard to prove that Hamiltonian (13) is connected with the non-Gaussian statistics. Dissipative and non-Hamiltonian systems can have the canonical Gibbs distribution as a solution of the stationary Liouville equations.⁴⁰ Using Ref. 40, it is easy to prove that some of fractional systems can have fractional Gibbs distribution

$$\rho(q, p) = \exp[\mathcal{F} - H_{\alpha,\beta}(q, p)]/kT, \tag{15}$$

as a solution of fractional Liouville equation.¹⁷

3. Fractional Average Values and Reduced Distributions

3.1. Fractional average values for configuration space

Let us derive the fractional generalization of average value of classical observable $A(q, p)$. For configuration space, the usual average value is

$$\langle A \rangle_1 = \int_{-\infty}^{\infty} A(x)\rho(x)dx, \tag{16}$$

and can be written

$$\langle A \rangle_1 = \int_{-\infty}^y A(x)\rho(x)dx + \int_y^{\infty} A(x)\rho(x)dx. \tag{17}$$

Using

$$(I_+^\alpha f)(y) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^y \frac{f(x)dx}{(y-x)^{1-\alpha}},$$

$$(I_-^\alpha f)(y) = \frac{1}{\Gamma(\alpha)} \int_y^{\infty} \frac{f(x)dx}{(x-y)^{1-\alpha}},$$

we rewrite Eq. (17) in the form

$$\langle A \rangle_1 = (I_+^1 A\rho)(y) + (I_-^1 A\rho)(y).$$

The fractional generalization of this equation is

$$\langle A \rangle_\alpha = (I_+^\alpha A\rho)(y) + (I_-^\alpha A\rho)(y). \tag{18}$$

We can rewrite Eq. (18) in the form

$$\langle A \rangle_\alpha = \frac{1}{2} \int_{-\infty}^{\infty} ((A\rho)(y-x) + (A\rho)(y+x))d\mu_\alpha(x), \tag{19}$$

where

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1}dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)}, \quad x^\alpha = \text{sgn}(x)|x|^\alpha. \tag{20}$$

Equation (18) defines the fractional generalization of the average value for coordinate space.

3.2. Fractional average values for phase space

Let us introduce some notations to define the fractional average value for phase space. Tilde operators

$$T_{x_k}f(\dots, x_k, \dots) = \frac{1}{2}(f(\dots, x'_k - x_k, \dots) + f(\dots, x'_k + x_k, \dots))$$

allows us to rewrite

$$\begin{aligned} &\frac{1}{4}(A(q' - q, p' - p, t)\rho(q' - q, p' - p, t) + A(q' + q, p' - p, t)\rho(q' + q, p' - p, t) \\ &+ A(q' - q, p' + p, t)\rho(q' - q, p' + p, t) + A(q' + q, p' + p, t)\rho(q' + q, p' + p, t)) \end{aligned}$$

in the simple form

$$T_q T_p (A(q, p, t)\rho(q, p, t)).$$

For k particle with coordinates q_{ks} and momenta p_{ks} , where $s = 1, \dots, m$, we define the operator

$$T[k] = T_{q_{k1}} T_{p_{k1}} \dots T_{q_{km}} T_{p_{km}}.$$

For the n -particle system phase space, we use

$$T[1, \dots, n] = T[1] \dots T[n].$$

Let us define the integral operators $\hat{I}_{x_k}^\alpha$ by

$$\hat{I}_{x_k}^\alpha f(x_k) = \int_{-\infty}^{+\infty} f(x_k) d\mu_\alpha(x_k), \tag{21}$$

then Eq. (19) has the form

$$\langle A \rangle_\alpha = \hat{I}_x^\alpha T_x A(x)\rho(x).$$

For k -particle we use the operator

$$\hat{I}^\alpha[k] = \hat{I}_{q_{k1}}^\alpha \hat{I}_{p_{k1}}^\alpha \dots \hat{I}_{q_{km}}^\alpha \hat{I}_{p_{km}}^\alpha,$$

such that

$$\hat{I}^\alpha[k]f(\mathbf{q}_k, \mathbf{p}_k) = \int f(\mathbf{q}_k, \mathbf{p}_k) d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k), \tag{22}$$

where $d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k)$ is an elementary $2m$ -dimensional phase volume

$$d\mu_\alpha(\mathbf{q}_k, \mathbf{p}_k) = (\alpha\Gamma(\alpha))^{-2m} dq_{k1}^\alpha \wedge dp_{k1}^\alpha \wedge \dots \wedge dq_{km}^\alpha \wedge dp_{km}^\alpha.$$

For the n -particle system, we use

$$\hat{I}^\alpha[1, \dots, n] = \hat{I}^\alpha[1] \cdots \hat{I}^\alpha[n].$$

Using the suggested notations, we can define the fractional generalization of the average value for n -particle by

$$\langle A \rangle_\alpha = \hat{I}^\alpha[1, \dots, n] T[1, \dots, n] A \rho_n. \tag{23}$$

In the simple case ($n = m = 1$), we have

$$\langle A \rangle_\alpha = \int_{-\infty}^\infty \int_{-\infty}^\infty d\mu_\alpha(q, p) T_q T_p A(q, p) \rho(q, p). \tag{24}$$

The fractional generalization of normalization condition¹⁷ can be written by

$$\langle 1 \rangle_\alpha = 1.$$

3.3. Reduced distribution functions

Let us consider a classical system with fixed number n of identical particles. Suppose k particle is described by the *dimensionless* generalized coordinates q_{ks} and generalized momenta p_{ks} , where $s = 1, \dots, m$. We use the notations $\mathbf{q}_k = (q_{k1}, \dots, q_{km})$ and $\mathbf{p}_k = (p_{k1}, \dots, p_{km})$. The state of this system is described by *dimensionless* n -particle distribution function ρ_n in the $2mn$ -dimensional phase space

$$\rho_n(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_n, \mathbf{p}_n, t). \tag{25}$$

We assume that function (25) is invariant under the permutations of identical particles:³⁴

$$\rho(\dots, \mathbf{q}_k, \mathbf{p}_k, \dots, \mathbf{q}_l, \mathbf{p}_l, \dots, t) = \rho(\dots, \mathbf{q}_l, \mathbf{p}_l, \dots, \mathbf{q}_k, \mathbf{p}_k, \dots, t).$$

Then the average values can be simplified.³⁴ Using the tilde distribution functions

$$\tilde{\rho}_n(\mathbf{q}, \mathbf{p}, t) = T[1, \dots, n] \rho_n(\mathbf{q}, \mathbf{p}, t), \tag{26}$$

we define

$$\tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = \tilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}^\alpha[2, \dots, n] \tilde{\rho}_n(\mathbf{q}, \mathbf{p}, t) \tag{27}$$

which is one-particle reduced distribution function. Obviously, that $\tilde{\rho}_1$ satisfies the normalization condition:¹⁷

$$\hat{I}^\alpha[1] \tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = 1. \tag{28}$$

Two-particle reduced distribution function $\tilde{\rho}_2$ is defined by the fractional integration of $\tilde{\rho}_n$ over all \mathbf{q}_k and \mathbf{p}_k , except $k = 1, 2$:

$$\tilde{\rho}_2(\mathbf{q}, \mathbf{p}, t) = \tilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \hat{I}^\alpha[3, \dots, n] \tilde{\rho}_n(\mathbf{q}, \mathbf{p}, t). \tag{29}$$

4. Liouville and Bogoliubov Equations for Fractional Systems

Let us consider the Hamilton's equations for n -particle system in the form

$$\frac{dq_{ks}^\alpha}{dt} = G_s^k(q^\alpha, p^\alpha), \quad \frac{dp_{ks}^\alpha}{dt} = ZF_s^k(q^\alpha, p^\alpha, t), \tag{30}$$

where Z is defined in Eq. (3). The evolution of ρ_n is described by the Liouville equation¹⁷ for fractional system

$$\frac{d\tilde{\rho}_n}{dt} + \Omega_\alpha \tilde{\rho}_n = 0. \tag{31}$$

This equation can be derived¹⁷ from the fractional normalization condition

$$\hat{I}^\alpha[1, \dots, n]\tilde{\rho}_n(\mathbf{q}, \mathbf{p}, t) = 1. \tag{32}$$

In Eq. (31) the derivative d/dt is a total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k,s=1}^{n,m} \frac{dq_{ks}}{dt} \frac{\partial}{\partial q_{ks}} + \sum_{k,s=1}^{n,m} \frac{dp_{ks}}{dt} \frac{\partial}{\partial p_{ks}}$$

that can be written for the fractional powers

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k,s=1}^{n,m} G_s^k \frac{\partial}{\partial q_{ks}^\alpha} + Z \sum_{k,s=1}^{n,m} F_s^k \frac{\partial}{\partial p_{ks}^\alpha}. \tag{33}$$

The α -omega function is

$$\Omega_\alpha = \sum_{k,s=1}^{n,m} (\{G_s^k, p_{ks}^\alpha\}_\alpha + Z\{q_{ks}^\alpha, F_s^k\}_\alpha), \tag{34}$$

where

$$\{A, B\}_\alpha = \sum_{k,s=1}^{n,m} \left(\frac{\partial A}{\partial q_{ks}^\alpha} \frac{\partial B}{\partial p_{ks}^\alpha} - \frac{\partial A}{\partial p_{ks}^\alpha} \frac{\partial B}{\partial q_{ks}^\alpha} \right). \tag{35}$$

Using Eqs. (33) and (34), we get Eq. (31) in the form

$$\frac{\partial \tilde{\rho}_n}{\partial t} = \Lambda_n \tilde{\rho}_n, \tag{36}$$

where Λ_n is Liouville operator:

$$\Lambda_n \tilde{\rho}_n = - \sum_{k,s=1}^{n,m} \left(\frac{\partial(G_s^k \tilde{\rho}_n)}{\partial q_{ks}^\alpha} + Z \frac{\partial(F_s^k \tilde{\rho}_n)}{\partial p_{ks}^\alpha} \right). \tag{37}$$

The Bogoliubov equations^{29–32} describe the evolution of the reduced distribution functions, and can be derived from the Liouville equation. In Ref. 18, we derive the first fractional Bogoliubov equation from Eq. (36):

$$\frac{\partial \tilde{\rho}_1}{\partial t} + \sum_{s=1}^m \frac{\partial(G_s^1 \tilde{\rho}_1)}{\partial q_{1s}^\alpha} + Z \sum_{s=1}^m \frac{\partial(F_s^{1e} \tilde{\rho}_1)}{\partial p_{1s}^\alpha} = (n-1)ZI(\tilde{\rho}_2). \tag{38}$$

Here $I(\tilde{\rho}_2)$ is a term with two-particle reduced distribution function,

$$I(\tilde{\rho}_2) = - \sum_{s=1}^m \frac{\partial}{\partial p_{1s}^\alpha} \hat{I}^\alpha[2] F_s^{12} \tilde{\rho}_2. \tag{39}$$

Equation (38) is called a *first Bogoliubov equation* for fractional systems.

The physical meaning of the term $I(\tilde{\rho}_2)$ is following: The term $I(\tilde{\rho}_2)d\mu_\alpha(\mathbf{q}, \mathbf{p})$ is a velocity of particle number change in $4m$ -dimensional elementary phase volume $d\mu_\alpha(\mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2, \mathbf{p}_2)$. This change is caused by the interactions between particles. If $\alpha = 1$, then we have the first Bogoliubov equation for non-Hamiltonian systems.

5. Transport Equation for Fractional Systems

Let us define the coordinate distribution (the density of number of particles) by the equation

$$n(\mathbf{q}, t) = \hat{I}^\alpha[\mathbf{p}] \tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t), \tag{40}$$

where $\hat{I}^\alpha[\mathbf{p}]$ is a fractional integration over the momenta

$$\hat{I}^\alpha[\mathbf{p}] = \prod_{s=1}^m \hat{I}_{p_s}^\alpha = \hat{I}_{p_1}^\alpha \dots \hat{I}_{p_m}^\alpha. \tag{41}$$

We can define the *local mean values* by

$$\langle A \rangle_{p,\alpha} = \langle A \rangle_{p,\alpha}(\mathbf{q}, t) = \frac{1}{n(\mathbf{q}, t)} \hat{I}^\alpha[\mathbf{p}] A(\mathbf{q}, \mathbf{p}) \tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t). \tag{42}$$

In the general case,

$$\langle A \rangle_{p,\alpha} \neq \langle A \rangle_\alpha, \tag{43}$$

and $\langle 1 \rangle_{p,\alpha} = 1$. The fractional average value $\langle A \rangle_\alpha$ is connected with the mean value $\langle A \rangle_{p,\alpha}$ by

$$\langle A \rangle_\alpha = \hat{I}^\alpha[\mathbf{q}] n(\mathbf{q}, t) \langle A \rangle_{p,\alpha},$$

where $\hat{I}^\alpha[\mathbf{q}]$ is a fractional integration over the coordinates $\mathbf{q} = (q_1, \dots, q_m)$.

Fractional analog of mean local velocity is

$$V_s(\mathbf{q}, t) = \frac{1}{n(\mathbf{q}, t)} \hat{I}^\alpha[\mathbf{p}] G_s(\mathbf{q}, \mathbf{p}) \rho_1(\mathbf{q}, \mathbf{p}, t), \tag{44}$$

i.e.

$$V_s(\mathbf{q}, t) = \langle G_s \rangle_{p,\alpha}, \tag{45}$$

where $G_s = G_s^1(\mathbf{q}, \mathbf{p})$ is defined by (30). We can consider G_s as a fractional generalization of the velocity that has the form

$$G_s = \frac{p_s^\alpha}{M}. \tag{46}$$

For the fractional generalization of kinetic energy of relative motion

$$\sum_{s=1}^m \frac{M}{2} (p_s^\alpha - V_s)^2, \tag{47}$$

we define density of this energy by

$$E(\mathbf{q}, t) = \frac{M}{2} \hat{I}^\alpha[\mathbf{p}] (\mathbf{G} - \mathbf{V})^2 \rho_1(\mathbf{q}, \mathbf{p}, t), \tag{48}$$

where $\mathbf{G} = \mathbf{p}^\alpha/m$, and $\mathbf{V} = \mathbf{V}(\mathbf{q}, t)$. The local temperature $T(\mathbf{q}, t)$ is defined by the mean kinetic energy of relative motion:

$$T(\mathbf{q}, t) = \frac{2E(\mathbf{q}, t)}{3k_B n(\mathbf{q}, t)}.$$

To derive Enskog transport equation for fractional systems, we multiply both sides of Eq. (38) by the observable $A(\mathbf{p})$, and integrate with respect to momenta. The first and second terms of left hand side of Eq. (38) are transformed by

$$\hat{I}[\mathbf{p}] A \frac{\partial \tilde{\rho}_1}{\partial t} = \frac{\partial}{\partial t} \hat{I}[\mathbf{p}] A \tilde{\rho}_1 = \frac{\partial}{\partial t} n(\mathbf{q}, t) \langle A \rangle_{p,\alpha}(\mathbf{q}, t), \tag{49}$$

$$\hat{I}[\mathbf{p}] A \frac{\partial (G_s \tilde{\rho}_1)}{\partial q_s^\alpha} = \frac{\partial}{\partial q_s^\alpha} \hat{I}[\mathbf{p}] A G_s \tilde{\rho}_1 = \frac{\partial}{\partial q_s^\alpha} n(\mathbf{q}, t) \langle A G_s \rangle_{p,\alpha}. \tag{50}$$

Integrating by part the third term of Eq. (38) and using the boundary condition

$$\lim_{p \rightarrow \pm\infty} \tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = 0, \tag{51}$$

we get

$$\begin{aligned} \hat{I}[\mathbf{p}] A \frac{\partial (F_s \tilde{\rho}_1)}{\partial p_s^\alpha} &= \hat{I}[\mathbf{p}] \frac{\partial}{\partial p_s^\alpha} A F_s \tilde{\rho}_1 - \hat{I}[\mathbf{p}] F_s \tilde{\rho}_1 \frac{\partial A}{\partial p_s^\alpha} \\ &= (A F_s \tilde{\rho}_1)_{-\infty}^{+\infty} - \hat{I}[\mathbf{p}] F_s \tilde{\rho}_1 \frac{\partial A}{\partial p_s^\alpha} = -n(\mathbf{q}, t) \left\langle F_s \frac{\partial A}{\partial p_s^\alpha} \right\rangle_{p,\alpha}. \end{aligned} \tag{52}$$

Then we use the usual assumption

$$\hat{I}[\mathbf{p}] A(\mathbf{p}) I(\tilde{\rho}_2) = 0. \tag{53}$$

for $A = M$, $A = p_s^\alpha$ and $A = \mathbf{p}^{2\alpha}$.

Finally, we obtain the Enskog transport equation for fractional systems:

$$\frac{\partial}{\partial t} (n(\mathbf{q}, t) \langle A \rangle_{p,\alpha}) + \frac{\partial}{\partial \mathbf{q}^\alpha} (n(\mathbf{q}, t) \langle A \mathbf{G} \rangle_{p,\alpha}) = n(\mathbf{q}, t) Z \left\langle \mathbf{F} \frac{\partial A}{\partial \mathbf{p}^\alpha} \right\rangle_{p,\alpha}, \tag{54}$$

where

$$\mathbf{F} \frac{\partial}{\partial \mathbf{p}^\alpha} = \sum_{s=1}^m F_s \frac{\partial}{\partial p_s^\alpha}, \quad \frac{\partial}{\partial \mathbf{q}^\alpha} (n(\mathbf{q}, t) \langle A \mathbf{G} \rangle_{p,\alpha}) = \sum_{s=1}^m \frac{\partial}{\partial q_s^\alpha} (n(\mathbf{q}, t) \langle A G_s \rangle_{p,\alpha}).$$

6. Gasdynamic Equation for Fractional Systems

Let us consider the special cases of transport equation (54) for

$$A = M, \quad A = p_s^\alpha = MG_s, \quad A = \frac{\mathbf{p}^{2\alpha}}{2M}.$$

If we use $A = M$, then Eq. (54) gives

$$\frac{\partial}{\partial t} \tilde{\rho}_M(\mathbf{q}, t) + \frac{\partial}{\partial q_s^\alpha} \tilde{\rho}_M(\mathbf{q}, t) \langle G_s \rangle_{p,\alpha}(\mathbf{q}, t) = 0, \tag{55}$$

where ρ_M is mass density

$$\tilde{\rho}_M(\mathbf{q}, t) = Mn(\mathbf{q}, t). \tag{56}$$

For $A = p_s^\alpha = MG_s$, we obtain

$$\frac{\partial}{\partial t} \tilde{\rho}_M V_l(\mathbf{q}, t) + \frac{\partial}{\partial q_s^\alpha} \tilde{\rho}_M \langle G_s G_l \rangle_{p,\alpha} = \tilde{\rho}_M(\mathbf{q}, t) Z \langle F_l \rangle_{p,\alpha}, \tag{57}$$

where we use Eqs. (46) and (56), and the relation

$$\frac{\partial G_l}{\partial p_s^\alpha} = M \delta_{ls}.$$

For $A = \mathbf{p}^{2\alpha}/2M$, we get

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\rho}_M(\mathbf{q}, t) \left\langle \frac{\mathbf{p}^{2\alpha}}{2M^2} \right\rangle_{p,\alpha} + \frac{\partial}{\partial q_s^\alpha} \tilde{\rho}_M(\mathbf{q}, t) \left\langle \frac{1}{2} G_l^2 G_s \right\rangle_{p,\alpha} \\ = \tilde{\rho}_M(\mathbf{q}, t) Z \langle F_s G_s \rangle_{p,\alpha}. \end{aligned} \tag{58}$$

Here $\langle F_s G_s \rangle_{p,\alpha}$ is a local mean value.

Let us define the deviation of velocity from its mean value by

$$C_s(\mathbf{q}, t) = G_s - V_s(\mathbf{q}, t) = G_s - \langle G_s \rangle_{p,\alpha}. \tag{59}$$

Substituting $G_s = V_s + C_s$ in the kinetic energy tensor $\langle G_s G_l \rangle_{p,\alpha}$, we get

$$\langle G_s G_l \rangle_{p,\alpha} = V_s V_s - \langle C_s C_l \rangle_{p,\alpha}, \tag{60}$$

where we use $\langle C_s \rangle_{p,\alpha} = 0$. From Eq. (60), we have

$$\left\langle \frac{\mathbf{p}^{2\alpha}}{2M} \right\rangle_{p,\alpha} = \frac{M\mathbf{V}^2}{2} + \frac{MC^2}{2}. \tag{61}$$

The tensor of internal stress

$$P_{sl} = \langle C_s C_l \rangle_{p,\alpha}, \tag{62}$$

can be represented as the sum

$$P_{sl}(\mathbf{q}, t) = \delta_{sl} P(\mathbf{q}, t) + \pi_{sl}(\mathbf{q}, t), \quad \pi_{ss}(\mathbf{q}, t) = 0,$$

where $\pi_{sl}(\mathbf{q}, t)$ is the tensor of viscous stress. Then

$$\begin{aligned} \frac{\rho_M}{2} \langle G_l^2 G_s \rangle_{p,\alpha} &= \frac{\rho_M}{2} \langle (V_l + C_l)^2 (V_s + C_s) \rangle_{p,\alpha} \\ &= \frac{\rho_M}{2} \langle (V_l^2 + C_l^2 + 2V_l C_l)(V_s + C_s) \rangle_{p,\alpha} \\ &= \frac{\rho_M}{2} (V_s^2 + \langle C_l^2 \rangle_{p,\alpha}) V_l + 2V_l P_{ls} + \frac{\rho_M}{2} \langle C_l^2 C_s \rangle_{p,\alpha}. \end{aligned} \quad (63)$$

Substituting Eq. (62) in Eq. (57), and substituting Eqs. (61) and (63) in Eq. (58), we get the equations for the functions $\rho_M(\mathbf{q}, t)$, $V_s(\mathbf{q}, t)$, $E(\mathbf{q}, t)$. From Eq. (55), we get the continuity equation for fractional systems:^{17,18}

$$\frac{\partial}{\partial t} \tilde{\rho}_M + \frac{\partial}{\partial q_s^\alpha} \tilde{\rho}_M V_s = 0, \quad (64)$$

which may be regarded as the equation of balance of ‘‘fractional matter’’. This matter can be described by fractional systems (30). In addition to ρ_M , the continuity equation (64) includes the density of momentum $\rho_M \mathbf{V}$. To obtain the equation for the density of momentum, we multiplied first fractional Bogoliubov equation by \mathbf{p}^α , and use fractional integration over \mathbf{p} . Taking advantage of the assumption (53) and the boundary condition (51), we get the equation for the components of the vector of density of momentum

$$\frac{\partial}{\partial t} \tilde{\rho}_M V_l + \frac{\partial}{\partial q_s^\alpha} (\tilde{\rho}_M V_s V_l + P_{sl}) = f_l, \quad (65)$$

where $f_l = f_l(\mathbf{q}, t) = Z \langle F_s \rangle_{p,\alpha}$.

Finally, to write down the equation of balance of fractional kinetic energy density, we multiplied first fractional Bogoliubov equation by \mathbf{p}^α , and use fractional integrated with respect to \mathbf{p} . Taking advantage of the assumption (53) and the boundary condition (51), we come to the equation for the density of kinetic energy of fractional systems:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \tilde{\rho}_M V^2 + E \right) + \frac{\partial}{\partial q_s^\alpha} \left(V_s \left[\frac{\tilde{\rho}_M V^2}{2} + E \right] + P_{sl} V_l + Q_s \right) = f_s V_s, \quad (66)$$

where

$$Q_s = \frac{\rho_M}{2} \langle C_l^2 C_s \rangle_{p,\alpha}.$$

Equations (64), (65), and (66) are the gasdynamic equations for fractional systems. Obviously, the set of this five equations is not closed. If we have $\pi_{sl} = 0$ and $Q_s = 0$, then these equations are

$$\frac{\partial}{\partial t} \tilde{\rho}_M + \frac{\partial}{\partial q_s^\alpha} \tilde{\rho}_M \langle G_s \rangle_{p,\alpha} = 0. \quad (67)$$

$$\frac{\partial}{\partial t} \tilde{\rho}_M V_l + \frac{\partial}{\partial q_s^\alpha} (\tilde{\rho}_M V_s V_l) = -\frac{\partial P}{\partial q_l^\alpha} + f_l, \quad (68)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \tilde{\rho}_M V^2 + E \right) + \frac{\partial}{\partial q_s^\alpha} \left(V_s \left[\frac{\tilde{\rho}_M V^2}{2} + E + P \right] \right) = f_s V_s, \quad (69)$$

and the set of equations is closed.

7. Conclusion

In this paper the gasdynamic equations for fractional systems are derived. In order to derive these equations, we use the first fractional Bogoliubov equation.¹⁸ Then we define the fractional generalization average values and the reduced distribution functions. The Enskog equation for fractional systems is considered. Gasdynamic equations (64), (65) and (66) can be considered as equations in the fractional space^{17,18} or for systems with non-Gaussian statistics.^{17,18}

Dissipative and non-Hamiltonian systems can have stationary states of the Hamiltonian systems.³⁹ Classical dissipative systems can have canonical Gibbs distribution as solutions of Liouville equations for the dissipative systems.^{14,40,41} Using the methods^{14,40} it is easy to find solutions for the Bogoliubov equations for fractional systems. Suggested Bogoliubov equation allows to formulate the dynamics for fractional generalization of quantum dissipative systems by methods suggested in Refs. 36–38.

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