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FOKKER–PLANCK EQUATION FOR FRACTIONAL SYSTEMS

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The normalization condition, average values, and reduced distribution functions can be generalized by fractional integrals. The interpretation of the fractional analog of phase space as a space with noninteger dimension is discussed. A fractional (power) system is described by the fractional powers of coordinates and momenta. These systems can be considered as non-Hamiltonian systems in the usual phase space. The generalizations of the Bogoliubov equations are derived from the Liouville equation for fractional (power) systems. Using these equations, the corresponding Fokker–Planck equation is obtained.

Keywords: Fokker–Planck equation; non-Hamiltonian systems; fractal; fractional integral.

1. Introduction

Fractional integrals and derivatives[1] have many applications in statistical mechanics and kinetics.[2,3] The generalization of the Fokker–Planck equation[4] can be used to describe kinetics in the fractal media. It is known that the Fokker-Planck equation can be derived from the Liouville and Bogoliubov equations.[5–7] The Liouville equation is obtained from the normalization condition and from the Hamilton equations. The Bogoliubov equations can be derived from the Liouville equation and from the definition of the average value. In this paper, the generalized Fokker–Planck equation is obtained from the Liouville and Bogoliubov equations for fractional (power) systems. For this aim, we use fractional generalizations of the normalization condition and the average values.[8–11]

In the paper, we suggest the physical interpretation of integrals of noninteger order. The fractional integral is considered as an integral on the fractal or noninteger-dimensional space. This interpretation is connected with the definition of noninteger dimension. We prove that fractional integration can be used to describe processes and systems on fractal. The physical values on fractals can be "averaged," and the distribution of the values on fractal can be replaced by some continuous distribution. To describe the distribution on the set with noninteger dimension, we use the fractional integrals. The order of the integral is equal to the fractal Hausdorff dimension of the set. The consistent approach to describe the distribution on fractal is connected with the mathematical definition of the integrals on fractals.[12–15] It was proved[12] that integrals on net of fractals can be approximated by fractional integrals. In Ref. 8–11, we proved that fractional integrals can be considered as integrals over the space with noninteger dimension up to a numerical factor. We use the well-known formulas of dimensional regularizations.[16] There is an interpretation that follows from the fractional measure of phase space,[8–11] which is used in the fractional integrals. The fractional phase space can be considered as a space that is described by the fractional powers of coordinates and momenta. Using this phase space, we can consider some of the non-Hamiltonian systems as generalized Hamiltonian systems.[8–11] The fractional systems can be described as exitations of the fractal medium.[8–11]

In Sec. 2, we consider the Hausdorff measure, the Hausdorff dimension, and the integration on fractals to fix notation and provide a convenient reference. The connections of the integration on fractals and the fractional integrals are discussed. The fractional average values and reduced distribution functions are defined. In Sec. 3, we derive Fokker–Planck equations from the Liouville equation for fractional (power) systems. A short conclusion is given in Sec. 4.

2. Integration on Fractal and Fractional Integration

2.1. *Hausdorff measure and Hausdorff dimension*

Fractals are measurable metric sets with a noninteger Hausdorff dimension. To define the Hausdorff measure and the Hausdorff dimension, we consider a measurable metric set (W, μ_H) with $W \subset R^n$. The elements of W are denoted by $x, y, z, \ldots$, and represented by n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$ such that W is embedded in R^n. The set W is restricted by the following conditions: (1) W is closed; (2) W is unbounded; (3) W is regular (homogeneous, uniform) with its points randomly distributed.

The diameter of a subset $E \subset W \subset R^n$ is

$$\mathrm{diam}(E) = \sup\{G(x,y) : x, y \in E\}\,,$$

where $G(x,y)$ is a metric function of two points x and $y \in W$.

Let us consider a set $\{E_i\}$ of subsets E_i such that $\mathrm{diam}(E_i) < \varepsilon \forall i$, and $W \subset \bigcup_{i=1}^{\infty} E_i$. Then, we define

$$\xi(E_i, D) = \omega(D)[\mathrm{diam}(E_i)]^D\,. \tag{1}$$

The factor $\omega(D)$ depends on the geometry of E_i. If $\{E_i\}$ is the set of all (closed or open) balls in W, then

$$\omega(D) = \frac{\pi^{D/2}2^{-D}}{\Gamma(D/2+1)}\,. \tag{2}$$

The Hausdorff dimension D of a subset $E \subset W$ is defined[17] by

$$D = \dim_H(E) = \sup\{d \in R : \mu_H(E,d) = \infty\} = \inf\{d \in R : \mu_H(E,d) = 0\}\,. \tag{3}$$

From definition (3), we obtain

(1) $\mu_H(E,d) = 0$ for $d > D = \dim_H(E)$;
(2) $\mu_H(E,d) = \infty$ for $d < D = \dim_H(E)$.

The Hausdorff measure μ_H of a subset $E \subset W$ is defined[17,18] by

$$\mu_H(E,D) = \omega(D) \lim_{\mathrm{diam}(E_i)\to 0} \inf_{\{E_i\}} \sum_{i=1}^{\infty} [\mathrm{diam}(E_i)]^D\,. \tag{4}$$

Note that $\mu_H(\lambda E, D) = \lambda^D \mu_H(E,D)$, where $\lambda > 0$, and $\lambda E = \{\lambda x, x \in E\}$.

2.2. *Function and integrals on fractal*

Let us consider the functions on W:

$$f(x) = \sum_{i=1}^{\infty} \beta_i \chi_{E_i}(x)\,, \tag{5}$$

where χ_E is the characteristic function of E: $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

The Lebesgue–Stieltjes integral for (5) is defined by

$$\int_W f d\mu = \sum_{i=1}^{\infty} \beta_i \mu_H(E_i)\,. \tag{6}$$

Therefore

$$\int_W f(x) d\mu_H(x) = \lim_{\mathrm{diam}(E_i)\to 0} \sum_{E_i} f(x_i)\xi(E_i,D)$$
$$= \omega(D) \lim_{\mathrm{diam}(E_i)\to 0} \sum_{E_i} f(x_i)[\mathrm{diam}(E_i)]^D\,. \tag{7}$$

It is possible to divide R^n into parallelepipeds

$$E_{i_1\cdots i_n} = \{(x_1,\ldots,x_n) \in W : x_j = (i_j - 1)\Delta x_j + \alpha_j, 0 \le \alpha_j \le \Delta x_j, j = 1,\ldots,n\}\,. \tag{8}$$

Then

$$d\mu_H(x) = \lim_{\mathrm{diam}(E_{i_1\cdots i_n})\to 0} \xi(E_{i_1\cdots i_n}, D)$$
$$= \lim_{diam(E_{i_1\cdots i_n})\to 0} \prod_{j=1}^{n} (\Delta x_j)^{D/n} = \prod_{j=1}^{n} d^{D/n} x_j\,. \tag{9}$$

The range of integration W can be parametrized by polar coordinates with $r = G(x, 0)$ and angle Ω. Then $E_{r,\Omega}$ can be thought of as a spherically symmetric covering around a center at the origin. In the limit, function $\xi(E_{r,\Omega}, D)$ gives

$$d\mu_H(r,\Omega) = \lim_{\mathrm{diam}(E_{r,\Omega})\to 0} \xi(E_{r,\Omega}, D) = d\Omega^{D-1} r^{D-1} dr \,. \tag{10}$$

Let us consider $f(x)$ that is symmetric with respect to some point $x_0 \in W$, i.e., $f(x) = \text{const.}$ for all x such that $G(x, x_0) = r$ for arbitrary values of r. Then the transformation

$$W \to W' : x \to x' = x - x_0 \tag{11}$$

can be performed to shift the center of symmetry. Since W is not a linear space, (11) need not be a map of W onto itself. Map (11) is measure preserving. Using (10), the integral over a D-dimensional metric space is defined by

$$\int_W f d\mu_H = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty f(r) r^{D-1} dr \,. \tag{12}$$

This integral is known in the theory of the fractional calculus.[1] The right Riemann–Liouville fractional integral is

$$I_-^D f(z) = \frac{1}{\Gamma(D)} \int_z^\infty (x-z)^{D-1} f(x) dx \,. \tag{13}$$

Equation (12) is reproduced by

$$\int_W f d\mu_H = \frac{2\pi^{D/2}\Gamma(D)}{\Gamma(D/2)} I_-^D f(0) \,. \tag{14}$$

Relation (14) connects the integral on fractal with the integral of fractional order. This result permits to apply different tools of the fractional calculus[1] for the fractal medium. As a result, the fractional integral can be considered as an integral on fractal up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$.

Note that the interpretation of fractional integral is connected with the fractional dimension.[8–11] This interpretation follows from the well-known formulas for dimensional regularizations.[16] The fractional integral can be considered as an integral in the noninteger-dimensional space up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$. It was proved[14] that the fractal space–time approach is technically identical with the dimensional regularization.

The integral defined in (7) satisfies the following properties:

(1) Linearity:

$$\int_X (af_1 + bf_2) d\mu_H = a \int_X f_1 d\mu_H + b \int_X f_2 d\mu_H \,, \tag{15}$$

where f_1 and f_2 are arbitrary functions; a and b are arbitrary constants.

(2) Translational invariance:

$$\int_X f(x + x_0) d\mu_H(x) = \int_X f(x) d\mu_H(x) \tag{16}$$

since $d\mu_H(x-x_0) = d\mu_H(x)$ as a consequence of homogeneity (uniformity).

(3) Scaling property:

$$\int_X f(ax)d\mu_H(x) = a^{-D}\int_X f(x)d\mu_H(x) \tag{17}$$

since $d\mu_H(x/a) = a^{-D}d\mu_H(x)$.

It has been shown[16] that conditions (15)–(17) define the integral up to normalization.[16]

2.3. *Multi-variable integration on fractal*

Integral (12) is defined for a single variable, and not multiple variables. It is useful for integrating spherically symmetric functions. This integral can be generalized for the multiple variables by using the product spaces and product measures.

Let us consider the measure spaces (W_k, μ_k, D) with $k = 1, 2, 3$, and form a Cartesian product of the sets W_k producing the space $W = W_1 \times W_2 \times W_3$. The definition of product measures and the application of the Fubini's theorem provides a measure for W as

$$(\mu_1 \times \mu_2 \times \mu_3)(W) = \mu_1(W_1)\mu_2(W_2)\mu_3(W_3)\,. \tag{18}$$

The integration over a function f on the product space is

$$\int f(\mathbf{r})d\mu_1 \times \mu_2 \times \mu_3 = \iiint f(x_1, x_2, x_3)d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3)\,. \tag{19}$$

In this form, the single-variable measure from (12) may be used for each coordinate x_k, which has an associated dimension α_k:

$$d\mu_k(x_k) = \frac{2\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)}|x_k|^{\alpha_k-1}dx_k\,, \quad k = 1, 2, 3\,. \tag{20}$$

The total dimension of $W = W_1 \times W_2 \times W_3$ is $D = \alpha_1 + \alpha_2 + \alpha_3$.

Let us reproduce the result (12) from (19). We take a spherically symmetric function $f(\mathbf{r}) = f(x_1, x_2, x_3) = f(r)$, where $r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. Equation (19) becomes

$$\begin{aligned}\int & d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3)f(x_1, x_2, x_3) \\ &= \frac{2\pi^{\alpha_1/2}}{\Gamma(\alpha_1/2)}\frac{2\pi^{\alpha_2/2}}{\Gamma(\alpha_2/2)}\frac{2\pi^{\alpha_3/2}}{\Gamma(\alpha_3/2)}\int dr\int d\phi\int d\theta\, J_3(r,\phi)r^{\alpha_1+\alpha_2+\alpha_3-3} \\ &\quad \times (\cos\phi)^{\alpha_1-1}(\sin\phi)^{\alpha_2+\alpha_3-2}(\sin\theta)^{\alpha_3-1}f(r)\,,\end{aligned} \tag{21}$$

where $J_3(r,\phi) = r^2\sin\phi$ is the Jacobian of the coordinate change.

To perform the integration in spherical coordinates (r, ϕ, θ), we use

$$\int_0^{\pi/2} \sin^{\mu-1}x\,\cos^{\nu-1}x\,dx = \frac{\Gamma(\mu/2)\Gamma(\nu/2)}{2\Gamma([\mu+\nu]/2)}\,, \tag{22}$$

where $\mu > 0$, $\nu > 0$. Then Eq. (21) becomes

$$\int d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3)f(r) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int f(r)r^{D-1}dr\,. \tag{23}$$

This equation describes integration over a spherically symmetric function in the D-dimensional space and reproduces result (12).

2.4. *Probability distribution on fractal*

The probability that is distributed in the three-dimensional Euclidean space is defined by

$$P_3(W) = \int_W \rho(\mathbf{r})dV_3\,, \tag{24}$$

where $\rho(\mathbf{r})$ is the density of probability distribution, and $dV_3 = dxdydz$ for the Cartesian coordinates.

If we consider the probability that is distributed on the measurable metric set W with the fractional Hausdorff dimension D, then the probability is defined by the integral

$$P_D(W) = \int_W \rho(\mathbf{r})dV_D\,, \tag{25}$$

where $D = \dim_H(W) = \alpha_1 + \alpha_2 + \alpha_3$, and

$$dV_D = d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3) = c_3(D,\mathbf{r})dV_3\,, \tag{26}$$

$$c_3(D,\mathbf{r}) = \frac{8\pi^{D/2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}|x|^{\alpha_1-1}|y|^{\alpha_2-1}|z|^{\alpha_3-1}\,. \tag{27}$$

There are many different definitions of fractional integrals.[1] For the Riemann–Liouville fractional integral, function $c_3(D,\mathbf{r})$ is

$$c_3(D,\mathbf{r}) = \frac{|x|^{\alpha_1-1}|y|^{\alpha_2-1}|z|^{\alpha_3-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\,, \tag{28}$$

where x, y, z are the Cartesian coordinates, and $D = \alpha_1 + \alpha_2 + \alpha_3$, $0 < D \leq 3$. As the result, we obtain the Riemann–Liouville fractional integral[1] in Eq. (25) up to numerical factor $8\pi^{D/2}$. Therefore, Eq. (25) can be considered as a fractional generalization of Eq. (24).

For $\rho(\mathbf{r}) = \rho(|\mathbf{r}|)$, we can use the Riesz definition of the fractional integrals.[1] Then

$$c_3(D,\mathbf{r}) = \frac{\Gamma(1/2)}{2^D\pi^{3/2}\Gamma(D/2)}|\mathbf{r}|^{D-3}\,. \tag{29}$$

Note that

$$\lim_{D\to 3-} c_3(D,\mathbf{r}) = (4\pi^{3/2})^{-1}\,. \tag{30}$$

Therefore, we suggest to use

$$c_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}|^{D-3} . \tag{31}$$

Definition (31) allows us to derive the usual integral in the limit $D \to (3-0)$.

For $D = 2$, Eq. (25) gives the fractal probability distribution in the volume. In general, it is not equivalent to the distribution on the two-dimensional surface. Equation (28) is equal (up to numerical factor $8\pi^{D/2}$) to the integral on the measurable metric set W with Hausdorff dimension $\dim_H(W) = D$. To have the usual dimensions of the physical values, we can use vector $\mathbf{r}$, and coordinates x, y, z as dimensionless values.

2.5. *Fractional average values*

To derive the fractional analog of the average value, we consider the fractional integral for function $f(x)$. If function $f(x)$ is equal to the distribution function $\rho(x)$, then we can derive the normalization condition. If function $f(x)$ is equal to the multiplication of distribution function $\rho(x)$ and classical observable $A(x)$, then we have the definition of the fractional average value.

The fractional generalization of the average value[8–11] can be presented by

$$\langle A \rangle_\alpha = (I_+^\alpha A\rho)(y) + (I_-^\alpha A\rho)(y) , \tag{32}$$

where

$$I_+^\alpha f = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{y} \frac{f(x)dx}{(y-x)^{1-\alpha}} , \quad I_-^\alpha f = \frac{1}{\Gamma(\alpha)} \int_{y}^{\infty} \frac{f(x)dx}{(x-y)^{1-\alpha}} . \tag{33}$$

For $\alpha = 1$, Eq. (32) gives the usual average value.

The fractional average value (32) can be written[8–11] as

$$\langle A \rangle_\alpha = \frac{1}{2} \int_{-\infty}^{\infty} [(A\rho)(y-x) + (A\rho)(y+x)] d\mu_\alpha(x) , \tag{34}$$

where

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1}dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)} . \tag{35}$$

Here, we use

$$x^\alpha = \mathrm{sgn}(x)|x|^\alpha , \tag{36}$$

where function $\mathrm{sgn}(x)$ is equal to $+1$ for $x \geq 0$, and -1 for $x < 0$.

Let us introduce notations to consider the fractional average value for phase space.

(1) The operator T_{x_k} is defined by

$$T_{x_k} f(\ldots, x_k, \ldots) = \frac{1}{2}(f(\ldots, x'_k - x_k, \ldots) + f(\ldots, x'_k + x_k, \ldots)) . \tag{37}$$

For k-particle, which is described by coordinates q_{ks} and momenta p_{ks} ($s = 1, \ldots, m$), the operator $T[k]$ is

$$T[k] = T_{q_{k1}} T_{p_{k1}} \cdots T_{q_{km}} T_{p_{km}} \,. \tag{38}$$

For the n-particle system, we define the operator $T[1, \ldots, n] = T[1] \cdots T[n]$.

(2) The operator $\hat{I}^{\alpha}_{x_k}$ is defined by

$$\hat{I}^{\alpha}_{x_k} f(x_k) = \int_{-\infty}^{+\infty} f(x_k) d\mu_{\alpha}(x_k) \,. \tag{39}$$

Then the fractional integral (34) can be rewritten in the form

$$\langle A \rangle_{\alpha} = \hat{I}^{\alpha}_{x} T_x A(x) \rho(x) \,.$$

The integral operator $\hat{I}^{\alpha}[k] = \hat{I}^{\alpha}_{q_{k1}} \hat{I}^{\alpha}_{p_{k1}} \cdots \hat{I}^{\alpha}_{q_{km}} \hat{I}^{\alpha}_{p_{km}}$ is

$$\hat{I}^{\alpha}[k] f(\mathbf{q}_k, \mathbf{p}_k) = \int f(\mathbf{q}_k, \mathbf{p}_k) d\mu_{\alpha}(\mathbf{q}_k, \mathbf{p}_k) \,, \tag{40}$$

where

$$d\mu_{\alpha}(\mathbf{q}_k, \mathbf{p}_k) = (\alpha \Gamma(\alpha))^{-2m} dq^{\alpha}_{k1} \wedge dp^{\alpha}_{k1} \wedge \cdots \wedge dq^{\alpha}_{km} \wedge dp^{\alpha}_{km} \,.$$

For the n-particle system, we use $\hat{I}^{\alpha}[1, \ldots, n] = \hat{I}^{\alpha}[1] \cdots \hat{I}^{\alpha}[n]$.

The fractional average values for the n-particle system is defined[8–11] by

$$\langle A \rangle_{\alpha} = \hat{I}^{\alpha}[1, \ldots, n] T[1, \ldots, n] A \rho_n \,. \tag{41}$$

In the simple case ($n = m = 1$), the fractional average value is

$$\langle A \rangle_{\alpha} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mu_{\alpha}(q, p) T_q T_p A(q, p) \rho(q, p) \,. \tag{42}$$

Note that the fractional normalization condition[8–11] is a special case of this definition of average values $\langle 1 \rangle_{\alpha} = 1$.

3. Fokker–Planck Equation from Liouville Equation

Let us consider a system with n identical particles and the Brownian particle. The distribution function of this system is denoted by $\rho_{n+1}(\mathbf{q}, \mathbf{p}, Q, P, t)$, where

$$\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \,, \quad \mathbf{q}_k = (q_{k1}, \ldots, q_{km}) \,,$$

$$\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \,, \quad \mathbf{p}_k = (p_{k1}, \ldots, p_{km})$$

are coordinates and momenta of the particles; $Q = (Q_s)$ and $P = (P_s)$ ($s = 1, \ldots, m$) are coordinates and momenta of Brownian particles. The fractional normalization condition[8–11] has the form

$$\hat{I}^{\alpha}[1, \ldots, n, n+1] \tilde{\rho}_{n+1} = 1 \,, \tag{43}$$

where

$$\tilde{\rho}_{n+1} = T[1, \ldots, n, n+1] \rho_{n+1}(\mathbf{q}, \mathbf{p}, Q, P, t) \,. \tag{44}$$

The reduced distribution function for the Brownian particle is

$$\tilde{\rho}_B(Q,P,t) = \hat{I}^\alpha[1,\dots,n]\tilde{\rho}_{n+1}(\mathbf{q},\mathbf{p},Q,P,t)\,. \tag{45}$$

The distribution $\tilde{\rho}_{n+1}$ satisfies the Liouville equation:[8–11]

$$\frac{\partial\tilde{\rho}_{n+1}}{\partial t} - i(L_n + L_B)\tilde{\rho}_{n+1} = 0\,, \tag{46}$$

where L_n and L_B are Liouville operators such that

$$-iL_n\rho = \sum_{k,s}^{n,m}\left(\frac{\partial(G_s^k\rho)}{\partial q_{ks}^\alpha} + \frac{\partial(F_s^k\rho)}{\partial p_{ks}^\alpha}\right), \tag{47}$$

$$-iL_B\rho = \sum_{k,s}^{n,m}\left(\frac{\partial(g_s\rho)}{\partial Q_s^\alpha} + \frac{\partial(f_s\rho)}{\partial P_s^\alpha}\right). \tag{48}$$

The forces F_s^k and f_s, and the velocities G_s^k and g_s are defined by the Hamilton equations of motion. For the kth particle,

$$\frac{dq_{ks}^\alpha}{dt} = G_s^k(\mathbf{q},\mathbf{p})\,, \quad \frac{dp_{ks}^\alpha}{dt} = F_s^k(\mathbf{q},\mathbf{p},Q,P)\,, \tag{49}$$

where $k = 1,\dots,n$. The Hamilton equations for the Brownian particle are

$$\frac{dQ_s^\alpha}{dt} = g_s(Q,P)\,, \quad \frac{dP_s^\alpha}{dt} = f_s(\mathbf{q},\mathbf{p},Q,P)\,. \tag{50}$$

For simplification, we suppose

$$G_s^k = p_{ks}^\alpha/m\,, \quad g_s = P_s^\alpha/M\,, \tag{51}$$

where $M \gg m$.

Let us use the boundary condition in the form

$$\lim_{t\to-\infty}\rho_{n+1}(\mathbf{q},\mathbf{p},Q,P) = \rho_n(\mathbf{q},\mathbf{p},Q,T)\rho_B(Q,P,t)\,, \tag{52}$$

where ρ_n is a canonical Gibbs distribution function

$$\rho_n(\mathbf{q},\mathbf{p},Q,T) = \exp\,\beta[\mathcal{F} - H(\mathbf{q},\mathbf{p},Q)]\,. \tag{53}$$

Here, $H(\mathbf{q},\mathbf{p},Q)$ is a Hamilton function

$$H(\mathbf{q},\mathbf{p},Q) = H_n(\mathbf{q},\mathbf{p}) + \sum_{k=1}^{n} U_B(\mathbf{q}_k,Q)\,, \tag{54}$$

where H_n is a Hamiltonian of the n-particle system, and U_B is an energy of interaction between particles and the Brownian particle. If we use Eqs. (49) and (51), then

$$H_n(\mathbf{q},\mathbf{p}) = \sum_{k,s}^{n,m}\frac{p_{ks}^{2\alpha}}{2m} + \sum_{k<l}U(\mathbf{q}_k,\mathbf{q}_l)\,. \tag{55}$$

The boundary condition (52) can be realized[19] by the infinitesimal source term in the Liouville equation

$$\frac{\partial \tilde{\rho}_{n+1}}{\partial t} - i(L_n + L_B)\tilde{\rho}_{n+1} = -\varepsilon(\tilde{\rho}_{n+1} - \tilde{\rho}_n \tilde{\rho}_B)\,. \tag{56}$$

Integrating (56) by $\hat{I}^\alpha[1,\ldots,n]$, we obtain the equation for the Brownian particle distribution

$$\frac{\partial \tilde{\rho}_B}{\partial t} + \sum_{s=1}^{m} \frac{\partial(g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} + \hat{I}^\alpha[1,\ldots,n] \sum_{s=1}^{m} \frac{\partial(f_s \rho_{n+1})}{\partial P_s^\alpha} = 0\,. \tag{57}$$

The formal solution[19] has the form

$$\tilde{\rho}_{n+1}(t) = \varepsilon \int_{-\infty}^{0} d\tau e^{\varepsilon\tau} e^{-i\tau(L_n+L_B)} \tilde{\rho}_B(t+\tau)\tilde{\rho}_n\,, \tag{58}$$

or

$$\tilde{\rho}_{n+1}(t) = \tilde{\rho}_B(t)\tilde{\rho}_n - \int_{-\infty}^{0} d\tau e^{\varepsilon\tau} e^{-i\tau(L_n+L_B)} \left(\frac{\partial}{\partial\tau} - i(L_n+L_B)\right) \tilde{\rho}_B(t+\tau)\tilde{\rho}_n\,. \tag{59}$$

Substituting (59) into (57), we get

$$\frac{\partial \tilde{\rho}_B}{\partial t} + \sum_{s=1}^{m} \frac{\partial(g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} + \sum_{s=1}^{m} \frac{\partial \rho_B}{\partial P_s^\alpha} \hat{I}^\alpha[1,\ldots,n](f_s \tilde{\rho}_n)$$
$$- \hat{I}^\alpha[1,\ldots,n] \sum_{s=1}^{m} \frac{\partial}{\partial P_s^\alpha} \int_{-\infty}^{0} d\tau\ e^{\varepsilon\tau} e^{-i\tau(L_n+L_B)}$$
$$\times \left(\frac{\partial}{\partial\tau} - i(L_n+L_B)\right) \tilde{\rho}_B(t+\tau)\tilde{\rho}_n = 0\,. \tag{60}$$

Note that $\hat{I}^\alpha[1,\ldots,n] f_s \tilde{\rho}_n$ can be considered as an average value of the force f_s. This average value for the canonical Gibbs distribution (53) is equal to zero. The last term can be simplified. Using

$$\frac{\partial \rho_n}{\partial Q_s^\alpha} = \frac{1}{kT} f_s^{(p)} \rho_n\,, \tag{61}$$

where $f_s^{(p)}$ is a potential force

$$f_s^{(p)} = -\frac{\partial U_B}{\partial Q_s^\alpha}\,, \tag{62}$$

we get

$$-iL_B \tilde{\rho}_{n+1} = \left(\frac{P_s f_s^{(p)t}}{MkT} \rho_B + \frac{\partial(g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} + \frac{\partial(f_s \tilde{\rho}_B)}{\partial P_s^\alpha}\right) \rho_n\,.$$

It can be proved by interactions that the term

$$\frac{\partial \tilde{\rho}_B}{\partial t} + \frac{\partial (g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} \tag{63}$$

in the integral of (60) does not contribute. Then

$$\frac{\partial \tilde{\rho}_B}{\partial t} + \sum_{s=1}^{m} \frac{\partial (g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} + \sum_{s=1}^{m} \frac{\partial}{\partial P_s^\alpha} \hat{I}^\alpha[1,\dots,n] \int_{-\infty}^{0} d\tau e^{\varepsilon\tau} f_s e^{-i\tau(L_n + L_B)} \tilde{\rho}_n$$

$$\cdot \left(\frac{\partial (f_{s'} \tilde{\rho}_B(t+\tau))}{\partial P_{s'}^\alpha} + \beta M^{-1} f_{s'} P_{s'} \tilde{\rho}_B(t+\tau) \right) = 0 \,. \tag{64}$$

This equation is a closed integro-differential equation for the reduced distribution function $\tilde{\rho}_B$. Note that force f_s can be presented in the form

$$f_s = f_s^{(p)} + f_s^{(n)} \,,$$

where $f_s^{(p)}$ is a potential force (62), and $f_s^{(n)}$ is a non-potential force that acts on the Brownian particle. For the equilibrium approximation $P \sim (MkT)^{1/2}$, $iL_B \sim M^{-1/2}$, $iL_n \sim m^{-1/2}$, and $M \gg m$, we can use the perturbation theory.

Using the approximation $\rho_B(t+\tau) = \rho_B(t)$ for Eq. (64), we obtain the Fokker–Planck equation for fractional power systems

$$\frac{\partial \tilde{\rho}_B}{\partial t} + \sum_{s=1}^{m} \frac{\partial (g_s \tilde{\rho}_B)}{\partial Q_s^\alpha} + \sum_{s=1}^{m} \frac{\partial}{\partial P_s^\alpha} \left(\frac{M}{\beta} \frac{\partial (\gamma_{ss'}^1 \tilde{\rho}_B(t))}{\partial P_{s'}^\alpha} + \gamma_{ss'}^2 P_{s'} \tilde{\rho}_B(t) \right) = 0 \,, \tag{65}$$

where

$$\gamma_{ss'}^1 = \beta M \hat{I}^\alpha[1,\dots,n] \int_{-\infty}^{0} d\tau e^{\varepsilon\tau} f_s e^{-i\tau L_n} f_{s'} \tilde{\rho}_n \,, \tag{66}$$

$$\gamma_{ss'}^2 = \beta M \hat{I}^\alpha[1,\dots,n] \int_{-\infty}^{0} d\tau e^{\varepsilon\tau} f_s e^{-i\tau L_n} f_{s'}^{(p)} \tilde{\rho}_n \,. \tag{67}$$

If $f_s = f_s^{(p)}$, then $\gamma_{ss'}^1 = \gamma_{ss'}^2$.

Let us consider the one-dimensional stationary Fokker–Planck equation (65) with

$$\partial (g_s \tilde{\rho}_B) / \partial Q_s^\alpha = 0 \,.$$

Then

$$\frac{\partial}{\partial P^\alpha} \left(\frac{M}{\beta} \frac{\partial (\gamma^1(P) \tilde{\rho}_B(t))}{\partial P^\alpha} + \gamma^2(P) P \tilde{\rho}_B(t) \right) = 0 \,. \tag{68}$$

Obviously, we get the relation

$$\frac{M}{\beta} \frac{\partial (\gamma^1(P) \tilde{\rho}_B(t))}{\partial P^\alpha} + \gamma^2(P) P \tilde{\rho}_B(t) = \text{const.} \tag{69}$$

Assuming that the constant is equal to zero, we get

$$\frac{\partial [\gamma^1(P) \tilde{\rho}_B(t)]}{\partial P^\alpha} = \frac{\beta \gamma^2(P) P}{M} \tilde{\rho}_B(t) \,, \tag{70}$$

or, in an equivalent form

$$\frac{\partial \ln[\gamma^1(P)\tilde{\rho}_B(t)]}{\partial P^\alpha} = \frac{\beta\gamma^2(P)P}{\gamma_1(P)M}. \tag{71}$$

The solution is

$$\ln[\gamma^1(P)\tilde{\rho}_B(t)] = \int \frac{\beta\gamma^2(P)P}{M\gamma_1(P)} dP^\alpha + \text{const.} \tag{72}$$

As the result, we obtain

$$\tilde{\rho}_B(t) = \frac{N}{\gamma^1(P)} \int \frac{\beta\gamma^2(P)P}{M\gamma_1(P)} dP^\alpha, \tag{73}$$

where the coefficient N is defined by the normalization condition. Equation (73) describes the solution of the stationary Fokker–Planck equation for the fractional (power) system. The special cases of (73) can be derived as done in Ref. 4.

4. Conclusion

In this paper, the fractional generalizations of the average value and the reduced distribution functions are used. The generalization of the Liouville and Bogoliubov equations are derived[8–11] from the fractional normalization condition. Using these equations, we obtain the Fokker–Planck equation for fractional (power) systems.

The Liouville, Bogoliubov, and Vlasov equations for fractional systems[8–11] can be considered as equations in the noninteger-dimensional phase space. For example, the systems that live on some fractals can be described by these equations. Note that the fractional systems can be presented as special non-Hamiltonian systems.[8–11] Fractional oscillators can be interpreted as elementary excitations of some fractal medium with noninteger mass dimension.[8–11] The fractional (power) systems are connected with the non-Gaussian statistics. Classical dissipative and non-Hamiltonian systems can have the canonical Gibbs distribution as a solution of the stationary equations.[20,21] Using the methods,[20,21] it is easy to prove that some fractional dissipative systems can have the fractional analog of the Gibbs distribution (non-Gaussian statistic) as a solution of the stationary equations for fractional systems.

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