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REVIEW OF SOME PROMISING FRACTIONAL PHYSICAL MODELS

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Fractional dynamics is a field of study in physics and mechanics investigating the behavior of objects and systems that are characterized by power-law nonlocality, power-law long-term memory or fractal properties by using integrations and differentiation of noninteger orders, i.e., by methods in the fractional calculus. This paper is a review of physical models that look very promising for future development of fractional dynamics. We suggest a short introduction to fractional calculus as a theory of integration and differentiation of noninteger order. Some applications of integro-differentiations of fractional orders in physics are discussed. Models of discrete systems with memory, lattice with long-range inter-particle interaction, dynamics of fractal media are presented. Quantum analogs of fractional derivatives and model of open nano-system systems with memory are also discussed.

Keywords: Fractional dynamics; fractional calculus; fractional models; systems with memory; long–range interaction; fractal media; open quantum systems.

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1. Introduction

In this paper we review some applications of fractional calculus and fractional differential equations in physics and mechanics. The interest in such applications has been growing continually during the last years. Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history[1–3] from 30 September 1695, when the derivatives of order $\alpha = 1/2$ has been described by Leibniz in a letter to L'Hospital. Therefore this date can be regarded as the birthday of fractional calculus. We can probably think that Joseph Liouville was the first in application of fractional calculus in physics.[4] The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Riemann, Abel, Riesz, Weyl.

All of us are familiar with derivatives and integrals, like first-order

$$f'(x) = D^1_x f(x) = \frac{d}{dx} f(x) \,, \quad (I^1 f)(x) = \int_0^x dx_1 f(x_1) \,.$$

and the nth order

$$f^{(n)}(x) = D^n_x f(x) = \frac{d^n}{dx^n} f(x) \,,$$

$$(I^n f)(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n f(x_n)\,,$$

where n is integer number $n = 1, 2, \ldots$, i.e., $n \in \mathbb{N}$.

Mathematicians consider the noninteger order of the integrals and derivatives from 1695,

$$f^{(\alpha)}(x) = D_x^\alpha f(x) = ?\,, \quad (I^\alpha f)(x) = ?\,,$$

where $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$.

At these moments there are international journals such as "Fractional Calculus and Applied Analysis", "Fractional Differential Calculus", "Communications in Fractional Calculus", which are dedicated entirely to the fractional calculus.

The first book dedicated specifically to fractional calculus is the book by Oldham and Spanier[3] published in 1974. There are two remarkably comprehensive encyclopedic-type monographs. The first such monograph is written by Samko, Kilbus and Marichev[5,6] and was published in Russian in 1987 and in English in 1993. In 2006 Kilbas, Srivastava and Trujillo published a very important and another remarkable book,[7] where one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. It should be noted the books on fractional differential equation by Podlubny[8] and an introduction to fractional calculus for physicists by Herrmann.[9] There exist mathematical monographs devoted to special questions of fractional calculus. For example, these include the book by McBride[10] published in 1979 (see also Ref. 11), the work by Kiryakova[12] of 1993. The fractional integrals and potentials are described in the monograph by Rubin,[13] the univalent functions, fractional calculus and their applications are described in the volume edited by Srivastava and Owa.[14] Fractional differentiation inequalities are described in the book by Anastassiou[15] published in 2009.

The physical applications of fractional calculus to describe complex media and processes are considered in the very interesting volume edited by Carpintery and Mainardi[16] published in 1997. Different physical systems are described in the papers of volumes edited by Hilfer[17] in 2000, and the edited volume of Sabatier, Agrawal and Tenreiro Machado[18] published in 2007. The most recent volumes on the subject of application of fractional calculus are the volumes edited by Luo and Afraimovich[19] in 2010, and by Klafter, Lim and Metzler[20] in 2011. The book by West, Bologna and Grigolini[21] published in 2003 is devoted to physical application of fractional calculus to fractal processes. The first book devoted to application of fractional calculus to chaos is the book by Zaslavsky[22] published in 2005. The interesting book by Mainardi[23] devoted to the application of fractional calculus in dynamics of viscoelastic materials. The books dedicated specifically to application of derivatives and integrals with noninteger orders in theoretical physics are the remarkable books by Uchaikin,[24,25] and the monographs by Tarasov.[26,27] We also note a new book by Uchaikin and Sibatov[28] devoted to fractional kinetics in solids.

Due to the fact that there are many books and reviews on application of fractional calculus to describe physical processes and systems, it is almost impossible in this review to cover all areas of current research in the field of fractional dynamics. Therefore we must choose some of the areas in this field. We have chosen areas of fractional dynamics that can be considered as the most perspective directions of research in my opinion. These areas are not related to a simple extension of equations with derivatives of integer order to noninteger. We consider fractional models that give relationships between different types of equations describing apparently the system and processes of various types. In addition, we think that these areas and models can give new prospects for a huge number of fundamentally new results in the construction of mathematical methods for the solution of physical problems, and in the description of new types of physical processes and systems.

As a first type of model, we consider discrete maps with memory that are equivalent to the fractional differential equations of kicked motions. These models are promising since an approximation for fractional derivatives of these equations of motion is not used. This fact allows us to study the fractional dynamics by computer simulations without approximations. It allows us to find and investigate a new type of chaotic motion and a new type of attractor.

As a second type of promising fractional models, we consider the discrete systems (or media) with long-range interaction of particles, and continuous limits of these systems such that equations of motion with long-range interaction are mapped into continuous medium equations with the fractional derivatives. As a result we have microscopic model for fractional dynamics of complex media.

The third type of models is related to the description of the fractional dynamics by microscopic models of open quantum systems which interacts with its environment. We give an example that demonstrate that time fractional dynamics and a fractional differential equation of motion can be connected with the interaction between the system and its environment with power-law spectral density.

We also consider fractional models that allow us to describe specific properties of fractal media dynamics; quantum analogs of fractional derivative with respect to coordinate and momentum; the importance of self-consistent formulation of fractal vector calculus and exterior calculus of differential forms that are not yet fully implemented. This review starts with a short introduction to the fractional calculus.

2. Derivatives and Integrals of Noninteger Orders

There are many different definitions of fractional integrals and derivatives of noninteger orders. The most popular definitions are based on the following.

(1) A generalization of Cauchy's differentiation formula;
(2) A generalization of finite difference;
(3) An application of the Fourier transform.

We should note that many usual properties of the ordinary derivative D^n are not realized for fractional derivative operators D^α. For example, a product rule, chain rule, semi-group property have strongly complicated analogs for the operators D^α.

2.1. *A generalization of Cauchy's differentiation formula*

Let G be an open subset of the complex plane $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a holomorphic function. Then we have the Cauchy's differentiation formula:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{n+1}}\, dz\,. \tag{1}$$

A generalization of (1) has been suggested by Sonin (1872) and Letnikov (1872) in the form:

$$D_x^\alpha f(x) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{\alpha+1}}\, dz\,, \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq -1, -2, -3, \ldots$. See Theorem 22.1 in the book by Samko, Kilbas and Marichev.[5,6] Expression (2) is also called Nishimoto derivative.[29] More correctly it should be called Sonin–Letnikov derivative.

2.2. *A generalization of finite difference*

It is well-known that derivatives of integer orders n can be defined by the finite differences. The differentiation of integer order n can be defined by:

$$D_x^n f(x) = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n}\,,$$

where Δ_h^n is a finite difference of integer order n that is defined by:

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh)\,. \tag{3}$$

The difference of noninteger order $\alpha > 0$ is defined by the infinite series:

$$\Delta_h^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh)\,, \tag{4}$$

where the binomial coefficients are:

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}\,.$$

The left-hand and right-hand sided Grünwald–Letnikov (1867, 1868) derivatives of order $\alpha > 0$ are defined by:

$$^{\mathrm{GL}}D_{x\pm}^\alpha f(x) = \lim_{h\to 0} \frac{\Delta_{\mp h}^\alpha f(x)}{h^\alpha}\,. \tag{5}$$

It is interesting that series (4) can be used for $\alpha < 0$ and Eq. (5) defines Grünwald–Letnikov fractional integral if:

$$|f(x)| < c(1+|x|)^{-\mu}\,, \quad \mu > |\alpha|\,.$$

Then (5) can be represented by:

$$^{\mathrm{GL}}D_{x\pm}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x\mp z)}{z^{\alpha+1}}\, dz$$

if $f(x) \in L_p(\mathbb{R})$, where $1 < p < 1/\alpha$ and $0 < \alpha < 1$.

2.3. A generalization by Fourier transform

If we define the Fourier transform operator $\mathcal{F}$ by:

$$(\mathcal{F}f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \,, \tag{6}$$

then the Fourier transform of derivative of integer order n is:

$$(\mathcal{F}D_x^n f)(\omega) = (i\omega)^n (\mathcal{F}f)(\omega) \,.$$

Therefore we can define the derivative of integer order n by:

$$D_x^n f(x) = \mathcal{F}^{-1}\{(i\omega)^n (\mathcal{F}f)(\omega)\} \,.$$

For $f(t) \in L_1(\mathbb{R})$, the left-hand and right-hand side Liouville fractional derivatives and integrals can be defined (see Theorem 7.1 in Refs. 5 and 6 and Theorem 2.15 in Ref. 7) by the relations:

$$(D_{\pm}^{\alpha} f)(x) = \mathcal{F}^{-1}\big((\pm i\omega)^{\alpha} (\mathcal{F}f)(\omega)\big) \,, \tag{7}$$

$$(I_{\pm}^{\alpha} f)(x) = \mathcal{F}^{-1}\left(\frac{1}{(\pm i\omega)^{\alpha}} (\mathcal{F}f)(\omega)\right) , \tag{8}$$

where $0 < \alpha < 1$ and

$$(\pm i\omega)^{\alpha} = |\omega|^{\alpha} \exp\left(\pm \mathrm{sgn}(\omega) \frac{i\alpha\pi}{2}\right) .$$

The Liouville fractional integrals (8) can be represented by:

$$(I_{\pm}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} z^{\alpha-1} f(x \mp z) dz \,. \tag{9}$$

The Liouville fractional derivatives (7) are:

$$(D_{\pm}^{\alpha} f)(x) = D_x^n (I_{\pm}^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^{\infty} z^{n-\alpha-1} f(x \mp z) dz \,, \tag{10}$$

where $n = [\alpha] + 1$.

We can define the derivative of fractional order α by:

$${}^{C}D_{\pm}^{\alpha} f(t) = I_{\pm}^{n-\alpha} (D_t^n f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\infty} z^{n-\alpha-1} D_x^n f(x \mp z) dz \,, \tag{11}$$

where $n = [\alpha]+1$. It is the Caputo derivative of order α.[7] For $x \in [a, b]$ the left-hand side Caputo fractional derivative of order $\alpha > 0$ is defined by:

$${}_{a}^{C}D_t^{\alpha} f(t) = {}_{a}I_t^{n-\alpha} D_t^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{d\tau D_{\tau}^n f(\tau)}{(t-\tau)^{\alpha-n+1}} \,, \tag{12}$$

where $n - 1 < \alpha < n$, and ${}_{a}I_t^{\alpha}$ is the left-hand side Riemann–Liouville fractional integral of order $\alpha > 0$ that is defined by:

$${}_{a}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}} \,, \quad (a < t) \,.$$

Note that the Riemann–Liouville fractional derivative has some notable disadvantages in applications such as nonzero of the fractional derivative of constants,

$$ {}_0D_t^\alpha C = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C \,, $$

which means that dissipation does not vanish for a system in equilibrium. The Caputo fractional differentiation of a constant results in zero

$$ {}_0^C D_t^\alpha C = 0 \,. $$

The desire to use the usual initial value problems

$$ f(t_0) = C_0 \,, \quad (D_t^1 f)(t_0) = C_1 \,, \quad (D_t^2 f)(t_0) = C_2, \ldots $$

lead to the application Caputo fractional derivatives instead of the Riemann–Liouville derivative.

The Riemann–Liouville and Caputo derivatives are connected.[7] Let $f(t)$ be a function for which the Caputo derivatives of order α exist together with the Riemann–Liouville derivatives. Then these fractional derivatives are connected by the relation:

$$ {}_a^C D_t^\alpha f(t) = {}_aD_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) \,. \tag{13} $$

The second term of the right-hand side of Eq. (13) regularizes the Caputo fractional derivative to avoid the potentially divergence from singular integration at $t = 0$.

2.4. *Some unusual properties of fractional derivatives*

Let us demonstrate the unusual properties of derivatives of noninteger orders by using the Riemann–Liouville derivatives.

(1) Semi-group property does not hold

$$ (D_{a+}^\alpha D_{a+}^\beta f)(x) = (D_{a+}^{\alpha+\beta} f)(x) - \sum_{k=1}^{[\beta]+1} (D_{a+}^{\beta-k} f)(a+) \frac{(x-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)} \tag{14} $$

for $f(x) \in L_1(a,b)$ and $(I_{a+}^{n-\alpha} f)(x) \in AC^n[a,b]$, (see Eq. (2.1.42) in Ref. 7).

As a consequence, in general we have:

$$ D_{a+}^\alpha D_{a+}^\alpha \neq D_{a+}^{2\alpha} \,. $$

(2) The derivative of the nonzero constant is not equal to zero

$$ (D_{a+}^\alpha 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \,. \tag{15} $$

(3) The initial conditions for differential equation with Riemann–Liouville derivative differ from the conditions for ordinary differential equations of the integer order:

$$ ({}_0D_t^{\alpha-k} x)(0+) = c_k \,, \quad k = 1, \ldots, n \,. \tag{16} $$

For example, conditions (16) for $1 < \alpha < 2$ give:

$$({}_0D_t^{\alpha-1}x)(0+) = c_1\,, \quad ({}_0D_t^{\alpha-2}x)(0+) = ({}_0I_t^{2-\alpha}x)(0+) = c_2\,.$$

(4) Representation in the form of an infinite series of derivatives of integer orders

$$(D^{\alpha}_{a+}f)(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} D^n_x f(x) \tag{17}$$

for analytic (expandable in a power series on the interval) functions on (a,b), (see Lemma 15.3 in Refs. 5 and 6).

(5) A generalization of the classical Leibniz rule

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

from integer n to fractional α contains an infinite series

$$(D^{\alpha}_{a+}(fg))(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} (D^{\alpha-k}_x f)(x) D^k_x g \tag{18}$$

for analytic functions on (a,b) (see Theorem 15.1 in Refs. 5 and 6). The sum is infinite and contains integrals of fractional order (for $k > [\alpha]+1$).

(6) The increasing complexity of the Newton–Leibniz equation for

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(x) = f(x) - \sum_{k=1}^{n} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} (D^{n-k}I^{n-\alpha}_{a+}f)(a)$$

for $f(x) \in L_1(a,b)$, $(I^{n-\alpha}f)(x) \in AC^n[a,b]$, (see Eq (2.1.39) in Ref. 7).

For $0 < \alpha \leq 1$, we have:

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(b) = f(b) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} (I^{1-\alpha}_{a+}f)(a)\,.$$

For $n \in \mathbb{N}$

$$(I^{n}_{a+}D^{n}_{a+}f)(x) = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (b-a)^k\,,$$

$$(I^{1}_{a+}D^{1}_{a+}f)(b) = f(b) - f(a)$$

(see Eq. (2.1.41) in Ref. 7).

(7) For the fractional derivatives there is an analogue of the exponent. The Mittag–Leffler function

$$E_{\alpha}[z] = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(k\alpha+1)}$$

is invariant with respect to the Riemann–Liouville

$$D^{\alpha}_{a+}E_{\alpha}[\lambda(x-a)^{\alpha}] = \lambda E_{\alpha}[\lambda(x-a)^{\alpha}]$$

(see Eq. (2.1.57) in Ref. 7).

3. Introduction to Fractional Dynamics

Fractional dynamics is a field in physics and mechanics, studying the behavior of objects and systems that are described by using integrations and differentiation of fractional orders, i.e., by methods of fractional calculus. Derivatives and integrals of noninteger orders are used to describe objects that can be characterized by the following properties.

(1) A power-law nonlocality;
(2) A power-law long-term memory;
(3) A fractal-type property.

3.1. *Fractional diffusion-wave equation*

In mathematics and physics the following equations are well-known, the diffusion equation:

$$\nabla^2 u(t, \mathbf{x}) = C_1 D_t^1 u(t, \mathbf{x}) \,,$$

and the wave equation:

$$\nabla^2 u(t, \mathbf{x}) = C_2 D_t^2 u(t, \mathbf{x}) \,.$$

We can consider a generalization of these equations such that it includes derivatives of noninteger order with respect to time. This generalization describes phenomena that can be characterized by diffusion and waves properties. The fractional diffusion-wave equation is the linear fractional differential equation obtained from the classical diffusion or wave equations by replacing the first- or second-order time derivatives by a fractional derivative (in the Caputo sense) of order α with $0 < \alpha < 2$,

$$\nabla^2 u(t, \mathbf{x}) = C_\alpha \, {}_0^C D_t^\alpha u(t, \mathbf{x}) \,.$$

The solutions of these fractional partial differential equations are described in the book[7] (see Sec. 6.1.2). This equation describes diffusion-wave phenomena,[33,34] which is also called the anomalous diffusion such that we have the superdiffusion for $1 < \alpha < 2$, and subdiffusion for $0 < \alpha < 1$. A more detailed description of these effects and phenomena can be found in the reviews.[35,36]

3.2. *Viscoelastic material*

If the force is immediately relaxed, then the deformation disappears. This property is called the elasticity. The elasticity is a physical property of materials which return to their original shape after they are deformed. The other well-known property is called the viscosity. The viscosity of a fluid is a measure of its resistance to gradual deformation by shear stress or tensile stress.

Mechanically, this behavior is represented with a spring of modulus E, which describes the instantaneous elastic response. The stress $\sigma(t)$ is proportional to the zeroth derivative of strain $\varepsilon(t)$ for elastic solids and to the first derivative of strain for viscous fluids.

The elastic solids are described by the Hooke's law:

$$\sigma(t) = E\varepsilon(t)\,,$$

where E is the elastic moduli.

The viscous fluids are described by the law of Newtonian fluids:

$$\sigma(t) = \eta D_t^1 \varepsilon(t)\,,$$

where η is the coefficient of viscosity.

There are materials that demonstrate these two properties (elasticity and viscosity) at the same time. These materials are called viscoelastic. To describe the fractional viscoelasticity Scott Blair (1947) uses the relation[30,31]

$$\sigma(t) = E_\alpha\, {}_0^C D_t^\alpha \varepsilon(t)\,,$$

where E_α is a constant.

If $F(x)$ is an acting force and x is the displacement, then Hooke's model of elasticity

$$F(x) = -kD_t^0 x(t)\,,$$

and Newton's model of a viscous fluid

$$F(x) = -kD_t^1 x(t)\,,$$

can be considered as particular cases of the relation

$$F(x) = -kD_t^\alpha x(t)\,.$$

This force describe the property that is called the fractional friction.

More complicated fractional models for viscoelasticity of materials are considered in the books by Rabotnov[32] and Mainardi.[23]

3.3. *Power-law memory and fractional derivatives*

A physical interpretation of equations with derivatives and integrals of noninteger order with respect to time is connected with the memory effects.

Let us consider the evolution of a dynamical system in which some quantity $A(t)$ is related to another quantity $B(t)$ through a memory function $M(t)$ by:

$$A(t) = \int_0^t M(t-\tau)B(\tau)d\tau\,. \tag{19}$$

This operation is a particular case of composition products suggested by Vito Volterra. In mathematics, Eq. (19) means that the value $A(t)$ is related with $B(t)$ by the convolution $A(t) = M(t) * B(t)$.

Equation (19) is a typical equation obtained for the systems coupled to an environment, where environmental degrees of freedom are being averaged. Let us note the memory functions for the case of the absence of the memory and the case of power-law memory.

The absence of the memory: For a system without memory, the time dependence of the memory function is:

$$M(t-\tau) = M(t)\delta(t-\tau)\,, \tag{20}$$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $A(t)$ is defined by $B(t)$ at the only instant t. In this case, the system loses all its values of quantity except for one: $A(t) = M(t)B(t)$. Using (19) and (20), we have:

$$A(t) = \int_0^t M(t)\delta(t-\tau)B(\tau)d\tau = M(t)B(t)\,. \tag{21}$$

Expression (21) corresponds to the well-known physical process with complete absence of memory. This process relates all subsequent values to previous values through the single current value at each time t.

Power-law memory: The power-like memory function is defined by:

$$M(t-\tau) = M_0(t-\tau)^{\varepsilon-1}\,, \tag{22}$$

where M_0 is a real parameter. It indicates the presence of the fractional derivative or integral. The integral representation is equivalent to a differential equation of the fractional order. Substitution of (22) into (19) gives the temporal fractional integral of order ε:

$$A(t) = \lambda I_t^{\varepsilon} B(t) = \frac{\lambda}{\Gamma(\varepsilon)} \int_0^t (t-\tau)^{\varepsilon-1} B(\tau)d\tau\,, \quad 0 < \varepsilon < 1\,, \tag{23}$$

where $\lambda = \Gamma(\varepsilon)M_0$. The parameter λ can be regarded as the strength of the perturbation induced by the environment of the system. The physical interpretation of the fractional integration is an existence of a memory effect with power-like memory function. The memory determines an interval $[0,t]$ during which $B(\tau)$ affects $A(t)$.

Equation (23) is a special case of relation for $A(t)$ and $B(t)$, where $A(t)$ is directly proportional to $B(t)$. In a more general case, the values $A(t)$ and $B(t)$ can be related by the equation:

$$f(A(t)\,, \quad M(t) * D_t^n B(t)) = 0\,, \tag{24}$$

where f is a smooth function. For dynamical systems relation (24) defines a memory effect. In this case (24) gives the relation $f(A(t), {}_0^C D_t^{\alpha} B(t)) = 0$ with Caputo fractional derivative. Relation (24) is a fractional differential equation.

4. Discrete Physical Systems with Memory

Discrete maps (universal, Chirikov-Taylor, rotator, Amosov, Zaslavsky, Henon) can be obtained from the correspondent equations of motion with a periodic sequence of delta-function-type pulses (kicks).

An approximation for derivatives of these equations is not used. This fact is used to study the evolution that is described by differential equations with periodic kicks.

Example: The universal map without memory

$$x_{n+1} = x_n + p_{n+1} T \,, \quad p_{n+1} = p_n - KTG[x_n] \tag{25}$$

is obtained from the differential equation of second-order with respect to time

$$D_t^2 x(t) + KG[x(t)] \sum_{k=1}^{\infty} \delta(t/T - k) = 0 \,, \tag{26}$$

where $T = 2\pi/\nu$ is the period, and K is an amplitude of the pulses.

If $G[x] = \sin(x)$, then we have the Chirikov–Taylor map.

For $G[x] = -x$ we have the Amosov system.

4.1. *Universal map with Riemann–Liouville type memory*

The Cauchy-type problem for the differential equations:

$$_0D_t^{\alpha} x(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right), \quad 1 < \alpha \leqslant 2 \tag{27}$$

with the initial conditions:

$$(_0D_t^{\alpha-1} x)(0+) = c_1 \,, \quad (_0D_t^{\alpha-2} x)(0+) = (_0I_t^{2-\alpha} x)(0+) = c_2 \,, \tag{28}$$

where $_0D_t^{\alpha-1}$ is the Riemann–Liouville derivative, is equivalent to the map equations in the form:

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{n} p_{k+1} V_{\alpha}(n-k+1) + \frac{c_2 T^{\alpha-2}}{\Gamma(\alpha-1)} (n+1)^{\alpha-2} \,, \tag{29}$$

$$p_{n+1} = p_n - KTG[x_n] \,, \quad (1 < \alpha \leqslant 2) \,, \tag{30}$$

where $p_1 = c_1$, and $V_{\alpha}(z) = z^{\alpha-1} - (z-1)^{\alpha-1}$, $(z \geqslant 1)$. The proof of this statement is given in Refs. 26 and 40. If $G[x] = \sin(x)$, then we have the Chirikov–Taylor map with memory. For $G[x] = -x$ we have the Amosov system with memory.

4.2. *Universal map with Caputo type memory*

The Cauchy-type problem for the differential equations:

$$D_t^1 x(t) = p(t) \,, \tag{31}$$

$$_0^C D_t^{\alpha-1} p(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right), \quad (1 < \alpha < 2) \,, \tag{32}$$

with the initial conditions $x(0) = x_0$, $p(0) = p_0$, where ${}_0^C D_t^{\alpha-1}$ is the Caputo derivative, is equivalent to the map equations in the form:

$$x_{n+1} = x_0 + p_0(n+1)T - \frac{KT^{\alpha}}{\Gamma(\alpha)} \sum_{k=1}^{n} (n+1-k)^{\alpha-1} G[x_k]\,, \tag{33}$$

$$p_{n+1} = p_0 - \frac{KT^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=1}^{n} (n+1-k)^{\alpha-2} G[x_k]\,. \tag{34}$$

The proof of this statement is given in Refs. 26 and 40. This map allows us to describe fractional maps with memory for dynamics with usual initial conditions.

4.3. *Kicked damped rotator map with memory*

Equation of motion:

$$ {}_0D_t^{\alpha} x - q\, {}_0D_t^{\beta} x = KG[x] \sum_{n=0}^{\infty} \delta(t - nT)\,, \tag{35}$$

where $q \in \mathbb{R}$, $1 < \alpha \leqslant 2$, $\beta = \alpha - 1$ and ${}_0D_t^{\alpha}$ is Riemann–Liouville derivative, can be represented in the form of the discrete map,

$$x_{n+1} = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n} p_{k+1} W_{\alpha}(q, T, n+1-k)\,, \tag{36}$$

$$p_{n+1} = e^{qT}(p_n + KG[x_n])\,, \tag{37}$$

where the function W_α is defined by:

$$W_{\alpha}(q, T, m+1) = T^{\alpha-1} \int_0^1 e^{-qTy} (m+y)^{\alpha-2} dy\,. \tag{38}$$

The proof of this statement is given in Ref. 26 (see also Refs. 38 and 41).

4.4. *New type of attractors*

The suggested maps with memory are equivalent to the correspondent fractional kicked differential equations.[37,38,40] An approximation for fractional derivatives of these equations is not used. This fact is used to study the evolution that is described by fractional differential equations. Computer simulations of the suggested discrete maps with memory prove that the nonlinear dynamical systems, which are described by the equations with fractional derivatives, exhibit a new type of chaotic motion and a new type of attractors. For example, the slow converging and slow diverging trajectories, ballistic trajectories and fractal-like sticky attractors, in the chaotic sea can be observed[39] for Chirikov–Taylor map with power-law memory (see also Refs. 39, 41–44).

5. Dynamics of Systems with Long-Range Interaction

Dynamics with long-range interaction has been the subject of continuing interest in different areas of science. The long-range interactions have been studied in discrete systems as well as in their continuous analogues.

The dynamics described by the equations with fractional space derivatives can be characterized by the solutions that have power-like tails.[7] Similar features were observed in the lattice models with power-like long-range interactions.[46–50] As it was shown,[51–59] the equations with fractional derivatives can be directly connected to chain and lattice models with long-range interactions.

Equations of motion of one-dimensional lattice system of interacting particles:

$$\frac{d^2 u_n(t)}{dt^2} = g \sum_{m=-\infty, m\neq n}^{+\infty} J(n,m)[u_n(t) - u_m(t)] + F(u_n(t)) \,, \tag{39}$$

where $u_n(t)$ are displacements from the equilibrium, $F(u_n)$ is the external on-site force and

$$J(n,m) = J(|n-m|) \,, \quad \sum_{n=1}^{\infty} |J(n)|^2 < \infty \,. \tag{40}$$

5.1. *Long-range interaction of power-law type*

We define a special type of interparticle interaction

$$\lim_{k\to 0} \frac{\hat{J}_\alpha(k) - \hat{J}_\alpha(0)}{|k|^\alpha} = A_\alpha \,, \quad \alpha > 0 \,, \quad 0 < |A_\alpha| < \infty \,, \tag{41}$$

where

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-ikn\Delta x} J(n) = 2 \sum_{n=1}^{\infty} J(n) \cos(kn\Delta x) \,. \tag{42}$$

This interaction is called the interaction of power-law type α.

As an example of the power-law type interaction, we can consider:

$$J(n-m) = \frac{1}{|n-m|^{\beta+1}} \,. \tag{43}$$

The other examples of power-law type interaction are considered in Refs. 26, 52 and 53.

Equations of motion (39) with the power-law interaction (43) give the following equations in the continuous limits:

(1) For $0 < \beta < 2$ ($\beta \neq 1$) we get the Riesz fractional derivative D_x^α of order $\alpha = \beta$:

$$\frac{\partial^2}{\partial t^2} u(x,t) - G_\alpha A_\alpha D_x^\alpha u(x,t) = F(u(x,t)) \,, \quad 0 < \alpha < 2 \,, \quad (\alpha \neq 1) \,. \tag{44}$$

(2) For $\beta > 2$ $(\beta \neq 3, 4, 5, \ldots)$ we get derivative of order $\alpha = 2$:

$$\frac{\partial^2}{\partial t^2} u(x,t) + G_\alpha \zeta(\alpha - 1) D_x^2 u(x,t) = F(u(x,t)), \quad \alpha > 2, \quad (\alpha \neq 3, 4, \ldots), \tag{45}$$

where $G_\alpha = g|\Delta x|^{\min\{\alpha;2\}}$ is the finite parameter.

(3) For $\beta = 1$ we get derivative of order $\alpha = 1$ and $\alpha = 2$ for $\beta = 3, 5, 7, \ldots$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - iG_1 \frac{\partial u(x,t)}{\partial x} = F(u(x,t)), \tag{46}$$

where $G_1 = \pi g \Delta x$ is the finite parameter.

(4) For $\beta = 3, 5, 7, \ldots$ $(\beta = 2m - 1$, where $m = 2, 3, 4, \ldots)$, we get equation with derivative of order $\alpha = 2$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - G_2 \frac{\partial^2 u(x,t)}{\partial x^2} = F(u(x,t)), \tag{47}$$

where

$$G_2 = \frac{(-1)^{m-1}(2\pi)^{2m-2}}{4(2m-2)!} B_{2m-2}\, g(\Delta x)^2 \tag{48}$$

are the finite parameters, and B_{2m-2} are Bernoulli numbers.

(5) For $\beta = 2k - 2$, where $k \in \mathbb{N}$, we have the logarithmic poles.

The effects of synchronization, breather-type and solution-type solutions for the systems with nonlocal interaction of power-law type $0 < \beta < 2$ $(\beta \neq 1)$ were investigated.[54–57] Nonequilibrium phase transitions in the thermodynamic limit for long-range systems are considered in Ref. 58. Statistical mechanics and dynamics of solvable models with long-range interactions are discussed in Ref. 60. Stationary states of fractional dynamics of systems with long-range interactions are discussed in Ref. 59. Fractional dynamics of systems with long-range space interaction and temporal memory is also considered in Refs. 57 and 59.

Fractional derivatives with respect to coordinates describe power-law nonlocal properties of the distributed system. Therefore the fractional statistical mechanics can be considered as special case of the nonlocal statistical mechanics.[66] As shown in the articles[52,53] the spacial fractional derivatives are connected with long-range interparticle interactions. We prove that nonlocal alpha-interactions between particles of crystal lattice give continuous medium equations with fractional derivatives with respect to coordinates. In the monographs by Vlasov,[66] a nonlocal statistical model of crystal lattice is suggested. Therefore we conclude[26] that the nonlocal and fractional statistical mechanics are directly connected with statistical dynamics of systems with long-range interactions.[60]

5.2. *Nonlocal generalization of the Korteweg–de Vries equation*

The Korteweg–de Vries equation is used in a wide range of physics phenomena, especially those exhibiting shock waves, travelling waves and solitons.[61] In the quantum mechanics certain physical phenomena can be explained by Korteweg–de Vries models. This equation is used in fluid dynamics, aerodynamics and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior and mass transport.

The continuous limits of the equations of lattice oscillations

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_1(n,m)[u_n^2 - u_m^2] + g_3 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_3(n,m)[u_n - u_m]\,,$$

give the nonlocal generalization of the Korteweg–de Vries equation:

$$\frac{\partial}{\partial t}u(x,t) - G_1 u(x,t)\frac{\partial}{\partial x}u(x,t) + G_3\frac{\partial^3}{\partial x^3}u(x,t) = 0$$

in the form

$$\frac{\partial}{\partial t}u(x,t) - G_{\alpha_1} u(x,t)\frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}}u(x,t) + G_{\alpha_3}\frac{\partial^{\alpha_3}}{\partial |x|^{\alpha_3}}u(x,t) = 0\,, \tag{49}$$

where $G_{\alpha_1} = g_1|\Delta x|^{\alpha_1}$ and $G_{\alpha_3} = g_3|\Delta x|^{\alpha_3}$ are finite parameters. Equation (49) is a fractional generalization of Korteweg–de Vries equation.[62]

The nonlinear power-law type interactions defined by $f(u) = u^2$ and $f(u) = u - gu^2$ for the discrete systems are used to derive the Burgers and Boussinesq equations and their fractional generalizations in the continuous limit. Note that a special case of this equation is suggested in Refs. 63 and 64.

5.3. *Nonlocal generalization of the Burgers and Boussinesq equations*

Let us consider examples of quadratic-nonlinear long-range interactions.[26,52,53]

We can consider the discrete systems that are described by the equations:

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_1(n-m)[u_n^2 - u_m^2] + g_2 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_2(n-m)[u_n - u_m]\,, \tag{50}$$

where $J_1(n-m)$ and $J_2(n-m)$ define interactions of power-law type with α_1 and α_2. If $\alpha_1 = 1$ and $\alpha_2 = 2$, then we get the well-known Burgers equation that is a nonlinear partial differential equation, which is used to describe boundary layer behavior, shock wave formation and mass transport. If $\alpha_2 = \alpha$, then we get the fractional Burgers equation that is suggested in Ref. 63. In the general case, the continuous limit gives the fractional Burgers equation in the form:

$$\frac{\partial}{\partial t}u(x,t) + G_{\alpha_1} u(x,t)\frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}}u(x,t) - G_{\alpha_2}\frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}}u(x,t) = 0\,. \tag{51}$$

We can consider the chain and lattice equations of the form

$$\frac{\partial^2 u_n(t)}{\partial t^2} = g_2 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_2(n,m)[f(u_n) - f(u_m)]$$

$$+ g_4 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_4(n,m)[u_n - u_m]\,, \tag{52}$$

where $f(u) = u - gu^2$, and $J_2(n-m)$ and $J_4(n-m)$ define the interactions of power-law types with α_2 and α_4. If $\alpha_2 = 2$ and $\alpha_4 = 4$, then in continuous limit we obtain the well-known nonlinear partial differential equation of forth-order that is called the Boussinesq equation. This equation was subsequently applied to problems in the percolation of water in porous subsurface strata, and it used to describe long waves in shallow water and in the analysis of many other physical processes. In the general case, the continuous limit gives the fractional Boussinesq equation of the form:

$$\frac{\partial^2}{\partial t^2} u(x,t) - G_{\alpha_2} \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} u(x,t) + g G_{\alpha_2} \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} u^2(x,t) + G_{\alpha_4} \frac{\partial^{\alpha_4}}{\partial x^{\alpha_4}} u(x,t) = 0\,. \tag{53}$$

Fractional generalization of Korteweg–de Vries, Burgers, Boussinesq equations can be used to describe properties of media with nonlocal interaction of particles.

6. Fractional Models of Fractal Media

Fractals are measurable metric sets with a noninteger Hausdorff dimension.[87,88] The main property of the fractal is noninteger Hausdorff dimension that should be observed on all scales. In real physical objects the fractal structure cannot be observed on all scales but only those for which $R_0 < R < R_m$, where R_0 is the characteristic scale of the particles, and R_m is the scale of objects. Real fractal media can be characterized by the asymptotic form for the relation between the mass $M(W)$ of a region W of fractal medium, and the radius R containing this mass:

$$M_D(W) = M_0 (R/R_0)^D\,, \quad R/R_0 \gg 1\,.$$

The number D is the mass dimension. The parameter D, does not depend on the shape of the region W, or on whether the packing of sphere of radius R_0 is close packing, a random packing or a porous packing with a uniform distribution of holes.

The fractality of medium means than the mass of fractal homogeneous medium in any region $W \subset \mathbb{R}^n$ increases more slowly than the n-dimensional volume of this region:

$$M_D(W) \sim \big(V_n(W)\big)^{D/n}\,.$$

As a result, we can define that fractal medium is a system or medium with non-integer physical (mass, charge, particle, ...) dimension.

To describe fractal media by fractional continuous model, we can use two different notions such as density of states $c_n(D,\mathbf{r})$ and distribution function $\rho(\mathbf{r})$.

(1) The function $c_n(D,\mathbf{r})$ is a density of states in the n-dimensional Euclidean space $\mathbb{R}^n$. The density of states describes the closely packed permitted states of

particles in the space $\mathbb{R}^n$. The expression $c_n(D, \mathbf{r})\, dV_n$ represents the number of states (permitted places) between V_n and $V_n + dV_n$.

(2) The function $\rho(\mathbf{r})$ is a distribution function for the n-dimensional Euclidean space $\mathbb{R}^n$. The distribution function describes a distribution of physical values (for example, the mass, probability, electric charge, number of particles) on a set of possible states in the space $\mathbb{R}^n$.

Note that some elementary models of fractal density of states and fractal distributions by open and closed boxes are suggested in Ref. 26.

6.1. *Homogeneity and fractality*

To describe the fractal medium, we use a continuous medium model. In this model the fractality and homogeneity properties can be realized in the following forms:

(1) Homogeneity: The local density of homogeneous fractal medium can be described by the constant density $\rho(\mathbf{r}) = \rho_0 = \text{const}$. This property means that the equations with constant density must describe the homogeneous media, i.e., if $\rho(\mathbf{r}) = \text{const}$ and $V(W_1) = V(W_2)$, then $M_D(W_1) = M_D(W_2)$.

(2) Fractality: The mass of the ball region W of fractal homogeneous medium obeys a power law relation $M \sim R^D$, where $0 < D < 3$, and R is the radius of the ball. If $V_n(W_1) = \lambda^n V_n(W_2)$ and $\rho(\mathbf{r}, t) = \text{const}$, then the fractality means that $M_D(W_1) = \lambda^D M_D(W_2)$.

These two conditions cannot be satisfied if the mass of a medium is described by integral of integer order. These conditions can be realized by the fractional equation:

$$M_D(W, t) = \int_W \rho(\mathbf{r}, t) dV_D\,, \quad dV_D = c_3(D, \mathbf{r}) dV_3\,, \tag{54}$$

where $\mathbf{r}$ is dimensionless vector variable.

6.2. *Balance equations for fractal media*

The equation of continuity (mass balance) for fractal media:

$$\left(\frac{d}{dt}\right)_D \rho = -\rho \nabla_k^D u_k\,. \tag{55}$$

The equation of momentum balance for fractal media:

$$\rho \left(\frac{d}{dt}\right)_D u_k = \rho f_k + \nabla_l^D p_{kl}\,. \tag{56}$$

The equation of energy balance for fractal media:

$$\rho \left(\frac{d}{dt}\right)_D e = c_3^{-1}(D, \mathbf{r}) c_2(d, \mathbf{r}) p_{kl} \nabla_l u_k + \nabla_k^D q_k\,. \tag{57}$$

Here D is a mass dimension of fractal medium and

$$\left(\frac{d}{dt}\right)_D = \frac{\partial}{\partial t} + c_3^{-1}(D, \mathbf{r}) c_2(d, \mathbf{r}) u_l \nabla_l\,, \quad \nabla_k^D A = c_3^{-1}(D, \mathbf{r}) \nabla_k (c_2(d, \mathbf{r}) A)\,.$$

These equations are proved in Ref. 107.

6.3. *Moment of inertia for fractal bodies*

The moments of inertia of fractal-homogeneous rigid ball:

$$I_z^{(D)} = \frac{2D}{3(D+2)} M_D R^2 \,, \quad \frac{I_z^{(D)}}{I_z^{(3)}} = 1 + \frac{2(D-3)}{3(D+2)} \,. \tag{58}$$

The moments of inertia of fractal-homogeneus rigid cylinder:

$$I_z^{(\alpha)} = \frac{\alpha}{\alpha+2} M_D R^2 \,, \quad \frac{I_z^{(\alpha)}}{I_z^{(q)}} = 1 + \frac{\alpha-2}{\alpha+2} \,. \tag{59}$$

The parameter α is a fractal mass dimension of the cross-section of cylinder ($1 < \alpha \leqslant 2$). This parameter can be easily calculated from the experimental data by box counting method for the cross-section of the cylinder.

The periods of oscillation for the Maxwell pendulum with fractal rigid cylinder:

$$\frac{T_0^{(\alpha)}}{T_0^{(2)}} = \sqrt{\frac{4(\alpha+1)}{3(\alpha+2)}} \,. \tag{60}$$

The deviation $T_0^{(\alpha)}$ from $T_0^{(2)}$ for $1 < \alpha \leqslant 2$ is no more that 6%. Equation (60) allows us to use an experimental determination of a fractal dimensional for fractal rigid body by measurements of periods of oscillations.

For a ball with mass M_D, radius R, and a mass fractal dimension D, we can consider the motion without slipping on an inclined plane with a fixed angle β to the horizon. The condition of rolling without slipping means that at each time point of the ball regarding the plane is stationary and the ball rotates on its axis. The center of mass of a homogeneous cylinder moves in a straight line. Using the law of energy conservation, we obtain the equation:

$$v(D) = \frac{3(D+2)}{5D+6} \, gt \sin\beta \,. \tag{61}$$

As a result, we have:

$$\frac{v(D)}{v(3)} = \frac{21(D+2)}{5(5D+6)} \,. \tag{62}$$

Note that the deviation of velocity $v(D)$ of fractal solid sphere from the velocity $v(3)$ of usual ball is less than 5%.

The suggested equations allows us to measure experimentally the fractional mass dimensions D of fractal materials by measuring the velocities. Note that this measured dimension D must be related to the fractal dimension that can be determined by the box counting method.

6.4. *Dipole and quadrupole moment of charged fractal distributions*

The fractional model can be used to describe fractal distribution of charges.[89,90] The distribution of charged particles is called a homogeneous one if all regions W

and W' with the equal volumes $V_D(W) = V_D(W')$ have the equal total charges on these regions, $Q_D(W) = Q_D(W')$. For charged particles that are distributed homogeneously over a fractal with dimension D, the electric charge Q satisfies the scaling law $Q(R) \sim R^D$, whereas for a regular n-dimensional Euclidean object we have $Q(R) \sim R^n$. This property can be used to measure the fractal dimension D of fractal distributions of charges. We consider this power-law relation as a definition of a fractal charge dimension. If all particles of a distribution are identical, then the charge dimension is equal to the mass dimension. In general, these dimensions can be considered as different characteristics of fractal distribution.

Let us consider the example of electric dipole moment for the homogeneous ($\rho(\mathbf{r}) = \rho_0$) fractal distribution of electric charges in the parallelepiped region

$$0 \le x \le A\,, \quad 0 \le y \le B\,, \quad 0 \le z \le C\,. \tag{63}$$

In the case of Riemann–Liouville fractional integral, we have the dipole moment $p_x^{(D)}$ in the form:

$$\begin{aligned} p_x^{(D)} &= \frac{\rho_0}{\Gamma^3(a)} \int_0^A dx \int_0^B dy \int_0^C dz x^a y^{a-1} z^{a-1} \\ &= \frac{\rho_0 (ABC)^a}{\Gamma^3(a)} \frac{A}{a^2(a+1)}\,, \end{aligned} \tag{64}$$

where $a = D/3$. The electric charge of parallelepiped region is defined by:

$$Q(W) = \rho_0 \int_W dV_D = \frac{\rho_0 (ABC)^a}{a^3 \Gamma^3(a)}\,. \tag{65}$$

Then the dipole moment for fractal distribution in parallelepiped is:

$$p_x^{(D)} = \frac{a}{a+1} Q(W) A\,. \tag{66}$$

By analogy with this equation,

$$p_y^{(D)} = \frac{a}{a+1} Q(W) B\,, \quad p_z^{(D)} = \frac{a}{a+1} Q(W) C\,. \tag{67}$$

Using $a/(a+1) = D/(D+3)$, we obtain:

$$p^{(D)} = \frac{2D}{D+3} p^{(3)}\,, \tag{68}$$

where $p^{(3)} = |\mathbf{p}^{(3)}|$ are the dipole moment for the usual three-dimensional homogeneous distribution. For example, the relation $2 \le D \le 3$ leads us to the inequality:

$$0.8 \le p^{(D)}/p^{(3)} \le 1\,. \tag{69}$$

These inequalities describe dipole moment of fractal distribution of charged particles in the parallelepiped region.

The example of electric quadrupole moment for the homogeneous ($\rho(\mathbf{r}) = \rho_0$) fractal distribution in the ellipsoid region W:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \le 1 \tag{70}$$

is considered in Refs. 26, 89 and 90. The fractional model of fractal media gives[89,90] the electric quadrupole moments for fractal ellipsoid

$$Q_{kk}^{(D)} = \frac{5D}{3D+6} Q_{kk}^{(3)} , \tag{71}$$

$$Q_{kl}^{(D)} = \frac{5\pi}{D+2} \frac{\Gamma^2(D/6+1/2)}{\Gamma^2(D/6)} Q_{kl}^{(3)} , \tag{72}$$

where $k \neq l$ and $k, l = 1, 2, 3$. For $2 < D < 3$, we get:

$$\frac{5}{6} < Q_{kk}^{(D)}/Q_{kk}^{(3)} < 1 , \tag{73}$$

$$0.6972 < Q_{kl}^{(D)}/Q_{kl}^{(3)} < 1 . \tag{74}$$

These inequalities describe values of the diagonal and nondiagonal elements of the electric quadrupole moments for fractal distribution of charged particles in ellipsoid region.

6.5. *Some applications of fractional models of fractal media*

In this section, we considered some fractional models to describe dynamics of fractal media. In general, the fractal medium cannot be considered as a continuous medium. There are points and domains that are not filled of particles. We consider the fractal media as special continuous media. We use the procedure of replacement of the medium with fractal mass dimension by some continuous model that uses the fractional integrals. The main notions that allow us to describe fractal media are a density of states and density of distributions. The fractional integrals are used to take into account the fractality of the media. Note that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor.[26]

The suggested fractional models of fractal media can have a wide application. This is due in part to the relatively small numbers of parameters that define a fractal medium of great complexity and rich structure. In many cases, the real fractal structure of matter can be disregarded and we can describe the medium by a fractional model, in which the fractional integration is used. The order of fractional integral is equal to the fractal physical dimension of the medium.

The fractional continuous model allows us to describe dynamics of fractal media and fractal distributions.[26,27,150] Applications of fractional models to describe fractal distributions of charges are considered in Refs. 26 and 90. We note that gravitational field of fractal distribution of particles and fields can be described by fractional continuous models[108] (see also Ref. 83). Using fractional integrals, the fractional generalization of the Chapman–Kolmogorov equation and the Fokker–Planck equation for fractal media are derived.[109] We note applications of fractional continuous models by Ostoja–Starzewski to the thermoelasticity,[91] and the thermomechanics,[92] the turbulence of fractal media,[93] the elastic and inelastic media with fractal geometries,[94] the fractal porous media[95] and the fractal solids.[96] The hydrodynamic accretion in fractal media[104–106] is considered by Roy and Ray by using a fractional continuous model.

7. Open System in Environment

The closed, isolated and Hamiltonian systems are idealizations that are not observable and therefore do not exist in the real world. As a rule, any system is always embedded in some environment and therefore it is never really closed or isolated. Frequently, the relevant environment is unobservable or it is unknown in principle. This would render the theory of open, non-Hamiltonian and dissipative quantum systems a fundamental generalization of the theory of closed Hamiltonian quantum systems. Now the open, dissipative and non-Hamiltonian quantum systems are of strong theoretical interest.[133–138]

7.1. *System and environment*

Let Q and P be the self-adjoint operators of coordinate and momentum of the system respectively, and q_k and p_k describe those of the environment.

The Hamiltonian H of the system is:

$$H_s = \frac{P^2}{2M} + V(Q)\,. \tag{75}$$

As a model of environment, we consider an infinite set of harmonic oscillators coupled to the system. The environment Hamiltonian is:

$$H_e = \sum_{n=1}^{N} \left(\frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2 q_n^2}{2} \right). \tag{76}$$

This model is called the independent-oscillator model, since the oscillators do not interact with each other.

The interaction between the system and the environment will be considered in the form:

$$H_i = -Q \sum_{n=1}^{N} C_n q_n + Q^2 \sum_{n=1}^{N} \frac{C_n^2}{2m_n \omega_n^2}\,, \tag{77}$$

where C_n are the coupling constants.

Note that the total Hamiltonian $H = H_s + H_e + H_i$ for the case $V(Q) = (M\Omega^2/2)Q^2$ is the well-known Caldeira–Leggett Hamiltonian.[68,69]

7.2. Equations of motion for open systems

Using total Hamiltonian $H = H_s + H_e + H_i$, we can derive Heisenberg equations for the system and the environment. For the system we have:

$$\frac{dQ}{dt} = \frac{1}{i\hbar}[Q, H] = M^{-1}P\,,$$
$$\frac{dP}{dt} = \frac{1}{i\hbar}[P, H] = -V'(Q) + \sum_{n=1}^{N}\left(C_n q_n - \frac{C_n^2}{m_n\omega_n^2}Q\right). \tag{78}$$

The Heisenberg equations for the environment are:

$$\frac{dq_n}{dt} = \frac{1}{i\hbar}[q_n, H] = m_n^{-1}p_n\,, \quad \frac{dp_n}{dt} = \frac{1}{i\hbar}[p_n, H] = -m_n\omega_n^2 q_n + C_n Q\,. \tag{79}$$

Eliminating the operators P and p_n, $n = 1, \ldots, N$, we can write Eqs. (78) and (79) in the form:

$$M\frac{d^2Q}{dt^2} + V'(Q) = \sum_{n=1}^{N}\left(C_n q_n - \frac{C_n^2}{m_n\omega_n^2}Q\right)\,, \tag{80}$$

$$m_n\frac{d^2q_n}{dt^2} + m_n\omega_n^2 q_n = C_n Q\,. \tag{81}$$

The solution of operator Eq. (81) has the form:

$$q_n(t) = q_n(0)\cos(\omega_n t) + \frac{p_n(0)}{m_n\omega_n}\sin(\omega_n t) + \frac{C_n}{m_n\omega_n}\int_0^t Q(\tau)\sin\omega_n(t-\tau)d\tau\,, \tag{82}$$

where $q_n(0)$ and $p_n(0)$ are the initial values of coordinate and momentum operators of the environment nth oscillators.

Using solution of (82) we can derive the equation:

$$M\frac{d^2Q}{dt^2} + \int_0^t \mathcal{M}(t-\tau)\frac{dQ(\tau)}{d\tau}d\tau + V'(Q) = F(t)\,, \tag{83}$$

where the function

$$\mathcal{M}(t) = \sum_{n=1}^{N}\frac{C_n^2}{m_n\omega_n^2}\cos(\omega_n t) \tag{84}$$

is called the memory kernel. The one-parameter operator function:

$$F(t) = \sum_{n=1}^{N}\left(C_n q_n(0)\cos(\omega_n t) + \frac{C_n p_n(0)}{m_n\omega_n}\sin(\omega_n t) - \frac{C_n^2}{m_n\omega_n^2}Q(0)\cos(\omega_n t)\right) \tag{85}$$

can be interpreted as a stochastic force since the initial states of the environment are uncertain and it can be determined by a distribution of the average values of $q_n(0)$ and $p_n(0)$.

7.3. *Quantum dynamics with memory*

The memory function $\mathcal{M}(t)$ describes dissipation if $\mathcal{M}(t)$ is positive definite and decreases monotonically. These conditions are achieved if $N \to \infty$ and if $C_n^2/(m_n\omega_n^2)$ and ω_n are sufficiently smooth functions of the index n.

For $N \to \infty$, the sum in Eq. (84) is replaced by the integral

$$\mathcal{M}(t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{J(\omega)}{\omega}\cos(\omega t)d\omega\,, \tag{86}$$

where $J(\omega)$ is spectral density. We assume that the oscillator environment contains an infinite number of oscillators with a continuous spectrum.

For the spectral density:

$$J(\omega) = \frac{\pi\omega}{2}\sum_{n=1}^{N}\frac{C_n^2}{m_n\omega_n^2}\delta(\omega-\omega_n)\,, \tag{87}$$

Eq. (86) gives the memory function (84). If we consider the Cauchy distribution $J(\omega) = a/(\omega^2+\lambda^2)$, then equation (86) gives the exponential memory kernel $\mathcal{M}(t) = (a/\lambda)e^{-\lambda t}$.

We can consider a power-law for the spectral density:

$$J(\omega) = A\omega^{\beta}\,, \quad 0<\beta<1\,, \tag{88}$$

where $A>0$. Note that density (88) leads to the power-law for the memory function $\mathcal{M}(t) \sim t^{-\beta}$. Equation (88) can be achieved by a different type of combinations of coupling coefficients $C(\omega)$ and density of states $g(\omega)$:

$$J(\omega) = \frac{\pi\omega}{2}g(\omega)C(\omega)\,.$$

Using the Fourier cosine-transform

$$\int_0^{\infty} x^{-\alpha}\cos(xy)dx = \frac{\pi}{2\Gamma(\alpha)\cos(\pi\alpha/2)}\,y^{\alpha-1}, \quad (0<\alpha<1)\,, \tag{89}$$

we get the equation:

$$M\frac{d^2Q}{dt^2} + \frac{A}{\sin(\pi\beta/2)}\,{}_0D_t^{\beta}Q + V'(Q) = F(t)\,, \tag{90}$$

where ${}_0D_t^{\beta}$ is the Caputo fractional derivative

$${}_0D_t^{\beta}Q(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{(D^1Q)(\tau)d\tau}{(t-\tau)^{\beta}}\,, \quad (0<\beta<1)\,. \tag{91}$$

As a simple example of quantum system, which is described by (90), we can consider the linear fractional oscillator that is an object of numerous investigations[110–114,117–122] because of different applications.

If $F(t) \neq 0$, then Eq. (90) can be considered as a fractional Langevin equation. We also can consider a general quantum analogs of fractional Langevin equation[123–126,128,130–132] that can be connected with the quantum Brownian motion that is considered by Lindblad.[140]

As a result, we obtain a fractional differential equation for operators $Q(t)$ from the interaction between the system and the environment with power-law spectral density.[67] The parameter α can be used to control quantum dynamics of nano-systems like individual atoms and molecules in an environment.[73–76] Quantum control is concerned with active manipulation of physical and chemical processes on the atomic and molecular scale. Controlled manipulation by atomic and molecular quantum systems has attracted a lot of research attention in recent years.[70–72] Note that the models to control of open quantum system dynamics is a very important subject of nanotechnology.

7.4. *Quantum analogs of fractional derivatives*

One of the way to derive a quantum description of physical systems is an application of a procedure of quantization to classical models. We can use the Weyl quantization to obtain quantum analogs of differential operator of noninteger orders with respect to coordinates.

The Weyl quantization π_W is defined by:

$$\pi_W\big(q_k A(q,p)\big) = \frac{1}{2}(Q_k\hat{A} + \hat{A}Q_k)\,, \quad \pi_W\big(p_k A(q,p)\big) = \frac{1}{2}\big(P_k\hat{A} + \hat{A}P_k\big)\,, \tag{92}$$

$$\pi_W\big(D^1_{p_k} A(q,p)\big) = -\frac{1}{i\hbar}(P_k\hat{A} - \hat{A}P_k)\,, \quad \pi_W\big(D^1_{p_k} A(q,p)\big) = \frac{1}{i\hbar}\big(Q_k\hat{A} - \hat{A}Q_k\big)\,, \tag{93}$$

for any $\hat{A} = A(Q,P) = \pi_W(A(q,p))$, where $k = 1,\ldots,n$, $Q_k = \pi_W(q_k)$ and $P_k = \pi_W(p_k)$.

Weyl quantization π_W maps[86] the differential operator $\mathcal{L}[q,p,D^1_q,D^1_p]$ on the function space and the superoperator $\mathcal{L}[L^+_Q, L^+_P, -L^-_P, L^-_Q]$ acting on the operator space, where,

$$L^-_A\hat{B} = \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A})\,, \quad L^+_A\hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})\,.$$

If $A(x)$ is an analytic function for $x \in (0,b)$, then the Riemann–Liouville fractional derivative can be represented in the form:

$${}_0D^\alpha_x A(x) = \sum_{n=0}^{\infty} a(n,\alpha) x^{n-\alpha} D^n_x A(x)\,, \tag{94}$$

where

$$a(n,\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)\Gamma(n-\alpha+1)}\,.$$

Equation (94) defines a fractional derivative on operator algebra.[86]

The Weyl quantization of the Riemann–Liouville fractional derivatives with respect to phase-space coordinates[45] gives:

$${}_0\mathcal{D}^\alpha_{Q_k} = \pi({}_0D^\alpha_{q_k}) = \sum_{n=0}^{\infty} a(n,\alpha)(L^+_{Q_k})^{n-\alpha}(-L^-_{P_k})^n\,, \tag{95}$$

$$ {}_0\mathcal{D}^{\alpha}_{P_k} = \pi({}_0D^{\alpha}_{p_k}) = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{P_k})^{n-\alpha}(L^{-}_{Q_k})^{n}\,, \tag{96} $$

where $L^{\pm}_{A}$ are defined by the equations:

$$ L^{-}_{A}\hat{B} = \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A})\,, \quad L^{+}_{A}\hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})\,. $$

For example, we have:

$$ {}_0\mathcal{D}^{\alpha}_{Q}Q^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}Q^{n-\alpha}\,, \quad {}_0\mathcal{D}^{\alpha}_{P}P^{n} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}P^{n-\alpha}\,, $$

where $n \geqslant 1$, and $\alpha \geqslant 0$.

It not only allows to consistently formulate the evolution of such quantum systems, but also to consider the dynamics of a wide class of quantum systems, such as the nonlocal non-Hamiltonian, dissipative and nonlinear systems. Quantum analogs of the nonlocal systems with regular and strange attractors can be described.[86] The regular quantum attractors can be considered as stationary states of non-Hamiltonian quantum systems. The condition given by Davies[139] defines the stationary state of non-Hamiltonian quantum system. An example, where the stationary state is unique and approached by all states for long time is considered by Lindblad[140] for Brownian motion of quantum harmonic oscillator. Spohn[141–143] derives sufficient condition for the existence of a unique stationary state for the non-Hamiltonian quantum system described by Lindblad equation. The stationary solution of the Wigner function evolution equation for non-Hamiltonian quantum system was discussed in Refs. 144 and 145. Quantum effects in the steady states of the dissipative map are considered in Ref. 146. Stationary pure states of quantum non-Hamiltonian systems are considered in Refs. 147 and 148. For classical non-Hamiltonian systems, stationary states are presented in Refs. 149–151.

8. Fractional Generalization of Vector Calculus

The fractional calculus has a long history from 1695. The history of fractional vector calculus is not so long. It has little more than ten years and can be reduced to small number of papers (about twenty articles) (see Ref. 65).

A consistent fractional vector calculus is important for application in the following research directions.

(1) Nonlocal statistical mechanics. It should be noted that Vlasov[66] book is entirely devoted to nonlocal statistical mechanics. The fractional derivatives in equations can be connected with a long-range power-law interparticle interactions in statistical mechanics.[52,60]

(2) Nonlocal electrodynamics,[77–80] where the spatial dispersion describes nonlocal properties of media.

(3) Nonlocal hydrodynamics and waves propagation in media with long-range interaction[81] (see also Sec. 8.16 in Ref. 26).

It is known that the theory of differential forms is very important in mathematics and physics.[82] The fractional differential forms can be interesting to formulate fractional generalizations of differential geometry, including symplectic, Kahler, Riemann and affine-metric geometries. These generalizations allow us to derive new rigorous results in modern theoretical physics and astrophysics[83] and in fractional generalization of relativistic field theory[152–166] in curved space–time. We assume that the fractional differential forms and fractional integral theorems for these forms can also be used to describe classical dynamics[84] and thermodynamics.

It is important to have fractional generalizations of the symplectic geometry, Lie and Poisson algebras, the concept of derivation on operator algebras. It allows to apply a generalization of algebraic structures of fractional calculus to classical and quantum mechanics. Note that the theory of operator algebras are very important in quantum theory.[85,86]

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