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Poiseuille equation for steady flow of fractal fluid

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Fractal fluid is considered in the framework of continuous models with noninteger dimensional spaces (NIDS). A recently proposed vector calculus in NIDS is used to get a description of fractal fluid flow in pipes with circular cross-sections. The Navier–Stokes equations of fractal incompressible viscous fluids are used to derive a generalization of the Poiseuille equation of steady flow of fractal media in pipe.

Keywords: Fractal fluid; Poiseuille equation; Navier–Stokes equation; steady flow.

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1. Introduction

A characteristic property of fractal materials and fluids is noninteger physical dimension such as mass and "particle" dimensions (see Refs. 1–3 and references therein). Continuous models of fractal distributions of particle, media and fields have been proposed in Refs. 4 and 5 in 2005 (see also Ref. 3) and then these models have been developed by Ostoja-Starzewski,[12–15] Balankin[16–19] and other scientists. These models are based on the notion of power-law density of states.[3] The continuous model of fractal media and materials are formulated by using an integral and differential operators on noninteger dimensional spaces (NIDS),[20–22] which was recently developed in Refs. 24 and 25.

It is well known that integral operators for NIDS are actively used in quantum field theories[20] for the dimensional regularization of ultraviolet divergences. NIDS generalizations of integration and the Laplacian are proposed by Stillinger in Ref. 21. The Stillinger approach[21] has been extended in Ref. 22 for multiple variables case by using product measure method. A new scalar Laplace operator for NIDS has also been suggested by Palmer and Stavrinou in Ref. 22. In Refs. 21 and 22, only the scalar Laplacian of scalar fields for NIDS has been proposed. A NIDS generalization of the vector Laplace operator[23] is not considered in Refs. 21 and

22. Problems of definition of NIDS differential operators such as gradient, divergence, curl operators are not discussed in Refs. 21 and 22. A possibility to use only the NIDS scalar Laplacian in the continuous models approach strongly restricts an application that these models do to fractal dynamics of fluids. This restriction leads us to the impossibility to consider equations for vector field $\mathbf{v}(\mathbf{r}, t)$ in fractal hydrodynamics.

Recently, a vector calculus for NIDS, where the NIDS differential operators of first-order (gradient, divergence), and the second-order operators (the scalar and vector Laplacians), have been proposed in Refs. 24 and 25. The suggested NIDS operations give us the tool to describe fractal fluids and materials by continuous models. For example, using the NIDS vector calculus, we formulate the fractal elasticity,[26] the fractal electrodynamics,[27,28] the heat transfer in fractal materials,[30] the acoustic waves in fractal media,[31] and the fractal hydrodynamics.[29]

In Ref. 29, NIDS continuous models of flow of fractal fluid have been proposed. The suggested generalizations of equations to describe flow of fractal fluid are based on the NIDS models of fractal media. Instead of the other approaches,[16–19,32] the suggested approach, which is based on the vector calculus for NIDS,[24,25] allows us to derive exact solutions of equations for flow of fractal fluids in pipes. The Poiseuille equations for fractal fluid and its solution have been first proposed by author in article.[29] In Refs. 16–19 and 32 and in the other articles, the solutions of equations of fractal fluid in pipes are not suggested.

Using the proposed NIDS generalization of the Navier–Stokes equations and the corresponding Poiseuille equations, in Ref. 29, we describe a flow of fractal fluid in a pipe with the internal radius R_1 and external radius R_2 and the boundary conditions $v_x(R_1) = 0$, $v_x(R_2) = 0$. Then, the flow in pipe without internal radius R is derived as a limit case $R_1 \to 0$ and $R_2 = R$, and the expression for $v_x(r)$ has been suggested. This expression is incorrect since it gives the velocity of flow at the center of this pipe ($r = 0$) equal to zero $v_x(0) = 0$. For pipe without internal radius ($R_1 = 0$), $v_x(0) = 0$ is physically incorrect value based on the boundary conditions $v_x(R_1) = v_x(0) = 0$ which is incorrect for this case. Note that the limit case $R_1 \to 0$ cannot be applied to the boundary condition $v_x(R_1) = 0$. For the pipes without internal radius ($R_1 = 0$), we should use the boundary condition in the form $v_x(0) = v_{\max}$.

In this article, we solve the fractal Poiseuille equation for isotropic incompressible viscous fractal fluid with this correct boundary condition $v_x(0) = v_{\max}$, which is correct for pipes without internal radius. The derived solution describes a physically correct expression for velocity $v_x(r)$ of flow of fractal fluids in pipes without internal radius.

2. Motivation of Study of Fractal Fluids

The study of fractal fluid flow is important for fractal biophysical models of blood flow in cardiovascular system,[6–8] complex dynamics of fractal fluids in hydrology,[9] and in dynamics of multi-phase media[10,11] with scale properties. It is known that the blood is composed of proteins, glucose, mineral ions, hormones, carbon dioxide, blood cells and other suspended particles. The blood as a multi-phase complex medium can demonstrate some properties of a fractal distribution of some blood components including bacteria, viruses and medicinal substances getting into the blood.

The blood channels, such as veins and arteries, cannot be considered as pipes with internal radius. Therefore, it is important to have a correct expression of the flow of viscous fractal fluid in pipes without internal radius.

A fluid can be considered as a fractal fluid if we have the power law $N_D(W) \sim R^D$ (or $M_D(W) \sim R^D$) for some scale region. The number D is called "particle" (mass) dimension. The fractal fluid is called homogeneous if this power-law does not depend on the translation of the region W. For homogeneous fractal fluid, any two regions W_1 and W_2 with the equal volumes $V_n(W_1) = V_n(W_2)$ have equal number of particles $N_D(W_1) = N_D(W_2)$.

We can consider a two-component medium where distribution of one component into another component is characterized by noninteger "particle" or mass dimension. This noninteger dimensional component can be considered as a fractal fluid. The fractal dimension can be determined experimentally by using labels with radioactive isotopes in particles of component, which is assumed fractal. Let us describe five possible implementations of fractal fluids in two-component medium.

(1) A simplest model of fractal fluid is a liquid which is distributed in an empty region with noninteger dimension $D < 3$. This model is a liquid analog of fractal porous solid medium.
(2) Fractal fluid can also be viewed as a two-phase medium composed of a gas and a liquid where the liquid phase is characterized by fractal dimension.
(3) An fractal emulsion, which is a mixture of two immiscible liquids, one of which (the dispersed phase) is fractally dispersed in the other (the continuous phase). In the emulsion, both the dispersed and the continuous phase are liquids, where we consider the dispersed phase as a fractal fluid.
(4) A fractal solution, which is a homogeneous mixture i composed of two liquid phase. In this case, we consider a fractal distribution of a solute dissolved in a nonfractal solvent. The solute is described as a fractal fluid.
(5) A fractal suspension, which is an internal phase (solid) that is fractally distributed in the external phase (fluid). Then, we consider a fractal suspension, where small solid particles are fractally distributed. For a liquid mixed with solid particles, the noninteger dimensions can be caused by a power-law distribution of solid particles by size or mass.

3. Differential Operators in Noninteger Dimensional Spaces

For simplification, the scalar fields φ and vector fields $\mathbf{v}$ function will be assumed to be independent of the angles $\varphi(\mathbf{r}, t) = \varphi(r, t)$, $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(r, t) = v_r(r, t)\, \mathbf{e}_r$, where

$\mathbf{e}_r = \mathbf{r}/r$ is the unit vector, and $v_r = v_r(r,t)$ is a radial part of $\mathbf{v}$. This means that we consider the case of only rotationally covariant functions which is analogous to simplification that is usually used for the NIDS integrations (see Sec. 4 of Ref. 20).

Consider a region W_D of fractal fluid with the mass dimensions $\dim(W_D) = D$ and the dimension $\dim(S_d) = d$ of the boundary $S_d = \partial W_D$. For these dimensions the relation $d = D - 1$ does not hold ($\dim(\partial W_D) \neq \dim(S_d) - 1$) in general. Therefore, we can use $\alpha_r = D - d$ as a dimension in the radial direction $\mathbf{e}_r$.

The NIDS gradient operator is defined[24] by the equation

$$\mathrm{Grad}_r^{D,d}\ \varphi(r) = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\, r^{\alpha_r-1}}\, \frac{\partial \varphi(r)}{\partial r}\, \mathbf{e}_r, \tag{1}$$

where $\varphi(\mathbf{r},t) = \varphi(r,t)$ is the scalar field. Applying integration in NIDS and corresponding Gauss's theorem, we give[24] the expression of the NIDS divergence in the form

$$\mathrm{Div}_r^{D,d}\ \mathbf{v}(r) = \frac{\pi^{(d+1-D)/2}\, \Gamma(D/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{D-d-1}} \frac{\partial v_r(r)}{\partial r} + \frac{d}{r^{D-d}}\, v_r(r) \right). \tag{2}$$

Equation (2) is rewritten by using α_r as

$$\mathrm{Div}_r^{D,d}\ \mathbf{v}(r) = \frac{\pi^{(1-\alpha_r)/2}\, \Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{\alpha_r-1}} \frac{\partial v_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\, v_r(r) \right). \tag{3}$$

Using the operators (1) and (2) for the fields $\varphi = \varphi(r)$ and $\mathbf{v} = v_r(r)\, \mathbf{e}_r$, we get[24] the NIDS Laplacians in the form

$${}^{S}\Delta_r^{D,d}\, \varphi = \mathrm{Div}_r^{D,d}\, \mathrm{Grad}_r^{D,d}\, \varphi, \quad {}^{V}\Delta_r^{D,d} \mathbf{v} = \mathrm{Grad}_r^{D,d}\, \mathrm{Div}_r^{D,d}\, \mathbf{v}. \tag{4}$$

The scalar NIDS Laplacian ${}^{S}\Delta_r^{D,d}$ for $d \neq D-1$ for the field $\varphi = \varphi(r)$ is

$${}^{S}\Delta_r^{D,d} \varphi = A(d,\alpha_r) \left(\frac{1}{r^{2\alpha_r-2}} \frac{\partial^2 \varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}} \frac{\partial \varphi}{\partial r} \right), \tag{5}$$

where

$$A(d,\alpha_r) = \frac{\Gamma((d+\alpha_r)/2)\, \Gamma(\alpha_r/2)}{\pi^{(2\alpha_r-1)/2}\, \Gamma((d+1)/2)}. \tag{6}$$

The vector NIDS Laplacian ${}^{V}\Delta_r^{D,d}$ for $d \neq D-1$ for vector field $\mathbf{v}(r) = v_r(r)\, \mathbf{e}_r$ has the form

$${}^{V}\Delta_r^{D,d} \mathbf{v} = A(d,\alpha_r) \left(\frac{1}{r^{2\alpha_r-2}} \frac{\partial^2 v_r}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}} \frac{\partial v_r}{\partial r} - \frac{d\, \alpha_r}{r^{2\alpha_r}}\, v_r \right) \mathbf{e}_r. \tag{7}$$

The NIDS differential operators (1), (2), (5) and (7) allow us to get equations of fractal fluid with boundary dimension $d \neq D-1$ in the framework of continuous models with NIDS.

4. Fractal Navier–Stokes Equations

Noninteger mass or "particle" dimensions are basic characteristics of fractal fluids.[3] In the region $W \subset \mathbb{R}^3$ of fractal fluid, the number of particles $N_D(W)$ or mass $M_D(W)$ increase more slowly than the volume $V_3(W)$ of W. In isotropic fractal fluid, the number of particles $N_D(W)$ in the ball region W with radius R, has the relation $N_D(W) = N_0(R/R_0)^D$, where R_0 is a characteristic scale, and D is the "particle" dimension. If particles have identical masses m_0, then $N_D(W) = N_0(R/R_0)^D$ gives the expression for the region mass $M_D(W) = M_0(R/R_0)^D$, where $M_0 = m_0\, N_0$, which means that mass dimension is equal to "particle" dimension. Fluid can be called a fractal fluids if the "particle" or mass dimension is a noninteger.

Using NIDS approach, the incompressible viscous fractal fluid can be described by the equations

$$\mathrm{Div}^{D,d}_r\, \mathbf{v}(r,t) = 0, \tag{8}$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{f} - \frac{1}{\rho}\,\mathrm{Grad}^{D,d}_r\, p + \nu\, {}^V\Delta^{D,d}_r \mathbf{v}, \tag{9}$$

where $\nu = \mu/\rho$ is kinematic viscosity, μ is dynamic viscosity μ, ρ is fluid density, $\mathbf{f}$ is vector field that describes a mass force, d/dt denotes the material derivative

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \Big(\mathbf{v}, \mathrm{Grad}^{D,d}_r\, \mathbf{v}\Big), \tag{10}$$

where the NIDS gradient $\mathrm{Grad}^{D,d}_r$, the NIDS divergence $\mathrm{Div}^{D,d}_r$, and the NIDS vector Laplace operator ${}^V\Delta^{D,d}_r$ are defined by Eqs. (1), (3) and (7). Equations (8)–(10) are the Navier–Stokes equations for fractal fluids. These equations describe incompressible viscous fractal fluids within NIDS approach.

For convenience of description, we use dimensionless variables $x/R_0 \to x$, $y/R_0 \to x$, $z/R_0 \to x$, $r/R_0 \to r$. Here R_0 is the characteristic length in fractal medium. Using these variables, density ρ and fields p, $\mathbf{v}$, f have usual "integer" physical dimensions.

The suggested Navier–Stokes equations can be applied to describe isotropic fractal fluids with spherical or cylindrical symmetry, i.e., when the fields p, $\mathbf{v}$, $\mathbf{f}$ are not dependent on the angles.

5. Poiseuille Equation for Fractal Fluid in Pipe

Let us describe a motion of an incompressible fractal fluid in pipe with circular cross-section. We will use the suggested continuous models with NIDS to consider equations of steady flow of fractal fluids in pipes. We assume that X-axis is the axis of this pipe. For laminar case, the flow of fractal fluid is along the X-axis at all points. Then the velocity field is a function of r only, i.e., $\mathbf{v}(\mathbf{r},t) = v_x(r)\,\mathbf{e}_x$. In this case, the continuity equation holds identically. For Y-axis and Z-axis, the suggested Navier–Stokes equation (9) give a constant pressure over circular cross-section of the pipe. Fractal Navier–Stokes equation (9) has the form

$$ {}^{S}\Delta^{D,d}_{r}\, v_x(r) = \frac{1}{\mu}\,\frac{dp}{dx}, \tag{11} $$

where $\mu = \rho\,\nu$, and $dp/dx = \text{const}$. Using is the pressure difference between Δp the ends of pipe and its length l, the pressure gradient dp/dx can be represented as $-\Delta p/l$.

Let us derive Poiseuille equation for fractal fluids from the Navier–Stokes equation (11). For the scalar field $\varphi_{\text{eff}}(r) = v_x(r)$, we can use Eqs. (1), (3) and (5), where $D \to D_x = D - \alpha_x$ and $d \to d_x = d - \alpha_x$, we can get corresponding equation. Substitution of (5) into the Navier–Stokes equation (11) that describes fractal fluid with $\alpha_r = D - d \neq 1$ in pipe, we get the equation

$$ A(d_x, \alpha_r) \left(\frac{1}{r^{2\alpha_r - 2}} \, \frac{\partial^2 v_x(r)}{\partial r^2} + \frac{d_x + 1 - \alpha_r}{r^{2\alpha_r - 1}} \, \frac{\partial v_x(r)}{\partial r} \right) - \frac{1}{\mu}\,\frac{dp}{dx} = 0, \tag{12} $$

where $A(d_x, \alpha_r)$ is defined by (6), $d_x = d - \alpha_x$, and α_x is dimension along the X-axis. Equation (12) with $\alpha_r = \alpha_x = 1$ gives

$$ \frac{\partial^2 v_x(r)}{\partial r^2} + \frac{D-2}{r}\,\frac{\partial v_x(r)}{\partial r} - \frac{1}{\mu}\,\frac{dp}{dx} = 0, \tag{13} $$

since $D = d + \alpha_r$. For $1 < D < 3$, the general solution of (12) has the form

$$ v_x(r) = C_1\, r^{\alpha_r - d_x} + C_2 + \frac{1}{2\,(d_x + \alpha_r)\,\alpha_r\, A(d_x, \alpha_r)\,\mu}\,\frac{dp}{dx}\, r^{2\alpha_r} \quad (1 < D < 3). \tag{14} $$

For fractal fluid in a pipe with the external radius R (and with zero internal radius), the constants C_1 and C_2 of the solution (14) can be derived from the boundary conditions

$$ v_x(0) = v_{\max}, \quad v_x(R) = 0. \tag{15} $$

These conditions give

$$ C_1 = 0, \quad C_2 = -\frac{1}{2\,(d_x + \alpha_r)\,\alpha_r\, A(d_x, \alpha_r)\,\mu}\,\frac{dp}{dx}\, R^{2\alpha_r}. \tag{16} $$

Substituting (16) into (14), we obtain

$$ v_x(r) = -\frac{1}{2\,(d_x + \alpha_r)\,\alpha_r\, A(d_x, \alpha_r)\,\mu}\,\frac{dp}{dx}\, R^{2\alpha_r} \left(1 - \left(\frac{r}{R}\right)^{2\alpha_r} \right) \quad (1 < D < 3). \tag{17} $$

This is the Poiseuille equation of isotropic fractal fluids. Nonfractal fluids are characterized by the integer dimensions $\alpha_r = 1$ and $d_x = 1$. In this case, Eq. (17) gives the standard Poiseuille equation

$$ v_x(r) = -\frac{1}{4\,\mu}\,\frac{dp}{dx}\, R^2 \left(1 - \left(\frac{r}{R}\right)^{2} \right). \tag{18} $$

Using the effective dynamic viscosity

$$ \mu_{\text{eff}}(\alpha_x, d_x) = \frac{1}{2}\,(d_x + \alpha_r)\,\alpha_r\, A(d_x, \alpha_r)\,\mu, \tag{19} $$

we represent Eq. (17) in the form

$$v_x(r) = -\frac{1}{4\,\mu_{\text{eff}}(\alpha_r, d_x)}\,\frac{dp}{dx}\,R^{2\alpha_r}\left(1-\left(\frac{r}{R}\right)^{2\alpha_r}\right) \quad (1 < D < 3), \tag{20}$$

where $\mu_{\text{eff}}(\alpha_r, d_x)$ for nonfractal case ($d_x = \alpha_r = 1$) is equal to μ. Note that Eq. (20) gives nonzero velocity at the center of the pipe

$$v_x(0) = -\frac{1}{4\,\mu_{\text{eff}}(\alpha_r, d_x)}\,\frac{dp}{dx}\,R^{2\alpha_r} \quad (1 < D < 3) \tag{21}$$

as opposed to incorrect limit expression $v_x(0) = 0$ of Ref. 29 for zero internal radius $R_1 = 0$. The plots of function (20) are presented by Figs. 1, 3, 5, 7 and 9, where $\mu = 1$ and $dp/dx = -1$.

In Ref. 29, it is suggested the equation

$$v_x(r) = -\frac{1}{4\,\mu_{\text{eff}}(\alpha_r, d_x)}\,\frac{dp}{dx}\,R^{2\alpha_r}\left(\left(\frac{r}{R}\right)^{\alpha_r - d_x} - \left(\frac{r}{R}\right)^{2\alpha_r}\right) \quad (1 < D < 3) \tag{22}$$

as a limit case of the equation for flow of fractal fluid in a pipe with the internal radius R_1 and external radius R_2, when $R_1 \to 0$ and $R_2 = R$. The main disadvantage of Eq. (22) is zero velocity $v_x(0) = 0$ at the center of the pipe $r = 0$. For comparison, see (21). For pipe without internal radius ($R_1 = 0$), Eq. (22) gives physically incorrect results. This is an incorrect expression of the limit case ($R_1 \to 0$) is obtained due to the use of the boundary condition $v_x(0) = 0$. The plots of function (22) are presented by Figs. 2, 4, 6, 8 and 10, where $\mu = 1$ and $dp/dx = -1$. We can see that the velocity of flow decreases when approaching the center of the pipe.

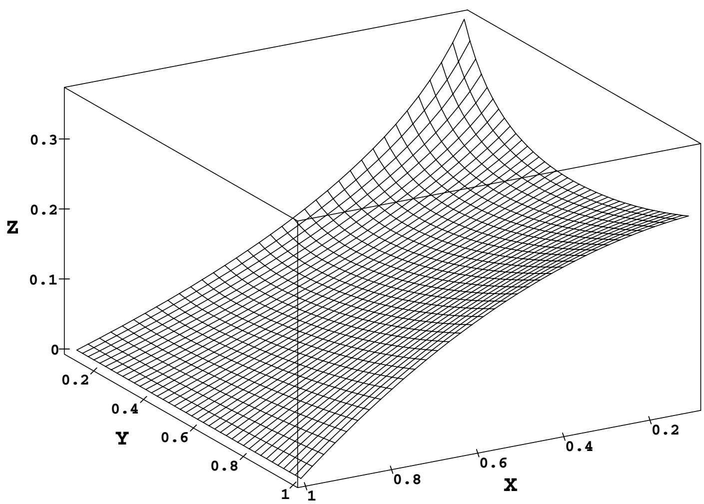

Fig. 1. Plot of the velocity function $z = v(r, \alpha_r)$ defined by (20) for the ranges $x = r/R \in [0.1; 1]$, $y = \alpha_r \in [0.1; 1]$, and $d_x = 0.9$, $dp/dx = -1$ and $\mu = 1$.

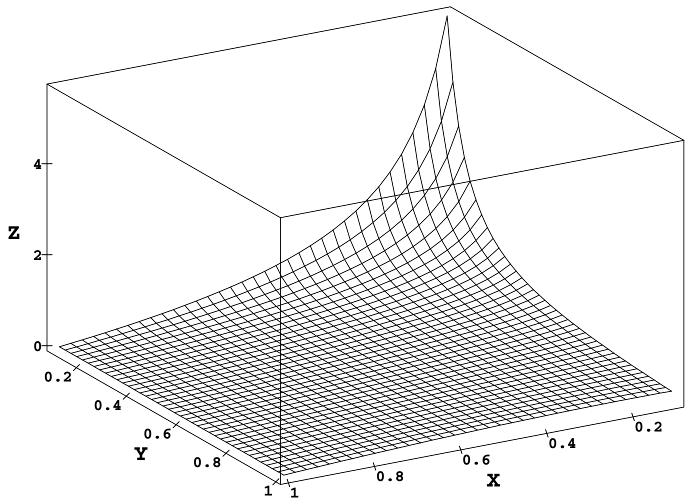

Fig. 2. Plot of the velocity function $z = v(r, \alpha_r)$ defined by (22) for the ranges $x = r/R \in [0.1; 1]$, $y = \alpha_r \in [0.1; 1]$, and $d_x = 0.9$, $dp/dx = -1$ and $\mu = 1$.

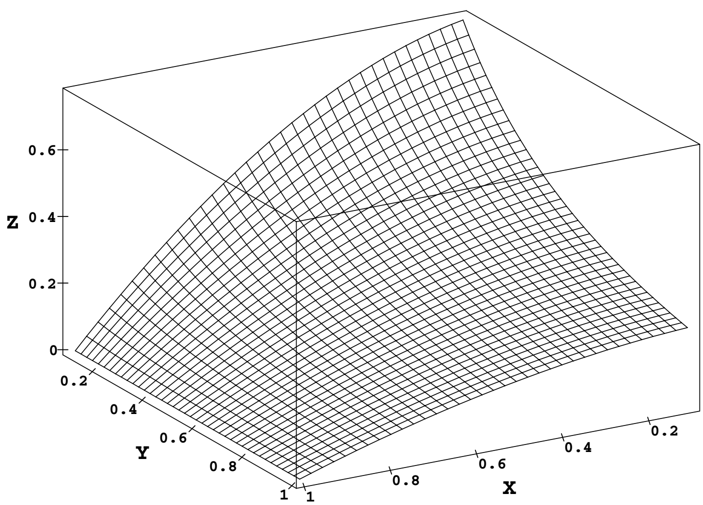

Fig. 3. Plot of the velocity function $z = v(r, d_x)$ defined by (20) for the ranges $x = r/R \in [0.1; 1]$, $y = d_x \in [0.1; 1]$, and $\alpha_r = 0.9$, $dp/dx = -1$ and $\mu = 1$.

If the radial dimension of fractal fluid is equal to one ($\alpha_r = 1$), then Eq. (20) gives the fractal Poiseuille equation in the form

$$v_x(r) = -\frac{1}{4\,\mu_{\mathrm{eff}}(D)}\,\frac{dp}{dx}\,R^2\,\left(1 - \left(\frac{r}{R}\right)^2\right), \tag{23}$$

where $1 < D < 3$ and the effective dynamic viscosity is

$$\mu_{\mathrm{eff}}(D) = \frac{D-1}{2}\,\mu. \tag{24}$$

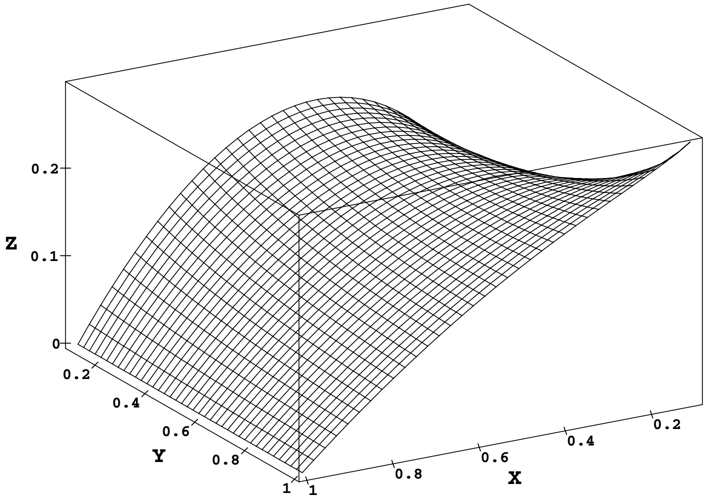

Fig. 4. Plot of the velocity function $z = v(r, d_x)$ defined by (22) for the ranges $x = r/R \in [0.1; 1]$, $y = d_x \in [0.1; 1]$, and $\alpha_r = 0.9$, $dp/dx = -1$ and $\mu = 1$.

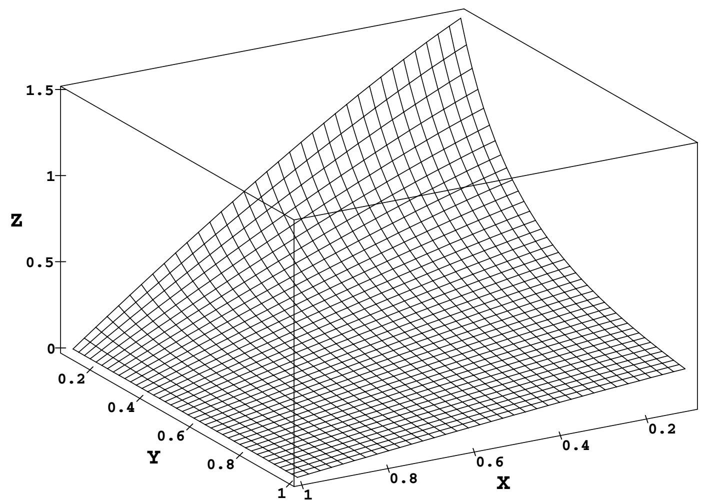

Fig. 5. Plot of the velocity function $z = v(r, d_x)$ defined by (20) for the ranges $x = r/R \in [0.1; 1]$, $y = d_x \in [0.1; 1]$, and $\alpha_r = 0.6$, $dp/dx = -1$ and $\mu = 1$.

In this case, the dependence of the velocity $v_x(r)$ on the distance r is the same as for the standard (nonfractal) case.

As a result, we obtain the following:

(1) For fractal fluids with $\alpha_r = 1$, the velocity distribution across the pipe is parabolic (23). This means that the behavior of fractal fluids with the radial dimension $\alpha_r = 1$ is similar to the behavior of nonfractal fluids with dynamic viscosity $\mu_{\text{eff}}(D)$.

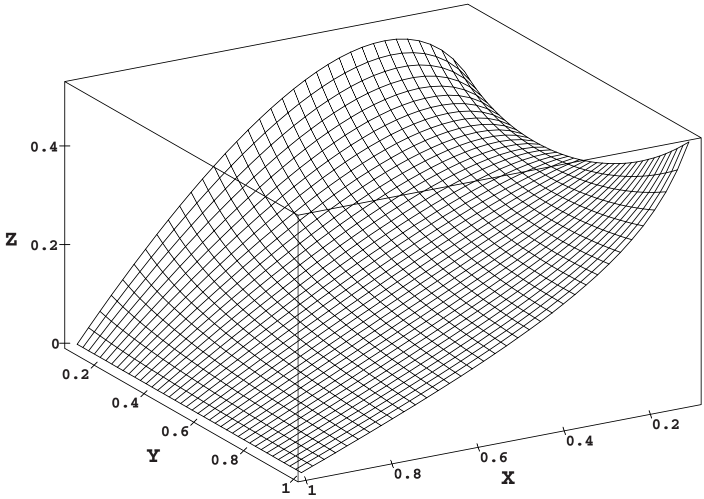

Fig. 6. Plot of the velocity function $z = v(r, d_x)$ defined by (22) for the ranges $x = r/R \in [0.1; 1]$, $y = d_x \in [0.1; 1]$, and $\alpha_r = 0.6$, $dp/dx = -1$ and $\mu = 1$.

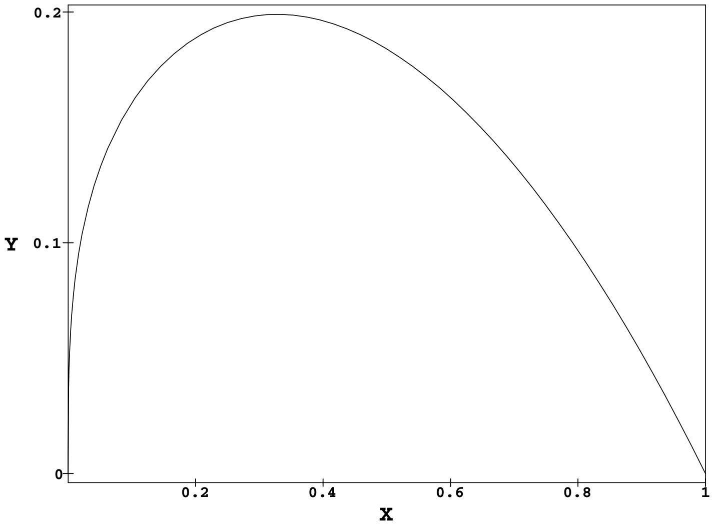

Fig. 7. Plot of the velocity function $y = v(r)$ defined by (20) for the ranges $x = r/R \in [0; 1]$, $\alpha_r = 1$, and $d_x = 0.7$, $dp/dx = -1$ and $\mu = 1$.

(2) For fractal fluids with $\alpha_r \neq 1$, we have noninteger power-law (20) and the effective dynamic viscosity in the form (19). See Figs. 1, 3, 5, 7 and 9.

The dimensions $\alpha_r < 1$ and $\alpha_x < 1$ describes the fractal fluids. We assume that $\alpha_x > 1$ describes a fractal turbulent flow, since the trajectories of the fluid particles can be considered as fractal curves with $\alpha_x > 1$ (the Koch curve has $\alpha_x = \ln(4)/\ln(3) \approx 1.26$).

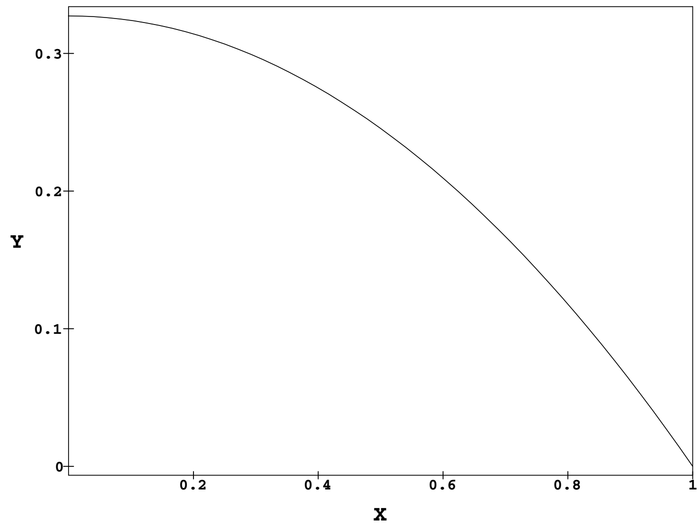

Fig. 8. Plot of the velocity function $y = v(r)$ defined by (22) for the ranges $x = r/R \in [0; 1]$, $\alpha_r = 1$, and $d_x = 0.7$, $dp/dx = -1$ and $\mu = 1$.

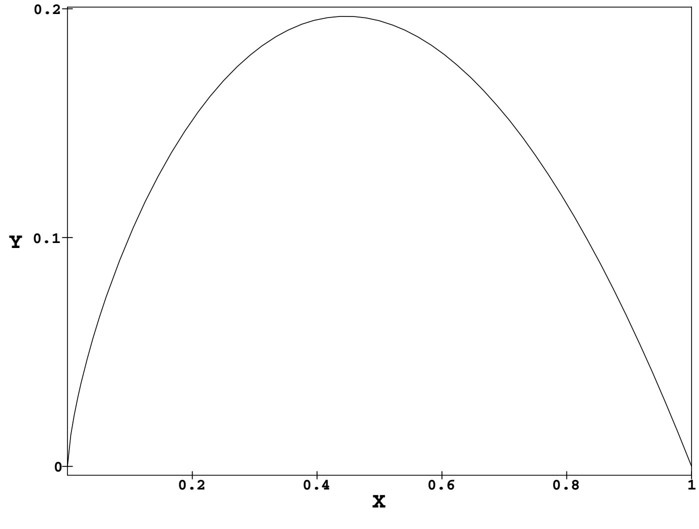

Fig. 9. Plot of the velocity function $y = v(r)$ defined by (20) for the ranges $x = r/R \in [0; 1]$, $\alpha_r = 1$, and $d_x = 0.3$, $dp/dx = -1$ and $\mu = 1$.

As a simple model of fractal fluid, we can consider a liquid, which is distributed in space $\mathbb{R}^3$ with mass dimension $D < 3$. Fractal fluids are liquid analogues of solid materials with fractal distribution of porous. In more general cases, we can consider two-component media, where distribution of one component (solid, liquid, gas) into another component (gas, empty space, fluid) is characterized by noninteger "particle" or mass dimension.

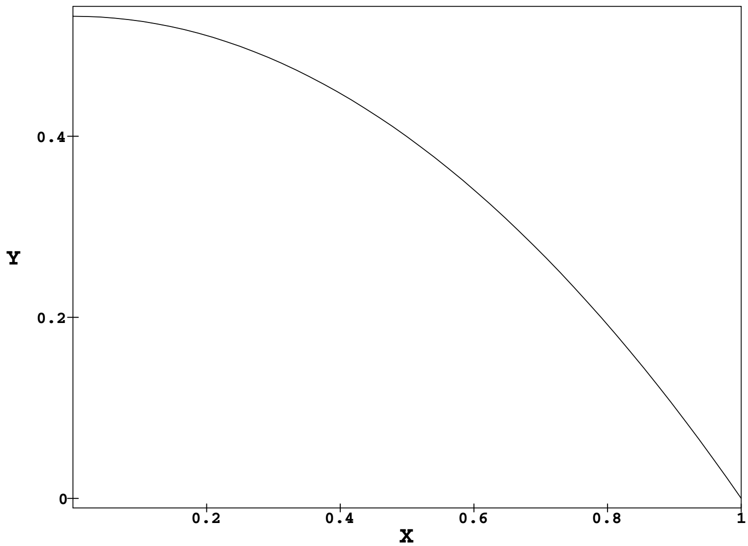

Fig. 10. Plot of the velocity function $y = v(r)$ defined by (22) for the ranges $x = r/R \in [0; 1]$, $\alpha_r = 1$, and $d_x = 0.3$, $dp/dx = -1$ and $\mu = 1$.

6. Conclusion

The Poiseuille equations for fractal fluid and its solution were first proposed in article.[29] Using the suggested NIDS calculus,[24,25] the flow of fractal fluids in a pipe with the internal radius R_1 and external radius R_2.

In this article, we solve the suggested Poiseuille equation for isotropic incompressible viscous fractal fluid with the boundary condition $v_x(0) = v_{\max}$. The derived solution describes a physically correct results for flow of fractal fluids in pipes without internal radius. The behavior of fractal fluids with the radial dimension $\alpha_r = 1$ is similar to the behavior of nonfractal fluids with some effective dynamic viscosity. For fractal fluids with $\alpha_r = 1$, the velocity distribution across the pipe is parabolic. For fractal fluids with $\alpha_r \neq 1$, we have noninteger power-law distribution across the pipe.

The suggested approach to describe flow of fractal fluid are based on the NIDS models of fractal media. Instead of the other approaches,[16–19,32] the suggested approach allows us to derive exact solutions of equations for flows of fractal fluids in pipes. We assume that proposed continuous models with NIDS and the suggested Navier–Stokes and Poiseuille equations of fractal fluid will be important in application in biophysics of blood flow in cardiovascular system, The blood as a multi-phase complex medium can demonstrate fractal properties of some blood components including bacteria, viruses and medicinal substances getting into the blood. The complex dynamics of fractal fluids in geophysics and hydrology can be described by the proposed approach to fractal hydrodynamics.

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References

1. J. Feder, *Fractals* (Plenum Press, New York, 1988).
2. K. F. Falconer, *The Geometry of Fractal Sets* (Cambridge University Press, Cambridge, 1985).
3. V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, New York, 2011).
4. V. E. Tarasov, *Phys. Lett. A* **336**, 167 (2005), arXiv:cond-mat/0506137.
5. V. E. Tarasov, *Ann. Phys.* **318**, 286 (2005), arXiv:physics/0602096.
6. Q. Y. Zhu *et al.*, *Int. J. Heat Mass Transf.* **54**, 1400 (2011).
7. E. Gabrys, M. Rybaczuk and A. Kedzia, *Chaos Solitons Fractals* **27**, 1 (2006).
8. G. Jayalalitha, V. S. Deviha and R. Uthayakumar, *Comput. Biol. Med.* **38**, 684 (2008).
9. D. A. Benson, M. M. Meerschaert and J. Revielle, *Adv. Water Resour.* **51**, 479 (2013).
10. S.-L. Soo, *Fluid Dynamics of Multiphase Systems* (Blaisdell Pub. Co., Massachusetts, 1967).
11. R. I. Nigmatulin, *Dynamics of Multiphase Media*, Vols. 1–2 (Hemisphere, New York, 1991).
12. P. N. Demmie and M. Ostoja-Starzewski, *J. Elast.* **104**, 187 (2011).
13. J. Li and M. Ostoja-Starzewski, *Int. J. Eng. Sci.* **49**, 1302 (2011).
14. M. Ostoja-Starzewski, *Z. Angew. Math. Phys.* **64**, 381 (2013), arXiv:1106.1491.
15. M. Ostoja-Starzewski *et al.*, *J. Appl. Math. Mech.* **94**, 373 (2014).
16. A. S. Balankin and B. E. Elizarraraz, *Phys. Rev. E* **85**, 025302 (2012).
17. A. S. Balankin and B. E. Elizarraraz, *Phys. Rev. E* **85**, 056314 (2012).
18. J. Li and M. Ostoja-Starzewski, *Phys. Rev. E* **88**, 057001 (2013).
19. A. S. Balankin and B. E. Elizarraraz, *Phys. Rev. E* **88**, 057002 (2013).
20. J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984).
21. F. H. Stillinger, *J. Math. Phys.* **18**, 1224 (1977).
22. C. Palmer and P. N. Stavrinou, *J. Phys. A* **37**, 6987 (2004).
23. P. Moon and D. E. Spencer, *J. Franklin Inst.* **256**, 551 (1953).
24. V. E. Tarasov, *Commun. Nonlinear Sci. Numer. Simul.* **20**, 360 (2015), arXiv:1503.02022.
25. V. E. Tarasov, *J. Math. Phys.* **55**, 083510 (2014), arXiv:1503.02392.
26. V. E. Tarasov, *C. R. Mecanique* **343**, 57 (2015), arXiv:1503.03060.
27. V. E. Tarasov, *Phys. Lett. A* **379**, 2055 (2015).
28. V. E. Tarasov, *Chaos Solitons Fractals* **81**, 38 (2015).
29. V. E. Tarasov, *Chaos Solitons Fractals* **67**, 26 (2014), arXiv:1503.02842.
30. V. E. Tarasov, *Int. J. Heat Mass Transf.* **93**, 427 (2016).
31. V. E. Tarasov, *Wave Motion* **63**, 18 (2016).
32. X.-J. Yang, D. Baleanu and J. A. T. Machado, *Math. Probl. Eng.* **2013**, 769724 (2013).