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Leibniz Rule and Fractional Derivatives of Power Functions

In this paper, we prove that unviolated simple Leibniz rule and equation for fractional-order derivative of power function cannot hold together for derivatives of orders $\alpha \neq 1$. To prove this statement, we use an algebraic approach, where special form of fractional-order derivatives is not applied. [DOI: 10.1115/1.4031364]

1 Introduction

Theory of fractional derivatives of noninteger orders [1–4], which has a long history [5,6], has wide applications in physics and mechanics, since it allows us to describe systems, media, and fields that are characterized by power-law nonlocality and memory of power-law type. Are known various types of fractional derivatives that are suggested by Riemann, Liouville, Riesz, Caputo, Grünwald, Letnikov, Sonin, Marchaud, Weyl, and some others scientists [1–3]. These fractional derivatives have a set of unusual properties. For example, all fractional derivatives violate the usual form of the Leibniz rule [7]. The correct form of a generalization of the Leibniz rule for fractional-order derivatives has been suggested by Liouville [8] in 1832 (see also Theorem 15.1 in Refs. [1,2]). Generalizations of the Leibniz rule for fractional derivatives are also derived by Osler in Refs. [9–12]. The unusual properties of the fractional derivatives allow us to describe unusual properties of materials and systems in physics and mechanics (for example, see Refs. [13–15] and references therein). Authors of some papers suggest new types of fractional derivatives and assume that the unviolated Leibniz rule and the equation for fractional-order derivative of power function hold together for these derivatives. In this paper, we prove that the Leibniz rule $\mathcal{D}^{\alpha}(f(x)\,g(x)) = (\mathcal{D}^{\alpha}f(x))\,g(x) + f(x)\,(\mathcal{D}^{\alpha}g(x))$ and the equation for fractional-order derivative of power function $\mathcal{D}^{\alpha}x^{\beta} = \Gamma(\beta+1)/\Gamma(\beta-\alpha+1)\,x^{\beta-\alpha}$ cannot hold together for derivatives of orders $\alpha \neq 1$. In our proof, we consider fractional-order derivatives $\mathcal{D}^{\alpha}$ of by using an algebraic approach, where special form of fractional derivatives is not important for our consideration.

2 Fractional Derivative of Power Functions

The well-known equation for the integer-order derivative of power function is

$$D^n x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}\, x^{\beta-n}, \quad (\beta > 0, \quad x \in \mathbb{R}_+) \tag{1}$$

where $D^n = d^n/dx^n$ is the derivative of integer-order $n \in \mathbb{N}$. For $n < \beta \in \mathbb{N}$, we can consider $x \in \mathbb{R}$ instead of $x \in \mathbb{R}_+$.

For the case $\beta = n - 1 - k$, where $k = 0$ or $k \in \mathbb{N}$, we should use the Euler's reflection formula such that

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} = \frac{\sin(\pi(n-\beta))}{\pi}\,\Gamma(n-\beta)\,\Gamma(\beta+1) \tag{2}$$

Equation for fractional-order derivative $\mathcal{D}^{\alpha}$ of power function x^{β} is usually considered in the form

$$\mathcal{D}^{\alpha}x^{\beta} = a(\alpha,\beta)\,x^{\beta-\alpha}, \quad (x \in \mathbb{R}_+, \quad \alpha > 0, \quad \beta > 0) \tag{3}$$

where the coefficient $a(\alpha,\beta)$ is a function of the parameters $\alpha > 0$ and $\beta > 0$. Note that we do not assume that Eq. (3) holds for all types of fractional derivatives. For example, the Hadamard and Hadamard type fractional derivatives of the power of the power function yield the same function, apart from a constant multiplication factor (see Eqs. (2.7.21)–(2.7.24) of Property 2.25 in Refs. [3,16]). At the same time, we should have a principle of correspondence, according to which expression for the derivative $\mathcal{D}^{\alpha}x^{\beta}$ should give Eq. (1) for $\alpha = n \in \mathbb{N}$.

Using the principle of correspondence with integer-order case, we assume that Eq. (3) with $\alpha = n \in \mathbb{N}$ should give Eq. (1). Therefore, the function $a(\alpha,\beta)$ should satisfy the following requirement.

Proposition 1. ("Correspondence principle I") *The coefficients $a(\alpha,\beta)$ of relation (3) for fractional derivative $\mathcal{D}^{\alpha}$ of power function x^{β} should satisfy the condition*

$$a(n,\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \quad (n \in \mathbb{N}) \tag{4}$$

For this reason, the fractional derivatives of power functions are defined such that relation (3) is usually considered with the coefficients

$$a(\alpha,\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}, \quad (\alpha > 0) \tag{5}$$

Note that we cannot use this form of coefficients $a(\alpha,\beta)$ for $\beta - \alpha + 1 = -k$, where $k = 0$ or $k \in \mathbb{N}$. In this case, we should use the Euler's reflection formula, such that Eq. (5) takes the form

$$a(\alpha,\beta) = \frac{\sin(\pi(\alpha-\beta))}{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+1) \tag{6}$$

3 Leibniz Rules

The Leibniz rule for first-order derivative has the simple form

$$D^n(f(x)\,g(x)) = (D^n f(x))\,g(x) + f(x)\,(D^n g(x)), \quad (n = 1) \tag{7}$$

The well-known equation for the integer-order derivative of positive integer-orders $n \in \mathbb{N}$ for the product of functions has the form

$$D^n(f(x)\,g(x)) = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\left(D^{n-k}f(x)\right)\left(D^k g(x)\right), \quad (n \in \mathbb{N}) \tag{8}$$

where the binomial coefficient can be represented in terms of the Gamma functions

$$\frac{n!}{(n-k)!\,k!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\,\Gamma(k+1)} \tag{9}$$

Using the principle of correspondence with integer-order case, it seems obvious that all generalizations of the Leibniz rule for

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noninteger orders α of derivatives should give expression (8) for $\alpha = n \in \mathbb{N}$. Therefore, we have the following requirement.

Proposition 2. ("Correspondence principle II") *Generalizations of the Leibniz rule for fractional derivatives $\mathcal{D}^\alpha$ of noninteger order $\alpha > 0$ should give the relation*

$$\mathcal{D}^\alpha(f(x)\,g(x)) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\,\Gamma(k+1)} \left(\mathcal{D}^{\alpha-k}f(x)\right)\left(D^k g(x)\right), \quad (\alpha = n \in \mathbb{N}) \tag{10}$$

for integer values of $\alpha = n \in \mathbb{N}$.

As a corollary of this statement, we can say that the Leibniz rule for derivative of noninteger order $\alpha \neq 1$ cannot have the simple form

$$\mathcal{D}^\alpha(f(x)\,g(x)) = (\mathcal{D}^\alpha f(x))\,g(x) + f(x)\,(\mathcal{D}^\alpha g(x)) \tag{11}$$

A violation of relation (11) is a characteristic property of all derivatives of integer-orders $n \in \mathbb{N}$ greater than one and for all types fractional-order derivatives with $\alpha > 0$.

In Ref. [7], theorem "No violation of the Leibniz rule. No fractional derivative" has been proved. In the proof, we consider fractional-order derivatives $\mathcal{D}^\alpha$ of by using an algebraic approach. Exact expression and definition of fractional derivatives are not important for this proof. The property of linearity and the Leibniz rule are used only.

Theorem. (No violation of the Leibniz rule. No fractional derivative) *If an operator $\mathcal{D}^\alpha_x$ satisfies the condition of $\mathbb{R}$-linearity*

$$\mathcal{D}^\alpha(c_1 f(x) + c_2 g(x)) = c_1(\mathcal{D}^\alpha f(x)) + c_2\,(\mathcal{D}^\alpha g(x)), \quad (c_1, c_2 \in \mathbb{R}) \tag{12}$$

and the Leibniz rule

$$\mathcal{D}^\alpha(f(x)\,g(x)) = (\mathcal{D}^\alpha f(x))\,g(x) + f(x)\,(\mathcal{D}^\alpha g(x)) \tag{13}$$

then this operator is the derivative of integer (first) order, which can be represented in the form $\mathcal{D}^\alpha = a(x)\,D^1$, where a(x) is a function on $\mathbb{R}$.

This theorem states that fractional derivatives of noninteger orders $\alpha \neq 1$ cannot satisfy the Leibniz rule (11). A correct form of the Leibniz rule for fractional-order derivatives should be obtained as a generalization of the Leibniz rule for integer-order derivatives (for example, see Sec. 2.7.2 of Ref. [17] and/or Ref. [9]). For example, the fractional generalization of the Leibniz rule for the Riemann–Liouville derivatives has the form of the infinite series

$$\mathcal{D}^\alpha(f(x)g(x)) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\,\Gamma(k+1)} \left(\mathcal{D}^{\alpha-k}f(x)\right)\left(D^k g(x)\right) \tag{14}$$

where the $f(x)$ and $g(x)$ are analytic functions for $x \in [a, b]$ (see Theorem 15.1 of Refs. [1,2]), $\mathcal{D}^\alpha$ is the Riemann–Liouville derivative; D^k is derivative of integer-order $k \in \mathbb{N}$. It should be noted that Eq. (14) contains an infinite sum. Moreover, the sum contains integrals of noninteger order for the values $k > [\alpha] + 1$. Equation (14) first suggested in Ref. [8] in 1832. Correct form of generalizations of the Leibniz rule for fractional derivatives is derived by Osler in Refs. [8–12]. For some remarks about the rule (14), see Theorem 2.18 and corresponding comments of Ref. [18].

4 Leibniz Rules for Power Functions

Let us consider the two power functions

$$f(x) = x^\beta, \quad g(x) = x^\gamma \quad (x \in \mathbb{R}_+) \tag{15}$$

where $\beta > 0$ and $\gamma > 0$. For β, $\gamma \geq \alpha > 0$, we consider functions (15) with $x \geq 0$ instead of $x > 0$. Using Eq. (3) for the functions (15), we have

$$\mathcal{D}^\alpha x^\beta = a(\alpha, \beta)\, x^{\beta-\alpha} \tag{16}$$

$$\mathcal{D}^\alpha x^\gamma = a(\alpha, \gamma)\, x^{\gamma-\alpha} \tag{17}$$

where $x \in \mathbb{R}_+$. Equations (16) and (17) give that the expression on the right side of the Leibniz rule (11) for the product (15) has the form

$$(\mathcal{D}^\alpha f(x))\,g(x) + f(x)\,(\mathcal{D}^\alpha g(x)) = (\mathcal{D}^\alpha x^\beta)\,x^\gamma + x^\beta\,(\mathcal{D}^\alpha x^\gamma) = (a(\alpha,\beta) + a(\alpha,\gamma))\,x^{\beta+\gamma-\alpha} \tag{18}$$

Using Eq. (3), the fractional-order derivative of product of functions (15) is

$$\mathcal{D}^\alpha(f(x)g(x)) = \mathcal{D}^\alpha x^{\beta+\gamma} = a(\alpha, \beta+\gamma)\, x^{\beta+\gamma-\alpha} \tag{19}$$

If the Leibniz rule (11) holds, then we should have the relation

$$a(\alpha, \beta+\gamma) = a(\alpha, \beta) + a(\alpha, \gamma) \tag{20}$$

for all $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.

As a result, we have the following requirement.

Proposition 3. *In order to the Leibniz rule (11) holds together with the equation for fractional-order derivative of power function (3), the relation (20) should be satisfied for all $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.*

For the coefficients (5), relation (20) can be written as

$$\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\beta+\gamma-\alpha+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} - \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} = 0 \tag{21}$$

For $\beta = \gamma = \alpha$, Eq. (21) has the form

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} - 2\,\Gamma(\alpha+1) = 0 \tag{22}$$

that can be rewritten as

$$\Gamma(2\alpha+1) - 2\,\Gamma^2(\alpha+1) = 0 \tag{23}$$

To illustrate relation (22), we give a plot of the function

$$Z(x) = \Gamma(2x+1) - 2\,\Gamma^2(x+1) \tag{24}$$

defined by the left-hand side of Eq. (23) The plot of the function is presented by Fig. 1 for range of $x = \alpha \in [0; 1.5]$.

If we use $0 < \alpha \leq 1$, $\beta = 1$, $\gamma = 1$, and $x \geq 0$, then Eq. (21) has the form

$$\frac{\Gamma(3)}{\Gamma(3-\alpha)} - \frac{2\,\Gamma(2)}{\Gamma(2-\alpha)} = 0 \tag{25}$$

Using $\Gamma(3-\alpha) = (2-\alpha)\Gamma(2-\alpha)$, and $\Gamma(n+1) = n!$, condition (25) can be rewritten as

$$\frac{2(\alpha-1)}{\Gamma(3-\alpha)} = 0 \tag{26}$$

This means that the unviolated Leibniz rule (11) holds only for $\alpha = 1$.

As a result, we have the following requirement.

Proposition 4. *In order to the Leibniz rule (11) holds, the relation (21) should be satisfied for all $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.*

It is easy to see that this requirement holds only for $\alpha = 1$. This means that the Leibniz rule (11) cannot be performed for derivatives of noninteger orders (and integer-orders $\alpha > 1$). It is important to emphasize that this proposition is true for any function space, if it includes power functions and we can consider fractional-order derivative of power function in the form (3).

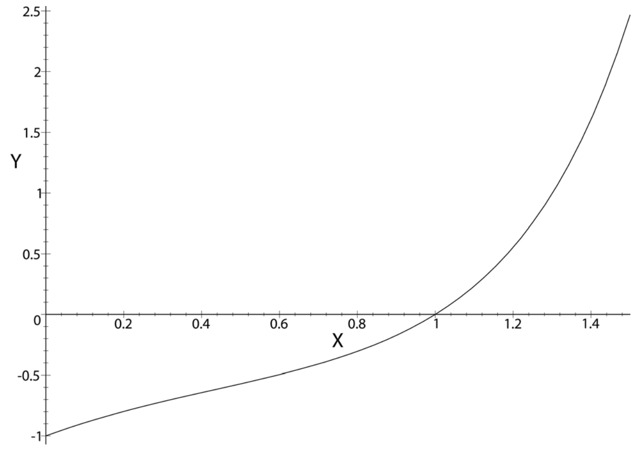

Fig. 1 Plot of the function $Z(x)$ (Eq. (24)) for the range $x = \alpha \in [0; 1.5]$

We should emphasize that violation of the Leibniz rule (11) is a characteristic property of fractional-order derivatives of all types [7,19] and derivatives of integer-orders $\alpha \neq 1$. In addition, the violation of the Leibniz rule (11) for fractional derivatives does not depend on the class of functions (in contrast to statements in Refs. [20–22]), if the relation (3) can be used.

Unfortunately, the unviolated Leibniz rule (11) and Eq. (3) for fractional-order derivative of power function are used together for so-called modified Riemann–Liouville derivatives and local fractional derivatives in Refs. [20–40].

Equation (3) and the Leibniz rule (11) are assumed for so-called modified Riemann–Liouville derivatives that are also called as Jumarie derivatives (see Eqs. (3.10) and (3.11) of Ref. [23]; Eqs. (4.2) and (4.3) of Ref. [26]; Eqs. (4.3) and (4.4) of Ref. [27]; Eqs. (13) and (14) of Ref. [29]; Eqs. (2.13) and (2.14) in Ref. [30]; "Simple rules" and Eq. (2) of Ref. [32]; and Simple rules of Eqs. (2.2) and (2.3) of Ref. [33]).

For so-called modified Riemann–Liouville derivatives, which are also called as Jumarie derivatives, Eq. (3) and the Leibniz rule (11) are assumed together (see Eq. (31) of Ref. [37], Eq. (5) of Ref. [38], and Proposition 2.4. of Ref. [40]).

It should be noted that statement [20], that the Leibniz rule in the form (11) holds for nondifferentiable functions is incorrect. The following statements are proved in Ref. [19].

PROPOSITION 5.

(1) *The Leibniz rule (11) for fractional derivatives of orders $\alpha \neq 1$ is not satisfied on a set of differentiable functions. The Leibniz rule (11) holds on a set of differentiable functions only for $\alpha = 1$.*

(2) *The Leibniz rule (11) for fractional derivatives of orders $\alpha \neq 1$ is not satisfied on a set of fractional–differentiable functions. Equation (3) for fractional-order derivative of power function and the Leibniz rule (11) on a set of fractional–differentiable functions hold together only for $\alpha = 1$.*

(3) *The Leibniz rule (11) cannot be used for nondifferentiable functions that are not fractional–differentiable.*

The violation of relation (11) is a characteristic property of all types fractional-order derivatives. In addition, relation (11) does not hold for derivatives of integer-orders $n \in \mathbb{N}$ greater than one. Moreover, unusual properties of fractional-order derivatives such as a violation of the usual Leibniz rule, a deformations of the usual chain rule [41,42], a violation of the semigroup property, and other characteristic properties [43,44] allow us to describe new unusual properties of complex media and physical systems. Note that a violation of the Leibniz rule in quantum theory is discussed in Ref. [45]. Therefore, these unusual properties are very important for application and attempts to remove a violation of the Leibniz rule for fractional-order derivatives can be described as "Throw out the baby with the bathwater" [46,47].

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