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Toward fractional gradient elasticity

Abstract: The use of an extension of gradient elasticity through the inclusion of spatial derivatives of fractional order to describe the power law type of non-locality is discussed. Two phenomenological possibilities are explored. The first is based on the Caputo fractional derivatives in one dimension. The second involves the Riesz fractional derivative in three dimensions. Explicit solutions of the corresponding fractional differential equations are obtained in both cases. In the first case, stress equilibrium in a Caputo elastic bar requires the existence of a nonzero internal body force to equilibrate it. In the second case, in a Riesz-type gradient elastic continuum under the action of a point load, the displacement may or may not be singular depending on the order of the fractional derivative assumed.

Keywords: fractional derivative; fractional elasticity; gradient elasticity.

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1 Introduction

The use of fractional derivatives and integrals [1–3] allows us to investigate the behavior of material processes and systems that are characterized by power law non-locality, power law long-term memory, and fractal properties. Fractional calculus has emerged as a powerful tool that has a wide range of applications in mechanics and physics (e.g., [4–13]).

Non-local effects in elasticity theory have been treated with two different approaches: the gradient elasticity theory (weak non-locality) and the integral elasticity theory (strong non-locality). The fractional calculus can then be used to establish a fractional generalization of non-local elasticity in two forms: the fractional gradient elasticity theory (weak non-locality) and the fractional integral elasticity theory (strong non-locality).

Some developments in framework and derivation of corresponding results for the fractional integral elasticity have been made in [14–16]. This has not been done, however, for gradient elasticity (for a recent review of the subject, one may consult [17, 18]). An extension of the phenomenological theory of gradient elasticity using the Caputo and Riesz spatial derivatives of non-integer order is suggested in the present article.

In Section 2, a phenomenological fractional generalization of one-dimensional gradient elasticity is discussed using the Caputo derivative to include gradient effects in the constitutive equation for the stress. The corresponding fractional differential equation for the displacement is solved analytically and expressed in terms of the Mittag-Leffler functions. In order that the stress field be equilibrated, the material should develop an internal force that should be added to the externally applied body force field.

In Section 3, a fractional generalization of gradient elasticity is discussed using the Riesz derivative (in particular, the fractional Laplacian in the Riesz form). Analytical solutions of the corresponding fractional differential equation are obtained for two cases: the sub-gradient and the super-gradient elasticities (in analogy to sub-diffusion and super-diffusion cases) for a continuum carrying a point load. Asymptotic expressions are derived for the displacement field near the point of application of the external load. They may or may not be singular depending on the order of the fractional derivative used.

2 Fractional gradient elasticity based on the Caputo derivative

In this section, we suggest a fractional generalization of the gradient elasticity model that includes the Caputo derivative of non-integer order. For this one-dimensional model, we derive a general solution of the corresponding fractional differential equation for the displacement. We demonstrate how to overcome the difficulties caused by the unusual properties of fractional derivatives. An

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alternative fractional gradient elasticity model for a three-dimensional case using the Riesz fractional derivative (in the form of a fractional Laplacian) is discussed in Section 3.

2.1 Fractional gradient elasticity equation

Let us consider the constitutive relation for a one-dimensional fractional gradient elasticity model that is based on the Caputo derivative in the form

$$\sigma(x)=E\varepsilon(x)\pm l_\beta^2 E\, {}^C D_{a+}^\beta \varepsilon(x), \tag{1}$$

where $\sigma(x)$ is the stress and $\varepsilon(x)$ is the strain, with the space variable x and the scale parameter l_β^2 being dimensionless. The symbol ${}^C D_{a+}^\beta$ is the Caputo derivative of order β with n-1<β<n. The ± sign is kept for generality, as various previous nonfractional gradient elasticity models use either sign (for a comprehensive of nonfractional gradient elasticity models, the reader may consult [18]). The left-sided Caputo fractional derivative [3] of order $\alpha>0$ for $x\in[a, b]$ is defined by

$${}^C D_{a+}^\alpha f(t)=I_{a+}^{n-\alpha} D_x^n f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{dz\, D_z^n f(z)}{(x-z)^{\alpha-n+1}}, \tag{2}$$

where $n-1<\alpha<n$ and I_{a+}^α is the left-sided Riemann-Liouville fractional integral of order $\alpha>0$ defined as

$$I_{a+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(z)dz}{(x-z)^{1-\alpha}}, \quad (z>a).$$

Then using the usual definition of the strain $\varepsilon(x)$ in terms of the displacement $u(x)$

$$\varepsilon(x)=D_x^1 u(x), \tag{3}$$

we obtain the fractional stress displacement equation in the form

$$\sigma(x)=E D_x^1 u(x)\pm l_\beta^2 E\, {}^C D_{a+}^{\beta+1} u(x). \tag{4}$$

In view of the fractional vector calculus framework, we can derive the fractional equation of equilibrium in the form

$$A_\alpha(x)\, {}^C D_{a+}^\alpha \sigma(x)+f(x)=0 \tag{5}$$

with the given functions $A_\alpha(x)$ and $f(x)$ denoting, as usual, the external body force field. The explicit form of the function $A_\alpha(x)$ is derived from the conservation law for non-local media using the fractional vector calculus [19]. Substitution of Eq. (4) into (5) gives

$${}^C D_{a+}^{\alpha+1} u(x)\pm l_\beta^2\, {}^C D_{a+}^\alpha\, {}^C D_{a+}^{\beta+1} u(x)+\tilde{f}(x)=0, \tag{6}$$

where

$$\tilde{f}(x)=E^{-1} A_\alpha^{-1}(x) f(x). \tag{7}$$

For the case $\alpha=1$, the governing fractional differential equation reads

$$D_x^2 u(x)\pm l_\beta^2 D_x^1\, {}^C D_{a+}^{\beta+1} u(x)+\tilde{f}(x)=0. \tag{8}$$

In general, we have $D_x^1\, {}^C D_x^{\beta+1}\neq {}^C D_x^{\beta+2}$.

2.2 Solution of the fractional gradient elasticity equation

Let us make use of the explicit form concerning the violation of the semigroup property for the Caputo derivative that gives the relationship between the product ${}^C D_{a+}^\alpha\, {}^C D_{a+}^\beta$ and the derivative ${}^C D_{a+}^{\alpha+\beta}$.

Using Eq. (2.4.6) in [3] of the form

$$({}^C D_{a+}^\alpha f)(x)=({}^{RL} D_{a+}^\alpha f)(x)-\sum_{k=0}^{n-1}\frac{(D^k f)(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \tag{9}$$

and using Property 2.1, Eq. (2.1.16), in [3],

$$I_{a+}^\alpha (x-a)^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta+\alpha}, \tag{10}$$

where $\alpha>0$ and $\beta>-1$, we obtain the relation

$${}^C D_{a+}^\alpha\, {}^C D_{a+}^\beta f(x)={}^C D_{a+}^{\alpha+\beta} f(x)+\sum_{k=0}^{n-1}\frac{f^{(k+n)}(a)}{\Gamma(n+k-\alpha-\beta)}(x-a)^{n+k-\alpha-\beta}, \tag{11}$$

where $0<\alpha\leq 1$, $n-1<\beta\leq n$. This relation explicitly shows a violation of the semigroup property for the Caputo derivative.

Using Eq. (11), we rewrite Eq. (6) in the form

$$({}^C D_{a+}^{\alpha+\beta+1} u)(x)\pm l_\beta^{-2}({}^C D_{a+}^{\alpha+1} u)(x)\pm f_{eff}(x)=0, \tag{12}$$

where $0<\alpha<1$, $1<\beta<2$ ($n=2$), or $2<\beta<3$ ($n=3$) and $f_{eff}(x)$ is an effective body force defined by

$$f_{eff}(x)=l_\beta^{-2}\sum_{k=0}^{n}\frac{u^{(k+n+1)}(a)}{\Gamma(n+k-\alpha-\beta)}(x-a)^{n+k-\alpha-\beta}+l_\beta^{-2}\tilde{f}(x). \tag{13}$$

Eq. (12) is a nonhomogeneous fractional differential equation with constant coefficients.

The solutions to equations of this type are given by theorem 5.16 in [3] (see also theorem 5.13 in [3] for the homogeneous case, $f_{eff}(x)=0$). To use these results, we assume that $a=0$. Let us consider the case $0<\alpha<1$, $1<\beta<2$

(i.e., $1<\alpha+1<2=m$, $2<\beta+1<3=n$). Then the solution of Eq. (12) has the form

$$u(x)=\int_0^x dz\, f_{eff}(z)(x-z)^{\alpha} E_{\alpha-\beta,\alpha+1}[\mp l_\beta^{-2}(x-z)^{\alpha-\beta}]+C_0 u_0(x) +C_1 u_1(x)+C_2 u_2(x), \quad (14)$$

where

$$u_0(x)=E_{\alpha-\beta,1}[\mp l_\beta^{-2} x^{\alpha-\beta}]\pm l_\beta^{-2} x^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}[\mp l_\beta^{-2} x^{\alpha-\beta}], \quad (15)$$

$$u_1(x)=x E_{\alpha-\beta,2}[\mp l_\beta^{-2} x^{\alpha-\beta}]\pm l_\beta^{-2} x^{\alpha-\beta+1} E_{\alpha-\beta,\alpha-\beta+2}[\mp l_\beta^{-2} x^{\alpha-\beta}], \quad (16)$$

$$u_2(x)=x^2 E_{\alpha-\beta,3}[\mp l_\beta^{-2} x^{\alpha-\beta}], \quad (17)$$

and

$$f_{eff}(x)=l_\beta^{-2}\sum_{k=0}^{2}\frac{u^{(k+3)}(0)}{\Gamma(2+k-\alpha-\beta)} x^{2+k-\alpha-\beta}+l_\beta^{-2}\tilde{f}(x). \quad (18)$$

The arbitrary real constants C_0, C_1, and C_2 in the case of the Caputo fractional derivatives are defined by the values of the integer-order derivatives $u(0)$, $u^{(1)}(0)$, and $u^{(2)}(0)$.

Here, $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [3], which is defined by

$$E_{\alpha,\beta}[z]=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha>0, \beta\in\mathbb{R}). \quad (19)$$

Note also that $E_{1,1}[z]=e^z$. The asymptotic behavior (see Eq. (1.8.27) in [3]) of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha} z^{(1-\beta)/\alpha}\exp(z^{1/\alpha}) -\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O(1/z^{N+1}) \quad (|z|\to\infty), \quad (20)$$

where $0<\alpha<2$.

Remark: It should be emphasized that the absence of the external force ($f(x)=0$) does not imply the vanishing of the effective force f_{eff}. In general, $f_{eff}(x)\neq 0$ for $f(x)=0$. Only in the case of commutativity of the Caputo fractional derivatives, i.e., if the semigroup property

$$({}^C D_{a+}^{\alpha}\, {}^C D_{a+}^{\beta} u)(x)=({}^C D_{a+}^{\alpha+\beta} u)(x)$$

is not violated, the vanishing of the external force $f(x)=0$ leads to the vanishing of the effective force $f_{eff}(x)=0$. It is easy to see that the semigroup property is satisfied if

$$u^{(3)}(0)=u^{(4)}(0)=u^{(5)}(0)=0. \quad (21)$$

If we consider Eq. (12) in the case $\alpha=1$, $1<\beta<2$, $f(x)=0$, and assume that condition (21) is satisfied, then solution (14) of Eq. (12) has the form

$$u(x)=C_0(E_{1-\beta,1}[\mp l_\beta^{-2} x^{1-\beta}]\pm l_\beta^{-2} x^{1-\beta} E_{1-\beta,2-\beta}[\mp l_\beta^{-2} x^{1-\beta}])+ +C_1(x E_{1-\beta,2}[\mp l_\beta^{-2} x^{1-\beta}]\pm l_\beta^{-2} x^{2-\beta} E_{1-\beta,3-\beta}[\mp l_\beta^{-2} x^{1-\beta}]) +C_2 x^2 E_{1-\beta,3}[\mp l_\beta^{-2} x^{1-\beta}]. \quad (22)$$

For this solution to be admissible, it should be checked if the function given by Eq. (22) satisfies the conditions $u^{(3)}(0)=u^{(4)}(0)=u^{(5)}(0)=0$. To verify these conditions, we use Eq. (1.8.22) of [3] in the form

$$\frac{d^n}{dz^n} E_{\alpha,\beta}[z]=n!\, E_{\alpha,\beta+n\alpha}^{n+1}[z], \quad (n\in\mathbb{N}). \quad (23)$$

The conditions given by Eq. (21) are not satisfied for the function given by Eq. (22). Thus, the solution is not admissible for the fractional one-dimensional gradient elasticity model considered herein. Therefore, we should take into account the effective force defined in Eq. (18) for the solution given by Eq. (14) to describe the fractional one-dimensional gradient elasticity model correctly.

3 Fractional gradient elasticity based on the Riesz derivative

An alternative fractional gradient elasticity model may be obtained using the Riesz fractional derivative. In this case, it turns out that a three-dimensional treatment is possible due to available results on the fractional Laplacian of the Riesz type. The corresponding fractional gradient elasticity governing equation can then be considered in the form

$$c_\alpha((-\Delta)^{\alpha/2} u)(\mathbf{r})+c_\beta((-\Delta)^{\beta/2} u)(\mathbf{r})=f(\mathbf{r}) \quad (\alpha>\beta), \quad (24)$$

where $\mathbf{r}\in\mathbb{R}^3$ and $r=|\mathbf{r}|$ are dimensionless and $(-\Delta)^{\alpha/2}$ is the Riesz fractional Laplacian of order α [3]. The coefficients (c_α, c_β) are phenomenological constants, and the rest of the symbols have their usual meaning. For $\alpha>0$ and suitable functions $u(\mathbf{r})$, $\mathbf{r}\in\mathbb{R}^3$, the Riesz fractional derivative can be defined [3] in terms of the Fourier transform $\mathcal{F}$ by

$$((-\Delta)^{\alpha/2} u)(\mathbf{r})=\mathcal{F}^{-1}(|\mathbf{k}|^{\alpha}(\mathcal{F}u)(\mathbf{k})), \quad (25)$$

where $\mathbf{k}$ denotes the wave vector. If $\alpha=4$ and $\beta=2$, we have the well-known equation of the gradient elasticity [18]:

$$c_2\Delta u(\mathbf{r})\text{-}c_4\Delta^2 u(\mathbf{r})+f(\mathbf{r})=0, \tag{26}$$

where

$$c_2=E, \quad c_4=\pm l^2 E. \tag{27}$$

Eq. (24) is the fractional partial differential equation that has the particular solution (section 5.5.1 in [3]) of the form

$$u(\mathbf{r})=\int_{\mathbb{R}^3} G^3_{\alpha,\beta}(\mathbf{r}\text{-}\mathbf{r}')f(\mathbf{r}')d^3\mathbf{r}', \tag{28}$$

where the Green-type function

$$G^3_{\alpha}(\mathbf{r})=\int_{\mathbb{R}^3}\frac{1}{c_\alpha|\mathbf{k}|^\alpha+c_\beta|\mathbf{k}|^\beta}e^{+i(\mathbf{k},\mathbf{r})}d^3\mathbf{k} \tag{29}$$

is given (see lemma 25.1 of [1, 2]) by the following equation:

$$G^3_{\alpha,\beta}(\mathbf{r})=\frac{1}{(2\pi)^{3/2}\sqrt{|\mathbf{r}|}}\int_0^\infty\frac{\lambda^{3/2}J_{1/2}(\lambda|\mathbf{r}|)}{c_\alpha\lambda^\alpha+c_\beta|\lambda|^\beta}d\lambda. \tag{30}$$

Here, $J_{1/2}(z)=\sqrt{2/(\pi z)}\sin(z)$ is the Bessel function of the first kind.

Let us consider, as an example, the W. Thomson (1848) problem [20] for the present model of fractional gradient elasticity. Determine the deformation of an infinite elastic continuum, when a concentrated force is applied to a small region of it. To solve this problem, we consider distances $|\mathbf{r}|$, which are large in comparison with the size of the region (neighborhood) of load application. In other words, we can suppose that the force is applied at a point. In this case, we have

$$f(\mathbf{r})=f_0\delta(\mathbf{r})=f_0\delta(x)\delta(y)\delta(z). \tag{31}$$

Then the displacement field $u(\mathbf{r})$ of fractional gradient elasticity has a simple form given by the particular solution

$$u(\mathbf{r})=f_0 G^3_\alpha(\mathbf{r}), \tag{32}$$

where $G^3_\alpha(z)$ is the Green's function given by Eq. (30). Therefore, the displacement field for the force applied at a point, Eq. (31), has the form

$$u(\mathbf{r})=\frac{1}{2\pi^2}\frac{f_0}{|\mathbf{r}|}\int_0^\infty\frac{\lambda\sin(\lambda|\mathbf{r}|)}{c_\alpha\lambda^\alpha+c_\beta\lambda^\beta}d\lambda \quad (\alpha>\beta). \tag{33}$$

For this solution of the fractional gradient elasticity equation (24) with $\alpha>\beta$, $0<\beta<2$, and $\alpha\neq 2$, with a point force $f(\mathbf{r})$ of the form given by Eq. (31), the asymptotic behavior is

$$u(\mathbf{r})\approx\frac{f_0\Gamma(2\text{-}\beta)\sin(\pi\beta/2)}{2\pi^2 c_\beta}\cdot\frac{1}{|\mathbf{r}|^{3\text{-}\beta}} \quad (|\mathbf{r}|\to\infty). \tag{34}$$

This asymptotic behavior $|\mathbf{r}|\to\infty$ does not depend on parameter α, and the corresponding asymptotic behavior for $|\mathbf{r}|\to 0$ does not depend on parameter β, where $\alpha>\beta$. The displacement field at large distances from the point of load application is determined only by term $(\text{-}\Delta)^{\beta/2}$, where $\beta<\alpha$. This can be interpreted as a fractional non-local "deformation" counterpart of the classical result based on the local Hooke's law. We note the existence of a maximum for the quantity $u(\mathbf{r})\cdot|\mathbf{r}|$ in the case $0<\beta<2<\alpha$.

From a mathematical point of view, there are two special cases: (i) fractional power law-weak non-locality with $\alpha=2$ and $0<\beta<2$; (ii) fractional power law-weak non-locality with $\alpha\neq 2$, $\alpha>\beta$, and $0<\beta<3$. It is thus useful to distinguish between the following two particular cases:

- Sub-gradient elasticity ($\alpha=2$ and $0<\beta<2$).
- Super-gradient elasticity ($\alpha>2$ and $\beta=2$).

In the sub-gradient elasticity model, the order of the fractional derivative is less than the order of the term related to the usual Hooke's law. The order of the fractional derivative in the super-gradient elasticity equation is larger than the order of the term related to the Hooke's law. The terms "sub-gradient" and "super-gradient" elasticity are used in analogy to the terms commonly used for anomalous diffusion [6–8]: sub-diffusion and super-diffusion.

3.1 Sub-gradient elasticity model

The fractional model of sub-gradient elasticity is described by Eq. (24) with $\alpha=2$ and $0<\beta<2$, i.e.,

$$c_2\Delta u(\mathbf{r})\text{-}c_\beta((\text{-}\Delta)^{\beta/2}u)(\mathbf{r})+f(\mathbf{r})=0, \quad (0<\beta<2). \tag{35}$$

The order of the fractional Laplacian $(\text{-}\Delta)^{\beta/2}$ is less than the order of the first term related to the usual Hooke's law. As a simple example, we can consider the square of the Laplacian, i.e., $\beta=1$. In general, parameter β defines the order of the power law non-locality that characterizes the elastic continuum. The particular solution of Eq. (35) for the point force problem at hand reads

$$u(\mathbf{r})=\frac{1}{2\pi^2}\frac{f_0}{|\mathbf{r}|}\int_0^\infty\frac{\lambda\sin(\lambda|\mathbf{r}|)}{c_2\lambda^2+c_\beta\lambda^\beta}d\lambda \quad (0<\beta<2). \tag{36}$$

The following asymptotic behavior for Eq. (36) can be derived using section 2.3.1 in [21] of the form

$$u(\mathbf{r}) = \frac{f_0}{2\pi^2|\mathbf{r}|}\int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{c_2\lambda^2 + c_\beta\lambda^\beta} d\lambda \approx \frac{C_0(\beta)}{|\mathbf{r}|^{3-\beta}} + \sum_{k=1}^{\infty} \frac{C_k(\beta)}{|\mathbf{r}|^{(2-\beta)(k+1)+1}} \quad (|\mathbf{r}|\to\infty), \tag{37}$$

where

$$C_0(\beta) = \frac{f_0}{2\pi^2 c_\beta}\Gamma(2-\beta)\sin\left(\frac{\pi}{2}\beta\right), \tag{38}$$

$$C_k(\beta) = -\frac{f_0 c_2^k}{2\pi^2 c_\beta^{k+1}}\int_0^\infty z^{(2-\beta)(k+1)-1}\sin(z)\,dz. \tag{39}$$

As a result, the displacement field generated by the force that is applied at a point in the elastic continuum with the fractional non-locality described by the fractional Laplacian $(-\Delta)^{\beta/2}$ with $0<\beta<2$ is given by

$$u(\mathbf{r}) \approx \frac{C_0(\beta)}{|\mathbf{r}|^{3-\beta}} \quad (0<\beta<2), \tag{40}$$

for large distances $|\mathbf{r}| \gg 1$.

3.2 Super-gradient elasticity model

The fractional model of super-gradient elasticity is described by Eq. (24), with $\alpha>2$ and $\beta=2$. In this case, we have

$$c_2\Delta u(\mathbf{r}) - c_\alpha((-\Delta)^{\alpha/2}u)(\mathbf{r}) + f(\mathbf{r}) = 0, \quad (\alpha>2). \tag{41}$$

The order of the fractional Laplacian $(-\Delta)^{\alpha/2}$ is greater than the order of the first term related to the usual Hooke's law. Parameter $\alpha>2$ defines the order of the power law non-locality of the elastic continuum. If $\alpha=4$, Eq. (41) reduced to Eq. (26). The case $3<\alpha<5$ can be considered to correspond as closely as possible ($\alpha\approx 4$) to the usual gradient elasticity model of Eq. (26).

The asymptotic behavior of the displacement field $u(|\mathbf{r}|)$ for $|\mathbf{r}|\to 0$ in the case of super-gradient elasticity is given by

$$u(\mathbf{r}) \approx \frac{f_0\Gamma((3-\alpha)/2)}{2^\alpha\pi^2\sqrt{\pi}c_\alpha\Gamma(\alpha/2)}\cdot\frac{1}{|\mathbf{r}|^{3-\alpha}}, \quad (2<\alpha<3), \tag{42}$$

$$u(\mathbf{r}) \approx \frac{f_0}{2\pi\alpha c_\beta^{1-3/\alpha} c_\alpha^{3/\alpha}\sin(3\pi/\alpha)}, \quad (\alpha>3). \tag{43}$$

Note that the above asymptotic behavior does not depend on parameter β and that the corresponding relation (42) does not depend on c_β. The displacement field $u(\mathbf{r})$ for short distances away from the point of load application is determined only by the term with $(-\Delta)^{\alpha/2}$ ($\alpha>\beta$), which can be considered as a fractional counterpart of the usual extra non-Hookean term of gradient elasticity.

A generalization of the phenomenological theory of gradient elasticity accomplished by including the Caputo and Riesz spatial derivatives of non-integer order is suggested in this article. Related lattice models with spatial dispersion of power law type as microscopic models of the fractional elastic continuum described by Eq. (24) were proposed in [22]. Using the approach suggested in [23, 24], Eq. (24) has been derived from the equations of lattice dynamics with power law spatial dispersion. We can point out that a phenomenological fractional gradient elasticity model can be obtained from different microscopic or lattice models. In addition, we note that the model of fractional gradient elastic continuum has an analogue in the plasma-like dielectric material with power law spatial dispersion [25]. It can be considered as a common or universal behavior of plasma-like and elastic materials in space by analogy with the universal behavior of low-loss dielectrics in time [26–28].

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