Map of discrete system into continuous

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Continuous limits of discrete systems with long-range interactions are considered. The map of discrete models into continuous medium models is defined. A wide class of long-range interactions that give the fractional equations in the continuous limit is discussed. The one-dimensional systems of coupled oscillators for this type of long-range interactions are considered. The discrete equations of motion are mapped into the continuum equation with the Riesz fractional derivative. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2337852]

I. INTRODUCTION

Equations which involve derivatives or integrals of noninteger order[1–5] have found many applications in recent studies in mechanics and physics.[6–11] Usually the fractional equations for dynamics or kinetics appear as some phenomenological models. Recently, the method to obtain fractional analogues of equations of motion was considered for sets of coupled particles with a long-range interaction.[12–14] Examples of systems with interacting oscillators, spins, or waves are used for numerous applications in physics, chemistry, biology.[15–26] Transfer from the equations of motion for discrete systems to the continuous media equation with fractional derivatives is an approximate procedure. Different applications of the procedure have already been used to derive fractional sine-Gordon and fractional wave Hilbert equation,[12,14] to study synchronization of coupled oscillators,[13] and for fractional Ginzburg-Landau equation.[13]

Long-range interaction has been the subject of great interest for a long time. Thermodynamics of the model of classical spins with long-range interactions has been studied in Refs. 15–17 and 19. An infinite one-dimensional Ising model with long-range interactions was considered by Dyson.[15] The d-dimensional classical Heisenberg model with long-range interaction is described in Refs. 16 and 19, and their quantum generalization can be found in Ref. 17. The long-range interactions have been widely studied in discrete systems on lattices as well as in their continuous analogues. Solitons in a one-dimensional lattice with the long-range Lennard-Jones-type interaction were considered in Ref 27. Kinks in the Frenkel-Kontorova model with long-range interparticle interactions were studied in Ref. 28. The properties of time periodic spatially localized solutions (breathers) on discrete chains in the presence of algebraically decaying interactions were considered in Refs. 24 and 25. Energy and decay properties of discrete breathers in systems with long-range interactions have also been studied in the framework of the Klein-Gordon,[22] and discrete nonlinear Schrödinger equations.[29] A remarkable property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails. Similar features were observed in the lattice models with power-like long-range interactions.[23–25,30–32,14] As it was shown in Refs. 13 and 14, analysis of the equations with fractional derivatives can provide results for the space asymptotics of their solutions.

The goal of this paper is to study a connection between the dynamics of system of particles with long-range interactions and the fractional continuous medium equations by using the transform operation. Here, we consider the one-dimensional lattice of coupled nonlinear oscillators. We make the transform to the continuous limit and derive the fractional equation which describes the

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dynamics of the oscillatory medium. We show how the continuous limit for the systems of oscillators with long-range interaction can be described by the corresponding fractional equation.

In Sec. II, the equations of motion for the system of oscillators with long-range interaction are considered. In Sec. III, the transform operation that maps the discrete equations into continuous medium equation is defined. In Sec. IV, the Fourier series transform of the equations of a system with long-range interaction is realized. In Sec. V, we consider a wide class of long-range interactions that can give the fractional equations in the continuous limit. In Sec. VI, the simple example of nearest-neighbor interaction is considered to demonstrate the application of the transform operation to the well-known case. In Sec. VII, the power-law long-range interactions with positive integer powers are considered. In Sec. VIII, the power-law long-range interactions with noninteger powers and the correspondent continuous medium equations are discussed. In Sec. IX, the nonlinear long-range interactions for the discrete systems are used to derive the Burgers, Korteweg-de Vries, and Boussinesq equations and their fractional generalizations in the continuous limit. In Sec. X, the fractional equations are obtained from the dispersion law for three-dimensional discrete system. The conclusion is given in Sec. XI.

II. EQUATIONS OF MOTION FOR INTERACTING OSCILLATORS

Consider a one-dimensional system of interacting oscillators that are described by the equations of motion,

$$\frac{\partial^2 u_n}{\partial t^2} = g\hat{\mathcal{I}}_n(u) + F(u_n), \tag{1}$$

where u_n are displacements from the equilibrium. The terms $F(u_n)$ characterize an interaction of the oscillators with the external on-site force. The term $\hat{\mathcal{I}}_n(u)$ is defined by

$$\hat{\mathcal{I}}_n(u) \equiv \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m)E(u_n,u_m), \tag{2}$$

and it takes into account the interaction of the oscillators in the system.

Examples:

1) If $J(n,m)=\delta_{n+1,m}-\delta_{n,m}$, and $E(u_n,u_m)=u_m$, then $\hat{\mathcal{I}}_n(u)=u_{n+1}-u_n=\Delta u_n$.

2) For $J(n,m)=\delta_{n+1,m}-2\delta_{n,m}+\delta_{n-1,m}$, and $E(u_n,u_m)=u_m$, we get

$$\hat{\mathcal{I}}_n(u) = u_{n+1} - 2u_n + u_{n-1} = \Delta^2 u_n .$$

3) We can consider the long-range interaction that is given by $J(n)=|n|^{-(1+\alpha)}$, where α is a positive real number. In this case, we have nonlocal coupling given by the power-law function. Constant α is a physical relevant parameter. Some integer values of α correspond to the well-known physical situations: Coulomb potential corresponds to $\alpha=0$, dipole-dipole interaction corresponds to $\alpha=2$, and the limit $\alpha\to\infty$ is for the case of nearest-neighbor interaction.

For the term (2) with $E(u_n,u_m)=u_m$, the translation invariance condition is

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m) = 0 \tag{3}$$

for all n. If (3) cannot be satisfied, we must define $E(u_n,u_m)=u_n-u_m$, and the interaction term (2) is

$$\hat{\mathcal{I}}_n(u) \equiv \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} J(n,m)[u_n - u_m]. \tag{4}$$

This interaction term is translation invariant. Note that the noninvariant terms lead to the divergences in the continuous limit (see the Appendix).

In this paper, we consider the wide class of interactions (4) that create a possibility to present the continuous medium equations with fractional derivatives. We also discuss the term (2) with $E(u_n,u_m)=f(u_n)-f(u_m)$ as nonlinear long-range interaction. As the examples, we consider $f(u)=u^2$ and $f(u)=u-gu^2$ that gives the Burgers, Korteweg-de Vries, and Boussinesq equations and their fractional generalizations in the continuous limit.

III. TRANSFORM OPERATION

In this section, we define the operation that transforms the system of equations for $u_n(t)$ into continuous medium equation for $u(x,t)$.

To derive a continuous medium equation, we suppose that $u_n(t)$ are Fourier coefficients of some function $\hat{u}(k,t)$. We define the field $\hat{u}(k,t)$ on $[-K/2,K/2]$ as

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-ikx_n} = \mathcal{F}_{\Delta}\{u_n(t)\}, \tag{5}$$

where $x_n=n\Delta x$, $\Delta x=2\pi/K$ is distance between oscillators, and

$$u_n(t) = \frac{1}{K}\int_{-K/2}^{+K/2} \mathrm{d}k\hat{u}(k,t)e^{ikx_n} = \mathcal{F}_{\Delta}^{-1}\{\hat{u}(k,t)\}. \tag{6}$$

These equations are the basis for the Fourier transform, which is obtained by transforming $u_n(t)$ from discrete variable to a continuous one in the limit $\Delta x \to 0$ ($K \to \infty$). The Fourier transform can be derived from (5), (6) in the limit as $\Delta x \to 0$. Replace the discrete $u_n(t)$ with continuous $u(x,t)$ while letting $x_n=n\Delta x=2\pi n/K \to x$. Then change the sum to an integral, and Eqs. (5) and (6) become

$$\tilde{u}(k,t) = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-ikx} u(x,t) = \mathcal{F}\{u(x,t)\}, \tag{7}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k e^{ikx}\tilde{u}(k,t) = \mathcal{F}^{-1}\{\tilde{u}(k,t)\}, \tag{8}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \tag{9}$$

and $\mathcal{L}$ denotes the passage to the limit $\Delta x \to 0$ ($K \to \infty$). Note that $\tilde{u}(k,t)$ is a Fourier transform of the field $u(x,t)$, and $\hat{u}(k,t)$ is a Fourier series transform of $u_n(t)$. The function $\tilde{u}(k,t)$ can be derived from $\hat{u}(k,t)$ in the limit $\Delta x \to 0$.

The procedure of the replacement of a discrete model by the continuous one is defined by the transform operation.

Definition 1: Transform operation $\hat{T}$ is a combination $\hat{T}=\mathcal{F}^{-1}\mathcal{L}\mathcal{F}_{\Delta}$ of the operations:

1) The Fourier series transform:

$$\mathcal{F}_{\Delta}: \quad u_n(t) \to \mathcal{F}_{\Delta}\{u_n(t)\} = \hat{u}(k,t). \tag{10}$$

2) The passage to the limit $\Delta x \to 0$:

$$\mathcal{L}: \quad \hat{u}(k,t) \to \mathcal{L}\{\hat{u}(k,t)\} = \tilde{u}(k,t). \tag{11}$$

3) The inverse Fourier transform:

$$\mathcal{F}^{-1}: \quad \tilde{u}(k,t) \to \mathcal{F}^{-1}\{\tilde{u}(k,t)\} = u(x,t). \tag{12}$$

The operation $\hat{T} = \mathcal{F}^{-1}\mathcal{L}\mathcal{F}_\Delta$ is called a transform operation, since it performs a transform of a discrete model of coupled oscillators into the continuous medium model.

Proposition 1: The transform operation $\hat{T}$ maps the function $F(u_n)$ into the function $F(u(x,t))$, i.e.,

$$\hat{T}F(u_n(t)) = F(u(x,t)), \tag{13}$$

where $u(x,t) = \hat{T}u_n(t)$, if the function F satisfies $\mathcal{L}F(u_n) = F(\mathcal{L}u_n)$.

Proof: The Fourier series transform leads to

$$\mathcal{F}_\Delta: \quad F(u_n) \to \mathcal{F}_\Delta F(u_n). \tag{14}$$

Note that $\mathcal{F}_\Delta F(u_n) \neq F(\mathcal{F}_\Delta u_n) = F(\hat{u}(k,t))$. The passage to the limit $\Delta x \to 0$ gives

$$\mathcal{L}: \quad \mathcal{F}_\Delta F(u_n) \to \mathcal{L}\mathcal{F}_\Delta F(u_n). \tag{15}$$

Then

$$\mathcal{L}\mathcal{F}_\Delta\{F(u_n)\} = \mathcal{F}\{\mathcal{L}F(u_n)\} = \mathcal{F}\{F(\mathcal{L}u_n)\} = \mathcal{F}\{F(u(x,t))\}, \tag{16}$$

where we use $\mathcal{L}\mathcal{F}_\Delta = \mathcal{F}\mathcal{L}$. The inverse Fourier transform get

$$\mathcal{F}^{-1}: \quad \mathcal{F}\{F(u(x,t))\} \to \mathcal{F}^{-1}\{\mathcal{F}\{F(u(x,t))\}\} = F(u(x,t)). \tag{17}$$

As the result, we prove (13).

IV. EQUATIONS FOR MOMENTUM SPACE

Let us consider a system of infinite numbers of oscillators with interparticle interaction that is described by (4). We suppose that $J(n,m)$ satisfies the condition

$$J(n,m) = J(n-m), \quad \sum_{n=1}^{\infty} |J(n)|^2 < \infty, \tag{18}$$

where $J(-n) = J(n)$.

Proposition 2: The Fourier series transform $\mathcal{F}_\Delta$ maps the equations of motion

$$\frac{\partial^2 u_n(t)}{\partial t^2} = g \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} J(n,m)[u_n - u_m] + F(u_n), \tag{19}$$

where u_n is the position of the nth oscillator, and F is an external on-site force, into

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\hat{u}(k,t) + \mathcal{F}_\Delta\{F(u_n)\}, \tag{20}$$

where

$$\hat{u}(k,t) = \mathcal{F}_\Delta\{u_n(t)\}, \quad \hat{J}_\alpha(k\Delta x) = \mathcal{F}_\Delta\{J(n)\},$$

and $\mathcal{F}_\Delta\{F(u_n)\}$ is an operator notation for the Fourier series transform of $F(u_n)$.

Proof: To derive the equation for the field $\hat{u}(k,t)$, we multiply Eq. (19) by $\exp(-ikn\Delta x)$, and summing over n from $-\infty$ to $+\infty$. Then

$$\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}\frac{\partial^2}{\partial t^2}u_n(t) = g\sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x}J(n,m)[u_n-u_m] + \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}F(u_n). \tag{21}$$

From

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_n(t), \tag{22}$$

the left-hand side of (21) gives

$$\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}\frac{\partial^2 u_n(t)}{\partial t^2} = \frac{\partial^2}{\partial t^2}\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_n(t) = \frac{\partial^2 \hat{u}(k,t)}{\partial t^2}. \tag{23}$$

The second term of the right-hand side of (21) is

$$\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}F(u_n) = \mathcal{F}_\Delta\{F(u_n)\}. \tag{24}$$

The first term of the right-hand side (r.h.s) of (21) is defined by the function $J(n,m)$. Let us introduce the notation

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-ikn\Delta x}J(n). \tag{25}$$

Using $J(-n)=J(n)$, the function (25) can be presented by

$$\hat{J}_\alpha(k\Delta x) = \sum_{n=1}^{+\infty} J(n)(e^{-ikn\Delta x}+e^{ikn\Delta x}) = 2\sum_{n=1}^{+\infty} J(n)\cos(k\Delta x). \tag{26}$$

From (26) it follows that

$$\hat{J}_\alpha(k\Delta x + 2\pi m) = \hat{J}_\alpha(k\Delta x), \tag{27}$$

where m is an integer.

The interaction term in (21) is

$$\sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x}J(n,m)[u_n-u_m] = \sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x}J(n,m)u_n - \sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x}J(n,m)u_m. \tag{28}$$

Using (22) and (25), the first term on the r.h.s. of (28) gives

$$\sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x}J(n,m)u_n = \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_n \sum_{\substack{m'=-\infty \\ m'\neq 0}}^{+\infty} J(m') = \hat{u}(k,t)\hat{J}_\alpha(0), \tag{29}$$

where we use (18) and $J(m'+n,n)=J(m')$, and

$$\hat{J}_\alpha(0) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} J(n) = 2\sum_{n=1}^{\infty} J(n). \tag{30}$$

For the second term on the r.h.s. of (28):

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m) u_m = \sum_{m=-\infty}^{+\infty} u_m \sum_{\substack{n=-\infty \\ n\neq m}}^{+\infty} e^{-ikn\Delta x} J(n,m)$$

$$= \sum_{m=-\infty}^{+\infty} u_m e^{-ikm\Delta x} \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} e^{-ikn'\Delta x} J(n') = \hat{u}(k,t)\hat{J}_\alpha(k\Delta x), \tag{31}$$

where we use $J(m,n'+m)=J(n')$.

As a result, Eq. (21) has the form

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\hat{u}(k,t) + \mathcal{F}_\Delta\{F(u_n)\}, \tag{32}$$

where $\mathcal{F}_\Delta\{F(u_n)\}$ is an operator notation for the Fourier series transform of $F(u_n)$.

V. ALPHA-INTERACTION

Let us consider the interaction term

$$\hat{\mathcal{I}}_n(u) \equiv \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m)[u_n - u_m], \tag{33}$$

where

$$J(n,m) = J(n-m) = J(m-n), \quad \sum_{n=1}^{\infty} |J(n)|^2 < \infty. \tag{34}$$

In Sec. IV, we prove that the Fourier series transform $\mathcal{F}_\Delta$ of (33) gives

$$\mathcal{F}_\Delta\{\hat{\mathcal{I}}_n(u)\} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\hat{u}(k,t), \tag{35}$$

where $\hat{u}(k,t)=\mathcal{F}_\Delta\{u_n(t)\}$, and

$$\hat{J}_\alpha(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-ikn} J(n) = 2\sum_{n=1}^{\infty} J(n)\cos(kn). \tag{36}$$

Definition 2: The interaction term (33) in the equation of motion (1) is called α-interaction if the function (36) satisfies the condition

$$\lim_{k\to 0} \frac{[\hat{J}_\alpha(k) - \hat{J}_\alpha(0)]}{|k|^\alpha} = A_\alpha, \tag{37}$$

where $\alpha>0$ *and* $0<|A_\alpha|<\infty$.

If the function $\hat{J}_\alpha(k)$ is given, then $J(n)$ can be defined by

$$J(n)=\frac{1}{\pi}\int_0^{\pi}\hat{J}_{\alpha}(k)\cos(nk)\mathrm{d}k. \tag{38}$$

The condition (37) means that $\hat{J}_{\alpha}(k)-\hat{J}_{\alpha}(0)=O(|k|^{\alpha})$, i.e.,

$$\hat{J}_{\alpha}(k)-\hat{J}_{\alpha}(0)=A_{\alpha}|k|^{\alpha}+R_{\alpha}(k), \tag{39}$$

for $k\to 0$, where

$$\lim_{k\to 0}R_{\alpha}(k)/|k|^{\alpha}=0. \tag{40}$$

Examples:

1) The first example of the α-interaction is

$$\hat{J}_{\alpha}(k)=A_{\alpha}|k|^{\alpha}.$$

Using (38), we obtain

$$J(n)=A_{\alpha}\left(\frac{(-1)^n\pi^{\alpha+1}}{\alpha+1}-\frac{(-1)^n\pi^{1/2}}{(\alpha+1)|n|^{\alpha+1/2}}L_1(\alpha+3/2,1/2,\pi n)\right), \tag{41}$$

where $L_1(\mu,\nu,z)$ is the Lommel function.[33]

2) The second example of the α-interaction is

$$J(n)=\frac{(-1)^n}{n^2}. \tag{42}$$

Using (Ref. 34, Sec. 5.4.2.12)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nk)=\frac{1}{4}\left(k^2-\frac{\pi^2}{3}\right),\quad |k|\leqslant\pi,$$

we get

$$\hat{J}_{\alpha}(k)=2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(kn)=\frac{1}{2}k^2-\frac{\pi^2}{6},\quad |k|\leqslant\pi. \tag{43}$$

Then we have $\alpha=2$, and

$$\hat{J}_{\alpha}(k)-\hat{J}_{\alpha}(0)=(1/2)k^2. \tag{44}$$

The inverse Fourier transform of this expression gives the coordinate derivatives of second order

$$\mathcal{F}^{-1}\{\hat{J}_{\alpha}(k)-\hat{J}_{\alpha}(0)\}=-\frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

3) For the interaction (42), we have $\alpha=2$ and the inverse Fourier transform of (44) gives the second-order derivative. At the same time, the interaction

$$J(n)=\frac{1}{n^2}$$

gives $\alpha=1$ and then the first-order coordinate derivative. It can be proved by using (Ref. 34, Sec. 5.4.2.12)

$$\hat{J}_{\alpha}(k) = 2\sum_{n=1}^{+\infty} \frac{\cos(kn)}{n^2} = \frac{1}{6}[3k^2 - 6\pi k + 2\pi^2], \quad (0 \leqslant k \leqslant 2\pi), \tag{45}$$

and $\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) \approx -\pi k$ for $k \to 0$. Therefore, the inverse Fourier transform leads to the derivative of first order.

4) For noninteger and odd numbers s,

$$J(n) = |n|^{-(s+1)}, \quad s > 0 \tag{46}$$

is an α-interaction.

For $0 < s < 2 (s \neq 1)$, we get

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) = 2\Gamma(-s)\cos(\pi s/2)|k|^s. \tag{47}$$

For $s = 1$,

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) = -(\pi/2)k. \tag{48}$$

For noninteger $s > 2$,

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) = -\zeta(\alpha - 1)k^2, \tag{49}$$

where $\zeta(z)$ is the Riemann zeta-function, The interaction (46) is considered in Sec. VII.

5) The other example is

$$J(n) = \frac{(-1)^n}{\Gamma(1 + \alpha/2 + n)\Gamma(1 + \alpha/2 - n)}. \tag{50}$$

Using the series (Ref. 34, Sec. 5.4.8.12)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(\beta + 1 + n)\Gamma(\beta + 1 - n)} \cos(nk) = \frac{2^{2\beta - 1}}{\Gamma(2\beta + 1)} \sin^{2\beta}\left(\frac{k}{2}\right) - \frac{1}{2\Gamma^2(\beta + 1)}, \tag{51}$$

where $\beta > -1/2$ and $0 < k < 2\pi$, we get

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) = \frac{2^{\alpha}}{\Gamma(\alpha + 1)} \sin^{\alpha}\left(\frac{k}{2}\right). \tag{52}$$

In the limit $k \to 0$, we obtain

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) \approx \frac{1}{\Gamma(\alpha + 1)} |k|^{\alpha}. \tag{53}$$

For noninteger α, the inverse Fourier transform of (53) gives the fractional Riesz derivative[1] of order α.

6) The α-interaction

$$J(n) = \frac{(-1)^n}{a^2 - n^2},$$

gives

$$\hat{J}(k) = \frac{\pi}{a \sin(\pi a)} \cos(ak) - \frac{1}{a^2}. \tag{54}$$

For $k \to 0$, we obtain

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) \approx \frac{a\pi}{2\sin(a\pi)} k^2. \tag{55}$$

The inverse Fourier transform of (55) leads to the coordinate derivative of second order.

7) For $J(n)=1/n!$, we use

$$\sum_{n=1}^{\infty} \frac{\cos(kn)}{n!} = e^{\cos k} \cos(\sin k), \quad |k| < \infty. \tag{56}$$

The passage to the limit $k \to 0$ gives

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) \approx -4ek. \tag{57}$$

Then $\alpha=1$, and we get the derivative of first order.

Proposition 3: The transform operation $\hat{T}$ maps the discrete equations of motion

$$\frac{\partial^2 u_n}{\partial t^2} = g \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} J(n,m)[u_n - u_m] + F(u_n) \tag{58}$$

with noninteger α-interaction into the fractional continuous medium equations:

$$\frac{\partial^2}{\partial t^2} u(x,t) - G_{\alpha} A_{\alpha} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x,t) - F(u(x,t)) = 0, \tag{59}$$

where $\partial^{\alpha}/\partial|x|^{\alpha}$ is the Riesz fractional derivative, and

$$G_{\alpha} = g|\Delta x|^{\alpha} \tag{60}$$

is a finite parameter.

Proof: The Fourier series transform $\mathcal{F}_{\Delta}$ of (58) gives (20). We will be interested in the limit $\Delta x \to 0$. Then Eq. (20) can be written as

$$\frac{\partial^2}{\partial t^2} \hat{u}(k,t) - G_{\alpha} \hat{\mathcal{T}}_{\alpha,\Delta}(k) \hat{u}(k,t) - \mathcal{F}_{\Delta}\{F(u_n(t))\} = 0, \tag{61}$$

where we use finite parameter (60), and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_{\alpha}|k|^{\alpha} - R_{\alpha}(k\Delta x)|\Delta x|^{-\alpha}. \tag{62}$$

Note that R_{α} satisfies the condition

$$\lim_{\Delta x \to 0} \frac{R_{\alpha}(k\Delta x)}{|\Delta x|^{\alpha}} = 0.$$

The expression for $\hat{\mathcal{T}}_{\alpha,\Delta}(k)$ can be considered as a Fourier transform of the operator (4). Note that $g \to \infty$ for the limit $\Delta x \to 0$, if G_{α} is a finite parameter.

In the limit $\Delta x \to 0$, Eq. (61) gets

$$\frac{\partial^2}{\partial t^2} \tilde{u}(k,t) - G_{\alpha} \hat{\mathcal{T}}_{\alpha}(k) \tilde{u}(k,t) - \mathcal{F}^{-1}\{F(u(x,t))\} = 0, \tag{63}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \quad \hat{\mathcal{T}}_{\alpha}(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_{\alpha}|k|^{\alpha}.$$

The inverse Fourier transform of (63) gives

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_\alpha \mathcal{T}_\alpha(x)u(x,t) - F(u(x,t)) = 0, \tag{64}$$

where $\mathcal{T}_\alpha(x)$ is an operator

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}. \tag{65}$$

Here, we have used the connection between the Riesz fractional derivative and its Fourier transform:[1]

$$|k|^\alpha \leftrightarrow -\frac{\partial^\alpha}{\partial |x|^\alpha}. \tag{66}$$

The properties of the Riesz derivative can be found in Refs. 1–4. Note that the Riesz derivative could be represented as

$$\frac{\partial^\alpha}{\partial |x|^\alpha}u(x,t) = -\frac{1}{2\cos(\pi\alpha/2)}(\mathcal{D}^\alpha_+ u(x,t) + \mathcal{D}^\alpha_- u(x,t)), \tag{67}$$

where $\alpha \neq 0,1,3,5,\ldots$, and $\mathcal{D}^\alpha_\pm$ are Riemann-Liouville left and right fractional derivatives defined by[1–4]

$$\begin{aligned}\mathcal{D}^\alpha_+ u(x,t) &= \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_{-\infty}^{x}\frac{u(\xi,t)\mathrm{d}\xi}{(x-\xi)^{\alpha-m+1}},\\ \mathcal{D}^\alpha_- u(x,t) &= \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_{x}^{\infty}\frac{u(\xi,t)\mathrm{d}\xi}{(\xi-x)^{\alpha-m+1}},\end{aligned} \tag{68}$$

where $m-1<\alpha<m$.

As the result, we obtain continuous medium equations (59) from (64) and (65).

VI. SIMPLE EXAMPLE OF NEAREST-NEIGHBOR INTERACTION

In this section, we demonstrate the application of transform operation to the well-known case:

$$J(n,m) = \delta_{n+1,m} - 2\delta_{n,m} + \delta_{n-1,m}, \tag{69}$$

where $\delta_{n,m}$ is the Kronecker symbol. Then the interaction term (2) has the form

$$\hat{\mathcal{I}}_n(u) = (u_{n+1} - u_n) - (u_n - u_{n-1}), \tag{70}$$

and describes the nearest-neighbor interaction. As the result, equations of motion (19) have the form

$$\frac{\partial^2 u_n}{\partial t^2} = g[u_{n+1} - 2u_n + u_{n-1}] + F(u_n). \tag{71}$$

The well-known result is the following.

Proposition 4: The transform operation $\hat{T}$ maps the equation of motion (71) into the continuous medium equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = G_2 \frac{\partial^2}{\partial x^2}u(x,t) + F(u), \tag{72}$$

where

$$G_2 = g(\Delta x)^2 \tag{73}$$

is a finite parameter.

Proof: To derive the equation for the field $\hat{u}(k,t)$, we multiply Eq. (71) by $\exp(-ikn\Delta x)$, and summing over n from $-\infty$ to $+\infty$. Then

$$\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}\frac{\partial^2}{\partial t^2}u_n(t) = g\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}[u_{n+1} - 2u_n + u_{n-1}] + \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}F(u_n). \tag{74}$$

The first term on the r.h.s. of (74) is

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}J(n,m)u_m &= \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}[u_{n+1} - 2u_n + u_{n-1}] \\ &= \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_{n+1} - 2\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_n + \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}u_{n-1} \\ &= \sum_{m'=-\infty}^{+\infty} e^{-ik(m-1)\Delta x}u_m - 2\hat{u}(k,t) + \sum_{s=-\infty}^{+\infty} e^{-ik(s+1)\Delta x}u_s \\ &= e^{ik\Delta x}\sum_{m'=-\infty}^{+\infty} e^{-ikm\Delta x}u_m - 2\hat{u}(k,t) + e^{-ik\Delta x}\sum_{s=-\infty}^{+\infty} e^{-iks\Delta x}u_s \\ &= e^{ik\Delta x}\hat{u}(k,t) - 2\hat{u}(k,t) + e^{-ik\Delta x}\hat{u}(k,t) = [e^{ik\Delta x} + e^{-ik\Delta x} - 2]\hat{u}(k,t) \\ &= 2[\cos(k\Delta x) - 1]\hat{u}(k,t) = -4\sin^2(k\Delta x)\hat{u}(k,t).\end{aligned} \tag{75}$$

As the result, we obtain

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = g\hat{J}_\alpha(k\Delta x)\hat{u}(k,t) + \mathcal{F}_\Delta\{F(u_n(t))\}, \tag{76}$$

where

$$\hat{J}_\alpha(k\Delta x) = -4\sin^2(k\Delta x). \tag{77}$$

For $\Delta x \to 0$, the asymptotics of the sine is

$$\sin(z) = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)!}z^{2m+1} \approx z - \frac{1}{6}z^3,$$

and (77) can be presented by

$$\hat{J}_\alpha(k\Delta x) \approx -(k\Delta x)^2 + \tfrac{1}{12}(k\Delta x)^4. \tag{78}$$

Using the finite parameter (73), the transition to the limit $\Delta x \to 0$ in Eq. (76) gives

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = -G_2k^2\tilde{u}(k,t) + \mathcal{F}^{-1}\{F(u)\}, \tag{79}$$

where we use $0 < |G_2| < \infty$. As the result, the inverse Fourier transform of (79) leads to the continuous medium equation (72).

VII. INTEGER POWER-LAW INTERACTION

Let us consider the power-law interaction (4) with

$$J(n) = |n|^{-(s+1)} \tag{80}$$

with positive integer number s.

Proposition 5: The power-law interaction (80) for the odd number s is α-interaction with $\alpha=1$ for $s=1$, and $\alpha=2$ *for $s=3,5,7,\ldots$. For even numbers s,* (80) *is not α-interaction. For odd number s, the transform operation $\hat{T}$ maps the equations of motion with the interaction (80) into the continuous medium equation with derivatives of first order for $s=1$, and the second order for other odd s.*

Proof: From (20), we get the equation for $\hat{u}(k,t)$ in the form

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + g[\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0)]\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{81}$$

where

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-ikn\Delta x}|n|^{-(1+s)}. \tag{82}$$

The function (82) can be presented by

$$\hat{J}_\alpha(k\Delta x) = \sum_{n=1}^{+\infty} \frac{1}{n^{1+s}}(e^{-ikn\Delta x} + e^{ikn\Delta x}) = 2\sum_{n=1}^{+\infty} \frac{1}{n^{1+s}} \cos(kn\Delta x). \tag{83}$$

Then we can use (Ref. 46, Sec. 5.4.2.12 and 5.4.2.7) the relations

$$\sum_{n=1}^{\infty} \frac{\cos(nk)}{n^2} = \frac{1}{12}(3k^2 - 6\pi k + 2\pi^2), \quad (0 \leqslant k \leqslant 2\pi), \tag{84}$$

$$\sum_{n=1}^{\infty} \frac{\cos(nk)}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2(2m)!} B_{2m}\left(\frac{k}{2\pi}\right), \quad (0 \leqslant k \leqslant 2\pi), \tag{85}$$

where $m=1,2,3,\ldots$, and $B_{2m}(z)$ are the Bernulli polynomials.[35] These polynomials are defined by

$$B_n(k) = \sum_{s=0}^{n} C_n^s B_s k^{n-s}, \tag{86}$$

where B_s are the Bernoulli numbers from

$$\frac{z}{e^z - 1} = \sum_{s=0}^{\infty} B_s \frac{z^s}{s!}, \quad (|z| < 2\pi). \tag{87}$$

For example,

$$B_2(k) = k^2 - k + 1/6, \quad B_4(k) = k^4 - 2k^3 + k^2 - 1/30. \tag{88}$$

Note $B_{2m-1}=0$ for $m=2,3,4,\ldots$.[35]

For $s=1$, we have

$$\hat{J}_{\alpha}(k\Delta x) - \hat{J}_{\alpha}(0) = \tfrac{1}{2}(k\Delta x)^2 - \pi k\Delta x \approx -\pi k\Delta x. \tag{89}$$

For $s=2m-1$ ($m=2,3,\ldots$), we have

$$\hat{J}_{\alpha}(k) = \frac{(-1)^{m-1}}{(2m)!}(2\pi)^{2m}B_{2m}\left(\frac{k}{2\pi}\right) \quad (0 \leqslant k \leqslant 2\pi). \tag{90}$$

Then

$$\hat{J}_{\alpha}(k\Delta x) - \hat{J}_{\alpha}(0) \approx \frac{(-1)^{m-1}(2\pi)^{2m-2}}{4(2m-2)!}B_{2m-2}(k\Delta x)^2. \tag{91}$$

For example, the interaction (80) with $s=3$ gives

$$\hat{J}_{\alpha}(k) - \hat{J}_{\alpha}(0) = -\frac{1}{48}[k^4 - 4\pi k^3 + 4\pi^2 k^2] \approx -\frac{\pi^2}{12}k^2. \tag{92}$$

For $s=0$, we have (Ref. 34, Sec. 5.4.2.9) the relation

$$\sum_{n=1}^{\infty} \frac{\cos(nk)}{n} = -\ln[2\sin(k/2)]. \tag{93}$$

Then, the limit $\Delta x \to 0$ gives

$$\hat{J}_{\alpha}(k\Delta x) \approx -\ln(k\Delta x) \to \infty. \tag{94}$$

For even numbers s,

$$|\hat{J}_{\alpha}(k\Delta x) - \hat{J}_{\alpha}(0)|/|k\Delta x|^s \to \infty \tag{95}$$

since the expression has the logarithmic poles.

The transition to the limit $\Delta x \to 0$ in Eq. (81) with $s=1$ gives

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} - G_1 k\tilde{u}(k,t) - \mathcal{F}^{-1}\{F(u(x,t))\} = 0, \tag{96}$$

where $G_1 = \pi g\Delta x$ is a finite parameter. The inverse Fourier transform of (96) leads to the continuous medium equation with coordinate derivative of first order:

$$\frac{\partial^2}{\partial t^2}u(x,t) - iG_1\frac{\partial}{\partial x}u(x,t) - F(u(x,t)) = 0. \tag{97}$$

This equation can be considered as the nonlinear Schrödinger equation.

The limit $\Delta x \to 0$ in Eq. (81) with $s=2m-1$ ($m=2,3,\ldots$) gives

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} + G_2 k^2\tilde{u}(k,t) - \mathcal{F}^{-1}\{F(u(x,t))\} = 0, \tag{98}$$

where

$$G_2 = \frac{(-1)^{m-1}(2\pi)^{2m-2}}{4(2m-2)!}B_{2m-2}g(\Delta x)^2$$

is a finite parameter. The inverse Fourier transform of (98) leads to the partial differential equation of second order:

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_2\frac{\partial^2}{\partial x^2}u(x,t) - F(u(x,t)) = 0. \tag{99}$$

This equation can be considered as a nonlinear wave equation.

VIII. NONINTEGER POWER-LAW INTERACTION

Let us consider the power-law interaction with

$$J(n) = |n|^{-(s+1)}, \tag{100}$$

where s is a positive noninteger number.

Proposition 6: The power-law interaction (100) with noninteger s is α-interaction with $\alpha=s$ for $0<s<2$, and $\alpha=2$ for $s>2$. For $0<s<2$ $(s\neq 1)$, the transform operation $\hat{T}$ maps the discrete equations with the interaction (100) into the continuous medium equation with fractional Riesz derivatives of order α. For $\alpha>2$ $(\alpha\neq 3,4,5,\ldots)$, the continuous medium equation has the coordinate derivatives of second order.

Proof: From (20), we obtain the equation for $\hat{u}(k,t)$ in the form

$$\frac{\partial^2\hat{u}(k,t)}{\partial t^2} + g[\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0)]\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{101}$$

where

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} e^{-ikn\Delta x}\frac{1}{|n|^{1+\alpha}}. \tag{102}$$

For fractional positive α, the function (102) can be presented by

$$\hat{J}_\alpha(k\Delta x) = \sum_{n=1}^{+\infty}\frac{1}{n^{1+\alpha}}(e^{-ikn\Delta x} + e^{ikn\Delta x}) = Li_{1+\alpha}(e^{ik\Delta x}) + Li_{1+\alpha}(e^{-ik\Delta x}), \tag{103}$$

where $Li_\beta(z)$ is a polylogarithm function. Using the series representation of the polylogarithm:[36]

$$Li_\beta(e^z) = \Gamma(1-\beta)(-z)^{\beta-1} + \sum_{n=0}^{\infty}\frac{\zeta(\beta-n)}{n!}z^n, \quad |z|<2\pi, \quad \beta\neq 1,2,3,\ldots, \tag{104}$$

we obtain

$$\hat{J}_\alpha(k\Delta x) = A_\alpha|\Delta x|^\alpha|k|^\alpha + 2\sum_{n=0}^{\infty}\frac{\zeta(1+\alpha-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n, \quad \alpha\neq 0,1,2,3,\ldots, \tag{105}$$

where $\zeta(z)$ is the Riemann zeta-function, $|k\Delta x|<2\pi$, and

$$A_\alpha = 2\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right). \tag{106}$$

From (105), we have

$$J_\alpha(0) = 2\zeta(1+\alpha).$$

Then

$$\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0) = A_\alpha|\Delta x|^\alpha|k|^\alpha + 2\sum_{n=1}^{\infty}\frac{\zeta(1+\alpha-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n, \tag{107}$$

where $\alpha \neq 0,1,2,3,\ldots$, and $|k\Delta x| < 2\pi$.

Substitution of (107) into Eq. (101) gives

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + gA_\alpha|\Delta x|^\alpha|k|^\alpha \hat{u}(k,t) + 2g\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0. \tag{108}$$

We will be interested in the limit $\Delta x \to 0$. Then Eq. (108) can be written in a simple form

$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) + G_\alpha \hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \quad \alpha \neq 0,1,2,\ldots, \tag{109}$$

where we use the finite parameter

$$G_\alpha = g|\Delta x|^{\min\{\alpha;2\}}, \tag{110}$$

and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = \begin{cases} A_\alpha|k|^\alpha - |\Delta x|^{2-\alpha}\zeta(\alpha-1)k^2, & 0<\alpha<2 \quad (\alpha\neq 1) \\ |\Delta x|^{\alpha-2}A_\alpha|k|^\alpha - \zeta(\alpha-1)k^2, & \alpha>2 \quad (\alpha\neq 3,4,\ldots). \end{cases} \tag{111}$$

The expression for $\hat{\mathcal{T}}_{\alpha,\Delta}(k)$ can be considered as a Fourier transform of the interaction operator (2). From (110), we see that $g\to\infty$ for the limit $\Delta x \to 0$, and finite value of G_α.

Note that (111) has a scale k_0:

$$k_0 = |A_\alpha/\zeta(\alpha-1)|^{1/(2-\alpha)}|\Delta x|^{-1} \tag{112}$$

such that the nontrivial expression $\hat{\mathcal{T}}_{\alpha,\Delta}(k) \sim |k|^\alpha$ appears only for $0<\alpha<2$, $(\alpha\neq 1)$, $k \ll k_0$.

The transition to the limit $\Delta x \to 0$ in Eq. (109) gives

$$\frac{\partial^2}{\partial t^2}\tilde{u}(k,t) + G_\alpha\hat{\mathcal{T}}_\alpha(k)\tilde{u}(k,t) - \mathcal{F}^{-1}\{F(u(x,t))\} = 0 \quad (\alpha \neq 0,1,2,\ldots), \tag{113}$$

where

$$\hat{\mathcal{T}}_\alpha(k) = \begin{cases} A_\alpha|k|^\alpha, & 0<\alpha<2, \quad \alpha\neq 1 \\ -\zeta(\alpha-1)k^2, & 2<\alpha, \quad \alpha\neq 3,4,\ldots. \end{cases} \tag{114}$$

The inverse Fourier transform to (113) is

$$\frac{\partial^2}{\partial t^2}u(x,t) + G_\alpha\mathcal{T}_\alpha(x)u(x,t) - F(u(x,t)) = 0 \quad \alpha \neq 0,1,2,\ldots, \tag{115}$$

where

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = \begin{cases} -A_\alpha\partial^\alpha/\partial|x|^\alpha & (0<\alpha<2, \quad \alpha\neq 1) \\ \zeta(\alpha-1)\partial^2/\partial|x|^2 & (\alpha>2, \quad \alpha\neq 3,4,\ldots). \end{cases}$$

Here, we have used the connection between the Riesz fractional derivative and its Fourier transform:[1]

$$|k|^{\alpha} \leftrightarrow -\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}, \quad k^2 \leftrightarrow -\frac{\partial^2}{\partial|x|^2}. \tag{116}$$

The properties of the Riesz derivative can be found in Refs. 1–4.

As the result, we obtain the continuous medium equations

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_{\alpha}A_{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t) = F(u(x,t)), \quad 0<\alpha<2 \quad (\alpha \neq 1), \tag{117}$$

and

$$\frac{\partial^2}{\partial t^2}u(x,t) + G_{\alpha}\zeta(\alpha-1)\frac{\partial^2}{\partial|x|^2}u(x,t) = F(u(x,t)), \quad \alpha>2 \quad (\alpha \neq 3,4,\dots). \tag{118}$$

Analogously, the continuous limit for the system

$$\frac{\partial u_n}{\partial t} = g\sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} |n-m|^{-\alpha-1}[u_n - u_m] + F(u_n) \tag{119}$$

gives the partial differential equations

$$\frac{\partial}{\partial t}u(x,t) - G_{\alpha}A_{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t) = F(u(x,t)), \quad 0<\alpha<2 \quad (\alpha \neq 1), \tag{120}$$

and

$$\frac{\partial}{\partial t}u(x,t) + G_{\alpha}\zeta(\alpha-1)\frac{\partial^2}{\partial|x|^2}u(x,t) = F(u(x,t)), \quad \alpha>2 \quad (\alpha \neq 3,4,\dots). \tag{121}$$

For $F(u)=0$, Eq. (120) is the fractional kinetic equation that describes the fractional superdiffusion.[37–39] If $F(u)$ is a sum of linear and cubic terms, then Eq. (120) has the form of the fractional Ginzburg-Landau equation.[40–44] A remarkable property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails.

IX. NONLINEAR LONG-RANGE INTERACTION

In this section, we consider the discrete equations with nonlinear long-range interaction:

$$\hat{\mathcal{I}}_n(u) = \sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} J_{\alpha}(n,m)[f(u_n) - f(u_m)], \tag{122}$$

where $f(u)$ is a nonlinear function of $u_n(t)$, and $J_{\alpha}(n,m)$ defines the α-interaction. As the example of $J_{\alpha}(n,m)=J_{\alpha}(n-m)$, we can use the functions

$$J_{\alpha}(n) = \frac{(-1)^n}{\Gamma(1+\alpha/2+n)\Gamma(1+\alpha/2-n)}. \tag{123}$$

We consider the interaction with $f(u)=u^2$ and $f(u)=u-gu^2$ that gives the Burgers, Korteweg-de Vries, and Boussinesq equations in the continuous limit for $\alpha=1,2,3,4$. If we use the fractional α in Eq. (123), we can obtain the fractional generalization of these equations.

Proposition 7: The Fourier series transform $\mathcal{F}_{\Delta}$ maps the equations of motion

$$\frac{\partial^2 u_n(t)}{\partial t^2} = g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_\alpha(n-m)[f(u_n)-f(u_m)] + F(u_n), \tag{124}$$

where F is an external on-site force, into

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\mathcal{F}_\Delta\{f(u_n)\} + \mathcal{F}_\Delta\{F(u_n)\}, \tag{125}$$

where $\hat{u}(k,t)=\mathcal{F}_\Delta\{u_n(t)\}$, *and* $\hat{J}_\alpha(k\Delta x)=\mathcal{F}_\Delta\{J(n)\}$.

If $J_\alpha(n)$ *defines the* α*-interaction, then the continuous limit* $\Delta x \to 0$ *and the inverse Fourier transform give*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = G_\alpha A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} f(u(x,t)) + F(u(x,t)), \tag{126}$$

where $G_\alpha = g|\Delta x|^\alpha$ *is a finite parameter.*

Proof: The Fourier series transform of the interaction term (122) can be presented as

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}\hat{\mathcal{I}}_n(u) &= \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m)[f(u_n)-f(u_m)] \\ &= \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m) f(u_n) - \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m) f(u_m).\end{aligned} \tag{127}$$

For the first term on the r.h.s. of (127):

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m) f(u_n) = \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} f(u_n) \sum_{\substack{m'=-\infty \\ m'\neq 0}}^{+\infty} J(m') = \mathcal{F}_\Delta\{f(u_n)\}\hat{J}_\alpha(0), \tag{128}$$

where we use $J(m'+n,n)=J(m')$. For the second term on the r.h.s. of (127):

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} e^{-ikn\Delta x} J(n,m) f(u_m) &= \sum_{m=-\infty}^{+\infty} f(u_m) \sum_{\substack{n=-\infty \\ n\neq m}}^{+\infty} e^{-ikn\Delta x} J(n,m) \\ &= \sum_{m=-\infty}^{+\infty} f(u_m) e^{-ikm\Delta x} \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} e^{-ikn'\Delta x} J(n') = \mathcal{F}_\Delta\{f(u_n)\}\hat{J}_\alpha(k\Delta x),\end{aligned} \tag{129}$$

where we use $J(m,n'+m)=J(n')$.

As the result, we obtain Eq. (125).

For the limit $\Delta x \to 0$, Eq. (125) can be written as

$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) - G_\alpha \hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{130}$$

where we use finite parameter $G_\alpha = g|\Delta x|^\alpha$, and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha - R_\alpha(k\Delta x)|\Delta x|^{-\alpha}. \tag{131}$$

Here, the function R_α satisfies the condition

$$\lim_{\Delta x \to 0} \frac{R_\alpha(k\Delta x)}{|\Delta x|^\alpha} = 0.$$

In the limit $\Delta x \to 0$, we get

$$\frac{\partial^2}{\partial t^2}\tilde{u}(k,t) - G_\alpha \hat{\mathcal{T}}_\alpha(k)\mathcal{F}^{-1}\{f(u(x,t))\} - \mathcal{F}^{-1}\{F(u(x,t))\} = 0, \tag{132}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \quad \hat{\mathcal{T}}_\alpha(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha.$$

The inverse Fourier transform of (132) gives

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_\alpha \mathcal{T}_\alpha(x) f(u(x,t)) - F(u(x,t)) = 0, \tag{133}$$

where $\mathcal{T}_\alpha(x)$ is an operator

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}. \tag{134}$$

As the result, we obtain the continuous medium equation (126).

Let us consider examples of quadratic-nonlinear long-range interactions.

1) The continuous limit of the lattice equations

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_1(n,m)[u_n^2 - u_m^2] + g_2 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_2(n,m)[u_n - u_m], \tag{135}$$

where $J_i(n)$ $(i=1,2)$ define the α_i-interactions with $\alpha_1=1$ and $\alpha_2=2$, gives the Burgers equation[45] that is a nonlinear partial differential equation of second order:

$$\frac{\partial}{\partial t}u(x,t) + G_1 u(x,t)\frac{\partial}{\partial x}u(x,t) - G_2 \frac{\partial^2}{\partial x^2}u(x,t) = 0. \tag{136}$$

It is used in fluid dynamics as a simplified model for turbulence, boundary layer behavior, shock wave formation, and mass transport. If we consider $J_2(n,m)$ with fractional $\alpha_2=\alpha$, then we get the fractional Burgers equation that is suggested in Ref. 46.

2) The continuous limit of the system of equations

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_1(n,m)[u_n^2 - u_m^2] + g_3 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_3(n,m)[u_n - u_m], \tag{137}$$

where $J_i(n)$ $(i=1,3)$ define the α_i-interactions with $\alpha_1=1$ and $\alpha_3=3$, gives Korteweg-de Vries (KdV) equation

$$\frac{\partial}{\partial t}u(x,t) - G_1 u(x,t)\frac{\partial}{\partial x}u(x,t) + G_3 \frac{\partial^3}{\partial x^3}u(x,t) = 0. \tag{138}$$

First formulated as part of an analysis of shallow-water waves in canals, it has subsequently been found to be involved in a wide range of physics phenomena, especially those exhibiting shock waves, traveling waves, and solitons. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport.

If we use noninteger α_i-interactions for $J_i(n)$, then we get the fractional generalization of the KdV equation.[47,48]

3) The continuous limit of

$$\frac{\partial^2 u_n(t)}{\partial t^2} = g_2 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_2(n,m)[f(u_n) - f(u_m)] + g_4 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_4(n,m)[u_n - u_m], \tag{139}$$

where

$$f(u) = u - gu^2,$$

and $J_i(n)$ define the α_i-interactions with $\alpha_2=2$ and $\alpha_4=4$, gives the Boussinesq equation that is a nonlinear partial differential equation of fourth order

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_2\frac{\partial^2}{\partial x^2}u(x,t) + gG_2\frac{\partial^2}{\partial x^2}u^2(x,t) + G_4\frac{\partial^4}{\partial x^4}u(x,t) = 0. \tag{140}$$

This equation was formulated as part of an analysis of long waves in shallow water. It was subsequently applied to problems in the percolation of water in porous subsurface strata. It also crops up in the analysis of many other physical processes.

X. FRACTIONAL DERIVATIVES FROM DISPERSION LAW

Let us consider the three-dimensional lattice that is described by the equations of motion

$$\frac{\partial u_{\mathbf{n}}}{\partial t} = g \sum_{\substack{\mathbf{m}=-\infty \\ \mathbf{m}\neq \mathbf{n}}}^{+\infty} J(\mathbf{n},\mathbf{m})[u_{\mathbf{n}} - u_{\mathbf{m}}] + F(u_{\mathbf{n}}), \tag{141}$$

where $\mathbf{n}=(n_1,n_2,n_3)$, and $J(\mathbf{n},\mathbf{m})=J(\mathbf{n}-\mathbf{m})=J(\mathbf{m}-\mathbf{n})$. We suppose that $u_{\mathbf{n}}(t)$ are Fourier coefficients of the function $\hat{u}(\mathbf{k},t)$:

$$\hat{u}(\mathbf{k},t) = \sum_{\mathbf{n}=-\infty}^{+\infty} u_{\mathbf{n}}(t)e^{-i\mathbf{k}\mathbf{r}_{\mathbf{n}}} = \mathcal{F}_{\Delta}\{u_{\mathbf{n}}(t)\}, \tag{142}$$

where $\mathbf{k}=(k_1,k_2,k_3)$, and

$$\mathbf{r}_{\mathbf{n}} = \sum_{i=1}^{3} n_i\mathbf{a}_i.$$

Here, $\mathbf{a}_i$ are translational vectors of the lattice. The continuous medium model can be derived in the limit $|\mathbf{a}_i|\to 0$.

To derive the equation for $\hat{u}(\mathbf{k},t)$, we multiply (141) by $\exp(-i\mathbf{k}\mathbf{r}_{\mathbf{n}})$, and summing over $\mathbf{n}$. Then, we obtain

$$\frac{\partial \hat{u}(\mathbf{k},t)}{\partial t} = g[\hat{J}_{\alpha}(0) - \hat{J}_{\alpha}(\mathbf{k}\mathbf{a})]\hat{u}(\mathbf{k},t) + \mathcal{F}_{\Delta}\{F(u_{\mathbf{n}})\}, \tag{143}$$

where $\mathcal{F}_{\Delta}\{F(u_{\mathbf{n}})\}$ is an operator notation for the Fourier series transform of $F(u_{\mathbf{n}})$, and

$$\hat{J}_\alpha(\mathbf{ka}) = \sum_{\mathbf{n}=-\infty}^{+\infty} e^{-i\mathbf{k}\mathbf{r}_\mathbf{n}} J(\mathbf{n}). \tag{144}$$

For the three-dimensional lattice, we define the α-interaction with $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, as an interaction that satisfies the conditions:

$$\lim_{k\to 0} \frac{[\hat{J}_\alpha(\mathbf{k}) - \hat{J}_\alpha(0)]}{|k_i|^{\alpha_i}} = A_{\alpha_i} \quad (i=1,2,3), \tag{145}$$

where $0<|A_{\alpha_i}|<\infty$. The conditions (145) mean that

$$\hat{J}_\alpha(0) - \hat{J}_\alpha(\mathbf{k}) = \sum_{i=1}^{3} A_{\alpha_1}|k_i|^{\alpha_i} + \sum_{i=1}^{3} R_{\alpha_i}(\mathbf{k}), \tag{146}$$

where

$$\lim_{k_i\to 0} R_{\alpha_i}(\mathbf{k})/|k_i|^{\alpha_i} = 0. \tag{147}$$

In the continuous limit ($|\mathbf{a}_i|\to 0$), the α-interaction in the three-dimensional lattice gives the continuous medium equations with the derivatives $\partial^{\alpha_1}/\partial x^{\alpha_1}$, $\partial^{\alpha_2}/\partial y^{\alpha_2}$, and $\partial^{\alpha_3}/\partial z^{\alpha_3}$.

Let us recall the appearance of the nonlinear parabolic equation.[49–52] Consider wave propagation in some media and present the wave vector $\mathbf{k}$ in the form

$$\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa} = \mathbf{k}_0 + \boldsymbol{\kappa}_\parallel + \boldsymbol{\kappa}_\perp, \tag{148}$$

where $\mathbf{k}_0$ is the unperturbed wave vector and subscripts ($\parallel$, $\perp$) are taken, respectively, to the direction of $\mathbf{k}_0$. A symmetric dispersion law

$$\omega(k) = \omega(\mathbf{k}) = \hat{J}_\alpha(\mathbf{ka}) - \hat{J}_\alpha(0) \tag{149}$$

for $\kappa=|\mathbf{k}-\mathbf{k}_0| \ll k_0=|\mathbf{k}_0|$ can be written as

$$\omega(k) = \omega(|\mathbf{k}|) = \omega(k_0 + [|\mathbf{k}| - k_0]) \approx \omega(k_0) + v_g(|\mathbf{k}| - k_0) + \tfrac{1}{2}v_g'(|\mathbf{k}| - k_0)^2, \tag{150}$$

where

$$v_g = \left(\frac{\partial\omega}{\partial k}\right)_{k=k_0}, \quad v_g' = \left(\frac{\partial^2\omega}{\partial k^2}\right)_{k=k_0}, \tag{151}$$

and

$$|\mathbf{k}| = |\mathbf{k}_0 + \boldsymbol{\kappa}| = \sqrt{(\mathbf{k}_0 + \kappa_\parallel)^2 + \kappa_\perp^2} \approx k_0 + \kappa_\parallel + \frac{1}{2k_0}\kappa_\perp^2. \tag{152}$$

Substitution of (152) into (150) gives

$$\omega(k) \approx \omega_0 + v_g\kappa_\parallel + \frac{v_g}{2k_0}\kappa_\perp^2 + \frac{v_g'}{2}\kappa_\parallel^2, \tag{153}$$

where $\omega_0=\omega(k_0)$. Expressions (143) and (153) in the dual space ("momentum representation") correspond to the following equation for $u=u(\mathbf{r},t)$ in the coordinate space:

$$i\frac{\partial u}{\partial t} = \omega_0 u - iv_g \frac{\partial u}{\partial x} - \frac{v_g}{2k_0}\Delta_\perp u - \frac{v_g'}{2}\Delta_\parallel u + F(u) \quad (154)$$

with respect to the field $u=u(t,x,y,z)$, where x is along $\mathbf{k}_0$, and we use the operator correspondence between the dual space and usual space-time:

$$\omega(k) \leftrightarrow i\frac{\partial}{\partial t}, \quad \kappa_\parallel \leftrightarrow -i\frac{\partial}{\partial x},$$
$$(\boldsymbol{\kappa}_\perp)^2 \leftrightarrow -\Delta_\perp = -\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}, \quad (\boldsymbol{\kappa}_\parallel)^2 \leftrightarrow -\Delta_\parallel = -\frac{\partial^2}{\partial x^2}. \quad (155)$$

Equation (154) is known as the nonlinear parabolic equation.[49–52] The change of variables from (t,x,y,z) to $(t,x-v_g t,y,z)$ gives

$$-i\frac{\partial u}{\partial t} = \frac{v_g}{2k_0}\Delta_\perp u + \frac{v_g'}{2}\Delta_\parallel u - \omega_0 u - F(u), \quad (156)$$

which is also known as the nonlinear Schrödinger equation.

Wave propagation in oscillatory medium with long-range interaction of oscillators can be easily generalized by rewriting the dispersion law (153), in the following way:

$$\omega(k) = \omega_0 + v_g \kappa_\parallel + G_\alpha(\boldsymbol{\kappa}_\perp^2)^{\alpha/2} + G_\beta(\boldsymbol{\kappa}_\parallel^2)^{\beta/2} \quad (1 < \alpha,\beta < 2) \quad (157)$$

with new finite constants G_α and G_β.

Using the connection between Riesz fractional derivative and its Fourier transform[1]

$$(-\Delta_\perp)^{\alpha/2} \leftrightarrow (\boldsymbol{\kappa}_\perp^2)^{\alpha/2} \quad (-\Delta_\parallel)^{\beta/2} \leftrightarrow (\boldsymbol{\kappa}_\parallel^2)^{\beta/2}, \quad (158)$$

we obtain from (157)

$$i\frac{\partial u}{\partial t} = -iv_g\frac{\partial u}{\partial x} + G_\alpha(-\Delta_\perp)^{\alpha/2}u + G_\beta(-\Delta_\parallel)^{\beta/2}u + \omega_0 u + F(u), \quad (159)$$

where $u=u(t,x,y,z)$. By changing the variables from (t,x,y,z) to (t,ξ,y,z), $\xi=x-v_g t$, and using

$$(-\Delta_\parallel)^{\beta/2} = \frac{\partial^\beta}{\partial|x|^\beta} = \frac{\partial^\beta}{\partial|\xi|^\beta}, \quad (160)$$

we obtain from (159)

$$i\frac{\partial u}{\partial t} = G_\alpha(-\Delta_\perp)^{\alpha/2}u + G_\beta(-\Delta_\parallel)^{\beta/2}u + \omega_0 u + F(u), \quad (161)$$

which can be called the fractional nonlinear parabolic equation. For $G_\beta=0$ and $F(u)=b|u|^2u$, we get the fractional Ginzburg-Landau equation.[40–44]

We may consider one-dimensional simplifications of Eq. (161), i.e.,

$$i\frac{\partial u}{\partial t} = G_\beta\frac{\partial^\beta u}{\partial|\xi|^\beta} + \omega_0 u + F(u), \quad (162)$$

where $u=u(t,\xi)$, $\xi=x-v_g t$, or

$$i\frac{\partial u}{\partial t} = G_\alpha\frac{\partial^\alpha u}{\partial|z|^\alpha} + \omega_0 u + F(u), \quad (163)$$

where $u=u(t,z)$.

Let us comment on the physical structure of (161). The first and second terms on the right-hand side are related to wave propagation in oscillatory medium with long-range interaction of oscillators. The term with $F(u)$ on the right-hand side of Eqs. (159) and (161) correspond to wave interaction due to the nonlinear properties of the media. Thus, Eq. (161) can describe fractal processes of self-focusing and related issues.

XI. CONCLUSION

One-dimensional system of long-range interacting oscillators serves as a model for numerous applications in physics, chemistry, biology, etc. Long-range interactions are important types of interactions for complex media. An interesting situation arises when we consider the wide class of α-interactions, where α is noninteger. A remarkable feature of these interactions is the existence of a transform operation that replaces the set of coupled individual oscillator equations by the continuous medium equation with the space derivative of noninteger order α. Such transform operation is an approximation that appears in the continuous limit. This limit allows us to consider different models in a unified way by applying tools of fractional calculus.[53,54]

Periodic space-localized oscillations, which arise in discrete systems, have been widely studied for short-range interactions. In the paper, the systems with long-range interactions were considered. The method to map the discrete equations of motion into the continuous fractional order differential equation is developed by the transform operation. It is known that the properties of a system with long-range interaction are very different from short-range one. The method of fractional calculus can be a new tool for the analysis of different lattice systems.

APPENDIX: DIVERGENCE OF NONINVARIANT INTERACTION TERM

Noninvariant interaction term leads to the infinity in the continuous medium equation. To demonstrate this property, we prove the following proposition.

Proposition 8. The α-interaction term

$$g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m) u_m, \tag{A1}$$

where $J(n,m)=|n-m|^{-(\alpha+1)}$ is not translation-invariant. The transform operation $\hat{T}$ of the term (A1) leads to the divergence of order $|\Delta x|^{-\alpha}$ in the continuous medium equations.

Let us prove this proposition for $0<\alpha<2$ $(\alpha\neq 1)$, and the following equations of motion

$$\frac{\partial^2 u_n}{\partial t^2} + g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m) u_m - F(u_n) = 0. \tag{A2}$$

Since

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m) = \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} |n-m|^{-(\alpha+1)} \neq 0,$$

then the interparticle interaction term in (A2) is noninvariant with respect to translations. To derive the equation for $\hat{u}(k,t)$, we multiply Eq. (A2) by $\exp(-ikn\Delta x)$, and summing over n. Then, we obtain

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + g\hat{J}_\alpha(k\Delta x)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{A3}$$

where $\hat{J}_\alpha(k)$ is defined by (102). Using (105), we present Eq. (A3) in the form

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + gA_\alpha|\Delta x|^\alpha|k|^\alpha\hat{u}(k,t) + 2g\zeta(\alpha+1)\hat{u}(k,t) + 2g\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{A4}$$

where ζ is the Riemann zeta-function and A_α is defined by (106). For the limit $\Delta x \to 0$ and $0 < \alpha < 2$ $(\alpha \neq 1)$, Eq. (A4) can be written as

$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) + G_\alpha A_\alpha|k|^\alpha\hat{u}(k,t) + 2g\zeta(\alpha+1)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{A5}$$

where $0 < \alpha < 2$, $\alpha \neq 1$, and $G_\alpha = g|\Delta x|^\alpha$ is a finite parameter. Note that $g \to \infty$ for $\Delta x \to 0$, if G_α is a finite. Therefore, the transition to the limit $\Delta x \to 0$ in Eq. (A5) gives the divergence term

$$\lim_{\Delta x \to 0} g\zeta(\alpha+1)\hat{u}(k,t) = \zeta(\alpha+1)G_\alpha\tilde{u}(k,t)\lim_{\Delta x\to 0}|\Delta x|^{-\alpha} \to \infty. \tag{A6}$$

To have the continuous model equations without divergences, we must consider $[u_m(t) - u_n(t)]$ instead of $u_m(t)$ in the interaction terms (A1).

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