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Weyl quantization of fractional derivatives

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The quantum analogs of the derivatives with respect to coordinates q_k and momenta p_k are commutators with operators P_k and Q_k. We consider quantum analogs of fractional Riemann–Liouville and Liouville derivatives. To obtain the quantum analogs of fractional Riemann–Liouville derivatives, which are defined on a finite interval of the real axis, we use a representation of these derivatives for analytic functions. To define a quantum analog of the fractional Liouville derivative, which is defined on the real axis, we can use the representation of the Weyl quantization by the Fourier transformation. © *2008 American Institute of Physics.*
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I. INTRODUCTION

It is well known that the derivatives with respect to coordinates q_k and momenta p_k can be represented as Poisson brackets by the equations

$$\frac{\partial}{\partial q_k}A(q,p) = -\{p_k, A(q,p)\},$$

$$\frac{\partial}{\partial p_k}A(q,p) = \{q_k, A(q,p)\}$$

for continuously differentiable functions $A(q,p) \in C^1(\mathbb{R}^{2n})$. Quantum analogs of these Poisson brackets are self-adjoint commutators. The Weyl quantization π gives

$$\pi(\{q_k, A(q,p)\}) = \frac{1}{i\hbar}[\pi(q_k), \pi(A)\}],$$

$$\pi(\{p_k, A(q,p)\}) = \frac{1}{i\hbar}[\pi(p_k), \pi(A)\}],$$

where $[A,B]=AB-BA$. As a result, we have that

$$L^-_{P_k} = -(1/i\hbar)[\pi(q_k),\cdot], \quad -L^-_{Q_k} = -(1/i\hbar)[\pi(q_k),\cdot] \tag{1}$$

can be considered as quantum analogs of derivatives $\partial/\partial q_k$ and $\partial/\partial p_k$. Then a quantum analog of a derivative of integer order n can be defined by the products of $L^-_{P_k}$ and $L^-_{Q_k}$. For example, a quantum analog of $\partial^2/\partial q_k \partial p_l$ has the form $-L^-_{P_k}L^-_{Q_l} = -(1/\hbar)^2[P_k,[Q_l,\cdot]]$. An analog of $\partial^n/\partial q_k^n$ is $(-1)^n(L^-_{P_k})^n$.

The theory of derivatives of noninteger order goes back to Leibniz, Liouville, Grunwald, Letnikov, and Riemann,[1–5] and the theory has found many applications in recent studies in physics (see, for example, Refs. 6–11 and references therein). The fractional derivative has different

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definitions,[1,3] and exploiting any of them depends on the kind of problems, initial (boundary) conditions, and specifics of the considered physical processes. The classical definition are the so-called Riemann–Liouville and Liouville derivative.[1] These fractional derivatives are defined by the same equations on a finite interval of $\mathbb{R}$ and of the real axis correspondently. Note that the Caputo and Riesz derivatives can be represented[1,3] through the Riemann–Liouville and Liouville derivatives. Therefore quantum analogs of the fractional Riemann–Liouville and Liouville derivatives allow us to derive quantum analogs for the Caputo and Riesz derivatives.

The quantum analogs of the derivatives $\partial/\partial q_k$ and $\partial/\partial p_k$ are commutators (1). What are quantum analogs of fractional Riemann–Liouville and Liouville derivatives? To obtain the quantum analogs of fractional Riemann–Liouville derivatives, which are defined on a finite interval of $\mathbb{R}$, we can use a representation of these derivatives for analytic functions. In this representation the Riemann–Liouville derivative is a series of derivatives of integer order. It allows us to use the correspondence between the integer derivatives and the self-adjoint commutators. To define a quantum analog of the fractional Liouville derivative, which is defined on the real axis $\mathbb{R}$, we can use the representation of the Weyl quantization by the Fourier transformation.

II. QUANTIZATION OF FRACTIONAL RIEMANN–LIOUVILLE DERIVATIVE

The fractional derivative ${}_aD_x^\alpha$ on $[a,b]$ in the Riemann–Liouville form is defined by the equation

$$ {}_0D_x^\alpha A(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{A(y)dy}{(x-y)^{\alpha-m+1}}, $$

where m is the first whole number greater than or equal to α. To simplify our equation, we use $a=0$. The derivative of powers n of x is

$$ {}_0D_x^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, \tag{2} $$

where $n \geq 1$ and $\alpha \geq 0$. Here $\Gamma(z)$ is a gamma function.

Let $A(x)$ be an analytic function for $x \in (0,b)$. The fractional Riemann–Liouville derivative on the interval $[0,b]$ can be presented (see Lemma 15.3 in Ref. 3) in the form

$$ {}_0D_x^\alpha A(x) = \sum_{n=0}^{\infty} a(n,\alpha) x^{n-\alpha} \frac{d^n A(x)}{dx^n}, $$

where

$$ a(n,\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)\Gamma(n-\alpha+1)}. $$

If $A(q,p)$ is an analytic function on $M \subset \mathbb{R}^{2n}$, then we can define the fractional derivatives

$$ {}_0D_{q_k}^\alpha A(q,p) = \sum_{n=0}^{\infty} a(n,\alpha) q_k^{n-\alpha} \frac{\partial^n}{\partial q_k^n} A(q,p), \tag{3} $$

$$ {}_0D_{p_k}^\alpha A(q,p) = \sum_{n=0}^{\infty} a(n,\alpha) p_k^{n-\alpha} \frac{\partial^n}{\partial p_k^n} A(q,p), \tag{4} $$

where $k=1,\ldots,N$. Using the operators

$$ L_{q_k}^+ A(q,p) = q_k A(q,p), \quad L_{p_k}^+ A(q,p) = p_k A(q,p), $$

$$L^{-}_{q_k}A(q,p) = \frac{\partial A(q,p)}{\partial p_k}, \quad L^{-}_{p_k}A(q,p) = -\frac{\partial A(q,p)}{\partial q_k},$$

Eqs. (3) and (4) can be rewritten in the forms

$${}_0D^{\alpha}_{q_k}A(q,p) = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{q_k})^{n-\alpha}(-L^{-}_{p_k})^{n}A(q,p),$$

$${}_0D^{\alpha}_{p_k}A(q,p) = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{p_k})^{n-\alpha}(L^{-}_{q_k})^{n}A(q,p).$$

As a result, the fractional derivatives are defined by

$${}_0D^{\alpha}_{q_k} = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{q_k})^{n-\alpha}(-L^{-}_{p_k})^{n}, \tag{5}$$

$${}_0D^{\alpha}_{q_k} = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{p_k})^{n-\alpha}(L^{-}_{q_k})^{n}. \tag{6}$$

The Weyl quantization π of q_k and p_k gives the operators

$$Q_k = \pi(q_k), \quad P_k = \pi(p_k).$$

The Weyl quantization of the operators $L^{\pm}_{q^k}$ and $L^{\pm}_{p^k}$ is defined[12,13] by the equation

$$\pi_W(L^{+}_{q_k}) = L^{+}_{Q_k}, \quad \pi_W(L^{-}_{q_k}) = L^{-}_{Q_k},$$

$$\pi_W(L^{+}_{p_k}) = L^{+}_{P_k}, \quad \pi_W(L^{-}_{p_k}) = L^{-}_{P_k},$$

where

$$L^{+}_{Q}A = \frac{1}{2}(QA + AQ), \quad L^{+}_{P}A = \frac{1}{2}(PA + AP), \tag{7}$$

$$L^{-}_{Q}A = \frac{1}{i\hbar}(QA - AQ), \quad L^{-}_{P}A = \frac{1}{i\hbar}(PA - AP). \tag{8}$$

As a result, the quantization of the fractional derivatives (5) and (6) gives the superoperators

$${}_0\mathcal{D}^{\alpha}_{Q_k} = \pi({}_0D^{\alpha}_{q_k}) = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{Q_k})^{n-\alpha}(-L^{-}_{P_k})^{n}, \tag{9}$$

$${}_0\mathcal{D}^{\alpha}_{P_k} = \pi({}_0D^{\alpha}_{p_k}) = \sum_{n=0}^{\infty} a(n,\alpha)(L^{+}_{P_k})^{n-\alpha}(L^{-}_{Q_k})^{n}. \tag{10}$$

Note that a superoperator is a rule that assigns to each operator exactly one operator (see, for example, Ref. 13). Equations (9) and (10) can be considered as definitions of the fractional derivation superoperators on an operator space (for example, on rigged operator Hilbert space[13]).

It is not hard to prove that

$${}_0\mathcal{D}^{\alpha}_{Q} Q^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} Q^{n-\alpha}, \quad {}_0\mathcal{D}^{\alpha}_{P} P^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} P^{n-\alpha},$$

where $n \geq 1$ and $\alpha \geq 0$.

III. QUANTIZATION OF FRACTIONAL LIOUVILLE DERIVATIVE

We are reminded of the formula for the Fourier transform $\tilde{A}(a)$ of some function $A(x)$:

$$\tilde{A}(a) = \mathcal{F}\{A(x)\} = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dx A(x) \exp\{-iax\},$$

which is valid for all $A(x)$ with

$$\int_{\mathbb{R}} dx |A(x)| < \infty.$$

If we require

$$\|A(x)\|_2 = \int_{\mathbb{R}} dx |A(x)|^2 < \infty,$$

then the Parseval formula $\|\tilde{A}\|_2 = \|A\|_2$ holds.

Let $\mathcal{F}$ be an extension of this Fourier transformation to a unitary isomorphism on $L_2(\mathbb{R})$. We define the operators

$$\mathcal{L} = \mathcal{F}^{-1} L(a) \mathcal{F}.$$

It is well defined if the function $L(a)$ is measurable. These operators form a commutative algebra. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be operators associated with the functions $L_1(a)$ and $L_2(a)$. If $\mathcal{L}_{12}$ is an operator associated with $L_{12}(a) = L_1(a) L_2(a)$, then

$$\mathcal{L}_{12} = \mathcal{L}_1 \mathcal{L}_2 = \mathcal{L}_2 \mathcal{L}_1.$$

As a result, we may present explicit formulas for a fractional derivative. If the Fourier transforms exist, then the operator D^{α}_{x} is

$$D^{\alpha}_{x} A(x) = \mathcal{F}^{-1} (ia)^{\alpha} \tilde{A}(a) = \mathcal{F}^{-1} (ia)^{\alpha} \mathcal{F} A(x),$$

where

$$(ia)^{\alpha} = |a|^{\alpha} \exp\left(\frac{\pi\alpha}{2} \operatorname{sgn}(a) \right).$$

In this paper, we use the following assumption, which is usually used.[3,14] The branch of z^{α} is so taken that $\mathrm{Re}(z^{\alpha}) > 0$ for $\mathrm{Re}(z) > 0$. This branch is a one-valued function in the z-plane cut along the negative real axis.

For $A(x) \in L_2(\mathbb{R})$, we have the integral representation

$$D^{\alpha}_{x} A(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} da dx' (ia)^{\alpha} A(x') \exp\{ia(x-x')\}. \tag{11}$$

Some elementary manipulations lead to the well-known Riemann–Liouville integral representation

$$D_x^\alpha A(x) = {}_{-\infty}D_x^\alpha A(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{-\infty}^{x} dx' \frac{A(x')}{(x-x')^{\alpha+1-m}}.$$

This form cannot be used to quantization. For the Weyl quantization, we consider representation (11).

Let $A(q,p)$ be a function of $L_2(\mathbb{R}^2)$ on the phase space $\mathbb{R}^2$. Then Eq. (11) can be presented in the form

$$D_q^\alpha D_p^\beta A(q,p) = \int_{\mathbb{R}^4} \frac{dadbdq'dp'}{(2\pi\hbar)^2} (ia)^\alpha (ib)^\beta A(q',p') e^{(i/\hbar)(a(q-q')+b(p-p'))}. \tag{12}$$

Using the Weyl quantization[13,15] of $A(q,p)$ in the form

$$A(Q,P) = \pi(A(q,p)) = \int_{\mathbb{R}^4} \frac{dadbdqdp}{(2\pi\hbar)^2} A(q,p) e^{(i/\hbar)(a(Q-qI)+b(P-pI))},$$

we obtain the following result of the Weyl quantization of (12):

$$\begin{aligned} D_Q^\alpha D_P^\beta A(Q,P) &= \pi(D_q^\alpha D_p^\beta A(q,p)) \\ &= \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^4} dadbdqdp (ia)^\alpha (ib)^\beta A(q,p) \exp\frac{i}{\hbar}(a(Q-qI) \\ &\quad + b(P-pI)). \end{aligned} \tag{13}$$

This equation can be considered as a definition of D_Q^α and D_P^β on a set of quantum observables.

Note that the general quantization[13,16] of $A(q,p)$ is defined by

$$A_F(Q,P) = \int_{\mathbb{R}^4} \frac{dadbdqdp}{(2\pi\hbar)^2} F(a,b) A(q,p) \exp\frac{i}{\hbar}(a(Q-qI)+b(P-pI)).$$

For the Weyl quantization, $F(a,b)=1$. If $F(a,b)=\cos(ab/2\hbar)$, then we have the Rivier quantization.[16] Equation (13) can be considered as a general quantization of $A(q,p)$ with the function

$$F(a,b) = (ia)^\alpha (ib)^\beta.$$

This function has zeros on the real a,b axis. It is clear that there is no dual operator basis.

IV. CONCLUSION

The quantum dynamics can be described by superoperators. A superoperator is a map that assigns to each operator exactly one operator. The natural description of the motion is in terms of the infinitesimal change in the system. In the equations of quantum systems the infinitesimal generators are defined by some derivation superoperators. A derivation is a linear map $\mathcal{D}$, which satisfies the Leibnitz rule $\mathcal{D}(AB)=(\mathcal{D}A)B+A(\mathcal{D}B)$ for all operators A and B. It is known that the superoperators $\mathcal{D}_P=(1/i\hbar)[Q,\cdot]$ and $\mathcal{D}_Q=(-1/i\hbar)[P,\cdot]$, which are used in equations of motion, are derivations of observables. For example, the quantum harmonic oscillator with the Hamiltonian $H=(1/2m)P^2+(m\omega^2/2)Q^2$ is described by the equation of motion $dA/dt=\mathcal{L}A$ with the infinitesimal generator $\mathcal{L}=-(1/m)L_P^+\mathcal{D}_Q+m\omega^2 L_Q^+\mathcal{D}_P$. We can consider fractional derivatives on a set of quantum observables as fractional powers $\mathcal{D}_Q^\alpha$ and $\mathcal{D}_P^\alpha$ of derivative superoperators $\mathcal{D}_Q$ and $\mathcal{D}_P$. Note that a fractional generalization of the Heisenberg equation is suggested in Ref. 17.

In this paper, a fractional generalization of the derivative superoperators on a set of quantum observables is suggested. A fractional power α of the superoperator $\mathcal{D}$ can be considered as a parameter to describe a measure of "screening" of environment. There exist the following special cases of the measure: (1) the absence of the environmental influence ($\alpha=0$), (2) the complete environmental influence ($\alpha=1$), and (3) the powerlike environmental influence ($0<\alpha<1$). As a result, a physical interpretation of fractional powers of the derivative superoperators $\mathcal{D}_Q$ and $\mathcal{D}_P$ can be a one-parameter description of a screening of interaction with environment.

Using the Weyl quantization and the representation of fractional derivative for analytic functions (see Lemma 15.3 in Ref. [3]) quantum analogs of the Riemann–Liouville and Liouville derivatives can be obtained. The Caputo and Riesz derivatives can be represented[1,3] through the Riemann–Liouville and Liouville derivatives. Therefore quantum analogs of the fractional Riemann–Liouville and Liouville derivatives allow us to derive quantum analogs for the Caputo and Riesz derivatives. Note that a quantum analog of a fractional derivative can be considered as a fractional power of a self-adjoint commutator.[17] Quantum analogs of fractional derivatives can give us a notion that allows one to consider quantum processes that are described by fractional differential equations at classical level (see, for example, Ref. 18 and references therein).

Note that the fractional equations of motion describe an anomalous diffusion.[6,9,10,19] It is known that the quantum Markovian equations (the Lindblad equations) are used to describe Brownian motion of quantum systems.[20] We can assume that the fractional power of derivative superoperators in the Lindblad equation can be used to describe anomalous processes and continuous time random walks in quantum systems.

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