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Discrete map with memory from fractional differential equation of arbitrary positive order

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Derivatives of fractional order with respect to time describe long-term memory effects. Using nonlinear differential equation with Caputo fractional derivative of arbitrary order $\alpha > 0$, we obtain discrete maps with power-law memory. These maps are generalizations of well-known universal map. The memory in these maps means that their present state is determined by all past states with power-law forms of weights. Discrete map equations are obtained by using the equivalence of the Cauchy-type problem for fractional differential equation and the nonlinear Volterra integral equation of the second kind.
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I. INTRODUCTION

Discrete maps are used for the study of evolution problems, possibly as a substitute of differential equations.[1–3] They lead to a much simpler formalism, which is particularly useful in numerical simulations. The universal discrete map is one of the most widely studied maps. It is a very important step in understanding the qualitative behavior of a wide class of systems described by differential equations. The derivatives of noninteger orders[4–6] are a natural generalization of the ordinary differentiation of integer order. Fractional differentiation with respect to time is characterized by power-law memory effects. The discrete maps with memory are considered in Refs. 7–13. It is important to connect fractional differential equations and discrete maps with memory. In Ref. 13, we prove that the discrete maps with memory can be derived from differential equations with fractional derivatives. The fractional generalization of the universal map was derived[13] from a differential equation with Riemann–Liouville fractional derivatives. The Riemann–Liouville derivative has some notable disadvantages in physical applications such as the hypersingular improper integral, where the order of singularity is higher than the dimension, and nonzero of the fractional derivative of constants, which would entail that dissipation does not vanish for a system in equilibrium. The desire to formulate initial value problems for physical systems leads to the use of Caputo fractional derivatives rather than Riemann–Liouville fractional derivative. In this paper, we obtain a discrete map with memory from differential equations with Caputo fractional derivative of arbitrary order $\alpha > 0$. The universal map with power-law memory is obtained by using the equivalence of the fractional differential equation and the Volterra integral equation. We reduce the Cauchy-type problem for the differential equations with the Caputo fractional derivative to a nonlinear Volterra integral equation of the second kind. The equivalence of this Cauchy-type problem and the correspondent Volterra equation was proved by Kilbas and Marzan in Refs. 14 and 15.

In Sec. II, differential equations with integer derivative and universal maps without memory are considered to fix notations and provide convenient references. In Sec. III, fractional differential equations with Caputo derivative and correspondent discrete maps with memory are considered. A

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fractional generalization of the universal map is obtained from kicked differential equations with the Caputo fractional derivative of arbitrary order $\alpha>0$. Finally, a short conclusion is given in Sec. IV.

II. UNIVERSAL MAP WITHOUT MEMORY

In this section, differential equations with derivative of second order and the universal map without memory are considered to fix notations and provide convenient references.

Let us consider the equation of motion,

$$D_t^2 x(t) + KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) = 0, \tag{1}$$

in which perturbation is a periodic sequence of delta-function-type pulses (kicks) following with period $T=2\pi/\nu$, K is an amplitude of the pulses, $D_t^2 = d^2/dt^2$, and $G[x]$ is some real-valued function. It is well known that this differential equation can be represented in the form of the discrete map,

$$x_{n+1} - x_n = p_{n+1} T, \quad p_{n+1} - p_n = -KTG[x_n]. \tag{2}$$

Equations (2) are called the universal map. For details, see, for example, Refs. 1–3.

Traditional method of derivation of the universal map equations from the differential equations is considered in Sec. 5.1 of Ref. 2. We use another method of derivation of these equations to fix notations and provide convenient references. It is easy to obtain the universal map by using the equivalence of the differential equation and the Volterra integral equation. The Cauchy-type problem for the differential equations,

$$D_t^1 x(t) = p(t), \tag{3}$$

$$D_t^1 p(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right), \tag{4}$$

with the initial conditions

$$x(0) = x_0, \quad p(0) = p_0 \tag{5}$$

is equivalent to the universal map equations of the form

$$x_{n+1} = x_0 + p_0 (n+1) T - KT^2 \sum_{k=1}^{n} G[x_k](n+1-k), \tag{6}$$

$$p_{n+1} = p_0 - KT \sum_{k=1}^{n} G[x_k]. \tag{7}$$

To prove this statement we consider the nonlinear differential equation (1) on a finite interval $[0, t_f]$ of the real axis, with the initial conditions (5). The Cauchy-type problem of the form (1) and (5) is equivalent to the Volterra integral equation,

$$x(\tau) = x_0 + p_0 t - K \sum_{k=1}^{\infty} \int_0^t d\tau G[x(t)] \delta\left(\frac{\tau}{T} - k\right)(t-\tau). \tag{8}$$

For $nT < t < (n+1)T$, we obtain

$$x(t) = x_0 + p_0 t - KT\sum_{k=1}^{n} G[x(kT)](t-kT). \quad (9)$$

Equations (9) and (3) give

$$p(t) = p_0 - KT\sum_{k=1}^{n} G[x(kT)]. \quad (10)$$

The solution of the left side of the $(n+1)$th kick,

$$x_{n+1} = x(t_{n+1}-0) = \lim_{\varepsilon\to 0+} x(T(n+1)-\varepsilon), \quad (11)$$

$$p_{n+1} = p(t_{n+1}-0) = \lim_{\varepsilon\to 0+} p(T(n+1)-\varepsilon), \quad (12)$$

where $t_{n+1}=(n+1)T$ gives the map equations (6) and (7). This ends the proof. Note that Eqs. (6) and (7) can be rewritten in the form (2). Using Eqs. (6) and (7), the differences $x_{n+1}-x_n$ and $p_{n+1}-p_n$ give Eqs. (2) of the universal map.

We note that Eqs. (2) with $G[x]=-x$ give the Anosov-type system,

$$x_{n+1} - x_n = p_{n+1}T, \quad p_{n+1} - p_n = KTx_n. \quad (13)$$

If $G[x]=\sin(x)$, then Eqs. (2) are

$$x_{n+1} - x_n = p_{n+1}T, \quad p_{n+1} - p_n = - KT\sin(x_n). \quad (14)$$

This map is known as the standard or Chirikov map.[1]

III. FRACTIONAL EQUATION AND UNIVERSAL MAP WITH MEMORY

In Ref. 13 we consider nonlinear differential equations with Riemann–Liouville fractional derivatives. The discrete maps with memory are obtained from these equations. The Riemann–Liouville fractional derivative has some notable disadvantages in physical applications such as the hypersingular improper integral, where the order of singularity is higher than the dimension, and nonzero of the fractional derivative of constants, which would entail that dissipation does not vanish for a system in equilibrium. The desire to formulate initial value problems for physical systems leads to the use of Caputo fractional derivatives[4,5] rather than Riemann–Liouville fractional derivative.

The left-sided Caputo fractional derivative[4,16–18] of order $\alpha>0$ is defined by

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{d\tau D_\tau^m f(\tau)}{(t-\tau)^{\alpha-m+1}} = {}_0I_t^{m-\alpha} D_t^m f(t), \quad (15)$$

where $m-1<\alpha<m$ and ${}_0I_t^{\alpha}$ is the left-sided Riemann–Liouville fractional integral of order $\alpha>0$, that is, defined by

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}} \quad (t>0). \quad (16)$$

The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desire order of fractional derivative. The Riemann–Liouville fractional derivative ${}_0D_t^{\alpha}$ is computed in the reverse order. Integration by part of (15) gives

$$ {}_0D_t^{\alpha}x(t) = {}_0^CD_t^{\alpha}x(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} x^{(k)}(0). \tag{17} $$

The second term in Eq. (17) regularizes the Caputo fractional derivative to avoid the potentially divergence from singular integration at $t=0$. In addition, the Caputo fractional differentiation of a constant results in zero ${}_0^CD_t^{\alpha}C=0$. The Riemann–Liouville fractional derivative of a constant need not be zero.[4]

If the Caputo fractional derivative is used instead of the Riemann–Liouville fractional derivative, then the initial conditions for fractional dynamical systems are the same as those for the usual dynamical systems. The Caputo fractional derivatives can be more applicable to dynamical systems than the Riemann–Liouville derivatives. Note that the Caputo fractional derivatives can be used to formulate a self-consisted fractional vector calculus.[19]

We consider the nonlinear differential equation of order α, where $0 \leq m-1 < \alpha \leq m$,

$$ {}_0^CD_t^{\alpha}x(t) = G[t,x(t)], \quad (0 \leq t \leq t_f), \tag{18} $$

involving the Caputo fractional derivative ${}_0^CD_t^{\alpha}$ on a finite interval $[0,t_f]$ of the real axis, with the initial conditions

$$ (D_t^k x)(0) = c_k, \quad k = 0, \dots, m-1. \tag{19} $$

Kilbas and Marzan[14,15] proved the equivalence of the Cauchy-type problem of the form (18) and (19) and the Volterra integral equation of second kind,

$$ x(t) = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau G[t,x(t)](t-\tau)^{\alpha-1}, \tag{20} $$

in the space $C^{m-1}[0,t_f]$.

The basic theorem regarding the nonlinear differential equation involving the Caputo fractional derivative states that the Cauchy-type problem (18) and (19) and the nonlinear Volterra integral equation (20) are equivalent in the sense that, if $x(t) \in C[0,t_f]$ satisfies one of these relations, then it also satisfies the other. In Refs. 14 and 15 (see also Ref. 4, Theorem 3.24.) this theorem is proven by assuming that a function $G[t,x]$ for any $x \in W \subset \mathbb{R}$ belong to $C_{\gamma}(0,t_f)$ with $0 \leq \gamma < 1$, $\gamma < \alpha$. Here $C_{\gamma}(0,t_f)$ is the weighted space of functions $f[t]$ given on $(0,t_f]$, such that $t^{\gamma}f[t] \in C(0,t_f)$.

Let us consider a generalization of Eq. (1) in the form of the fractional differential equation,

$$ {}_0^CD_t^{\alpha}x(t) + KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) = 0 \quad (m-1 < \alpha < m), \tag{21} $$

where ${}_0^CD_t^{\alpha}$ is the Caputo fractional derivative, with the initial conditions

$$ D_t^s x(0) = x_0^{(s)} \quad (s = 0,1, \dots, m-1). \tag{22} $$

Using $x^{(s)}(t)=D_t^s x(t)$, $s=0,1,\dots,m-1$, Eq. (21) can be rewritten in the Hamilton form.

Theorem: *The Cauchy-type problem for the fractional differential equations,*

$$ D_t^1 x^{(s)}(t) = x^{(s+1)}(t) \quad (s = 0,1, \dots, m-2), \tag{23} $$

$$ {}_0^CD_t^{\alpha-m+1} x^{(m-1)}(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) \quad (m-1 < \alpha < m), \tag{24} $$

with the initial conditions

$$x^{(s)}(0) = x_0^{(s)} \quad (s = 0, 1, \ldots, m-1) \tag{25}$$

is equivalent to the discrete map equations,

$$x_{n+1}^{(s)} = \sum_{k=0}^{m-s-1} \frac{x_0^{(k+s)}}{k!}(n+1)^k T^k - \frac{KT^{\alpha-s}}{\Gamma(\alpha-s)} \sum_{k=1}^{n} (n+1-k)^{\alpha-1-s} G[x_k], \tag{26}$$

Proof: Using the Kilbas–Marzan result for Eq. (18) with the function

$$G[t,x(t)] = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right),$$

we obtain that the Cauchy-type problem (21) and (22) is equivalent to the Volterra integral equation of second kind,

$$x(t) = \sum_{k=0}^{m-1} \frac{x_0^{(k)}}{k!} t^k - \frac{K}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t d\tau (t-\tau)^{\alpha-1} G[x(\tau)] \delta\left(\frac{\tau}{T} - k\right), \tag{27}$$

in the space of continuously differentiable functions $x(t) \in C^{m-1}[0, t_f]$.

If $nT < t < (n+1)T$, then Eq. (27) gives

$$x(t) = \sum_{k=0}^{m-1} \frac{x_0^{(k)}}{k!} t^k - \frac{KT}{\Gamma(\alpha)} \sum_{k=1}^{m} (t-kT)^{\alpha-1} G[x(kT)]. \tag{28}$$

Using the variables (23), Eq. (28) gives

$$x^{(s)}(t) = \sum_{k=0}^{m-1-s} \frac{x_0^{(k+s)}}{k!} t^k - \frac{KT}{\Gamma(\alpha-s)} \sum_{k=1}^{n} (t-kT)^{\alpha-1-s} G[x(kT)], \tag{29}$$

where $s = 0, 1, \ldots, m-1$, $nT < t < (n+1)T$, $m-1 < \alpha < m$, and we use $\Gamma(z) = (z-1)\Gamma(z-1)$. The solution of the left side of the $(n+1)$th kick (11) and (12) can be represented by Eqs. (26), where we use the condition of continuity $x^s(t_n+0) = x^s(t_n-0)$, $s = 0, 1, \ldots, m-2$.

This ends the proof. ■

Equations (26) define a generalization of the universal map. This map is derived from a fractional differential equation with Caputo derivatives without any approximations. The main property of the suggested map is a long-term memory that means that their present state depends on all past states with a power-law form of weights.

If $G[x] = \sin(x)$, then Eqs. (26) define a generalization of standard map. For $G[x] = -x$, we have Anosov-type system with memory.

In the case of $1 < \alpha < 2$, $m = 2$, we have the following universal map with memory:

$$x_{n+1} = x_0 + p_0(n+1)T - \frac{KT^{\alpha}}{\Gamma(\alpha)} \sum_{k=1}^{n} (n+1-k)^{\alpha-1} G[x_k], \tag{30}$$

$$p_{n+1} = p_0 - \frac{KT^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=1}^{n} (n+1-k)^{\alpha-2} G[x_k], \tag{31}$$

where $x_n = x_n^{(0)}$ and $p_n = x_n^{(1)}$. If $\alpha = m = 2$, then Eqs. (26) give the universal map of the form (6) and (7) that is equivalent to Eqs. (2). As a result, the usual universal map is a special case of this universal map with memory.

IV. CONCLUSION

Equations for discrete maps with memory are suggested. The maps with power-law memory describe fractional dynamics of complex physical systems. The suggested map with memory is generalizations of well-known universal map. These maps are equivalent to the correspondent fractional kicked differential equations. To derive the map equations an approximation for fractional derivatives is not used. We obtain a discrete map with memory from fractional differential equation by using the equivalence of the Cauchy-type problem and the nonlinear Volterra integral equation of the second kind.

Fractional differentiation with respect to time is characterized by power-law memory effects that correspond to intrinsic dissipative processes in the physical systems. Therefore, the universal maps with memory have regular and strange attractors for some values of parameters K and α. The suggested universal maps with memory demonstrate a chaotic behavior with a new type of attractors. Numerical simulations of the universal map with memory prove that the nonlinear dynamical systems, which are described by the equations with fractional derivatives, exhibit a new type of chaotic motion. For some regions of parameters K and α these universal maps with memory demonstrate a new type of regular and strange attractors. The universal maps with power-law memory can be used to describe properties of regular and strange attractors of the fractional differential equations with kicks.

[1] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[2] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics: From the Pendulum to Turbulence and Chaos* (Harwood Academic, New York, 1988).
[3] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).
[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
[5] I. Podlubny, *Fractional Differential Equations* (Academic, San Diego, 1999).
[6] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications* (Gordon and Breach, New York, 1993).
[7] A. Fulinski and A. S. Kleczkowski, Phys. Scr. **35**, 119 (1987).
[8] E. Fick, M. Fick, and G. Hausmann, Phys. Rev. A **44**, 2469 (1991).
[9] K. Hartwich and E. Fick, Phys. Lett. A **177**, 305 (1993).
[10] M. Giona, Nonlinearity **4**, 911 (1991).
[11] J. A. C. Gallas, Physica A **198**, 339 (1993).
[12] A. A. Stanislavsky, Chaos **16**, 043105 (2006).
[13] V. E. Tarasov and G. M. Zaslavsky, J. Phys. A **41**, 435101 (2008).
[14] A. A. Kilbas and S. A. Marzan, Dokl. Akad. Nauk **399**, 7 (2004).
[15] A. A. Kilbas and S. A. Marzan, Differentsial'nye Uravneniya **41**, 82 (2005).
[16] M. Caputo, Geophys. J. R. Astron. Soc. **13**, 529 (1967).
[17] M. Caputo and F. Mainardi, Pure Appl. Geophys. **91**, 134 (1971).
[18] *Fractals and Fractional Calculus in Continuum Mechanics*, edited by A. Carpinteri and F. Mainardi (Springer, Wien, 1997).
[19] V. E. Tarasov, Ann. Phys. **323**, 2756 (2008).