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Uncertainty relation for non-Hamiltonian quantum systems

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General forms of uncertainty relations for quantum observables of non-Hamiltonian quantum systems are considered. Special cases of uncertainty relations are discussed. The uncertainty relations for non-Hamiltonian quantum systems are considered in the Schrödinger-Robertson form since it allows us to take into account Lie-Jordan algebra of quantum observables. In uncertainty relations, the time dependence of quantum observables and the properties of this dependence are discussed. We take into account that a time evolution of observables of a non-Hamiltonian quantum system is not an endomorphism with respect to Lie, Jordan, and associative multiplications.
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I. INTRODUCTION

The uncertainty relation states a fundamental limit on the standard deviation values of quantum observables, such as position and momentum. The uncertainty relation is a basic inequality of quantum mechanics. It was introduced by Heisenberg[1] for the coordinate Q and momentum P in the from of an approximate relation $\Delta Q \Delta P \sim \hbar$, where $\hbar$ is the Planck constant. This relation for operators Q and P in the form of inequality was rigorously proved by Kennard,[2]

$$\Delta Q\ \Delta P \geq \frac{\hbar}{2}, \tag{1}$$

where ΔQ and ΔP are the standard deviations of the coordinate Q and momentum P, which are defined by

$$\Delta A = \sqrt{\langle (A - \langle A \rangle\, I)^2 \rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}.$$

Inequality (1) is called the Heisenberg's uncertainty relation. Robertson[3] extended this inequality to arbitrary pair of quantum observables X and Y,

$$\Delta X\ \Delta Y \geq \frac{1}{2} |\, \langle XY - YX \rangle\, |. \tag{2}$$

Schrödinger[4] and Robertson[5] prove the following more strong inequality, which is a generalization of the Heisenberg-type uncertainty relation (2) for two quantum observables

$$(\Delta X)^2\ (\Delta Y)^2 \geq \frac{1}{4}\Big(\langle XY - YX \rangle^2 + \langle XY + YX \rangle^2 \Big). \tag{3}$$

Generalized Heisenberg-type and Schrödinger-Robertson-type uncertainty relations are obtained for two arbitrary operators both in the case of pure and of mixed states by several authors.[6–10] Note that uncertainty relations for open quantum systems are considered by Ingarden,[11] Sandulescu and Scutaru[12,13] for an example of quantum harmonic oscillator with linear fraction. Note that general properties of uncertainty relations for non-Hamiltonian quantum systems are not described at present time (see Chap. 19 of Ref. 14). It is connected with the fact that dynamics of quantum observables of non-Hamiltonian systems is not an endomorphism of Lie, Jordan, and associative

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operator algebras.[14] In non-Hamiltonian dynamics, the time evolution cannot be considered as an endomorphism of algebraic structures on a set of quantum observables.

II. MOTIVATION IN THE FORM OF AN EXAMPLE OF NON-HAMILTONIAN QUANTUM SYSTEM

In this section, we give some motivation in the form of an example of a relevant non-Hamiltonian quantum system before launching into the details of the derivation of uncertainty relations for quantum observables of such type systems. As the example we consider a damped linear quantum oscillator (see Refs. 12 and 20 and Sec. 15.10 in Ref. 14). The same consideration can be realized for quantum systems that are described in Refs. 21–23. The basic assumption is that the general form of equations for observables of quantum non-Hamiltonian systems given by the Lindblad equation.[18] Another simple condition imposed to the operators H, V_k is that they are functions of the basic operators of the one-dimensional quantum system Q and P of such kind that the obtained model is exactly solvable.[12,20] This condition implies that $V_k = V_k(Q, P)$ are at most the first degree polynomials in Q and P, and $H = H(Q, P)$ is at most a second degree polynomial in Q and P. These assumptions are of the same kind as those made in classical dynamics when one takes the friction force proportional to the velocity.

The equation for quantum observables Q and P of this model are

$$\frac{dQ}{dt} = \frac{1}{m}P + \mu Q - \lambda Q, \quad \frac{dP}{dt} = -m\omega^2 Q - \mu P - \lambda P. \tag{4}$$

The solution of these equations has the form (for details, see Sec. 15.10 in Ref. 14),

$$Q_t = e^{-\lambda t}\Big(\cosh(\nu t) + \frac{\mu}{\nu}\sinh(\nu t)\Big)Q + \frac{1}{m\nu}e^{-\lambda t}\sinh(\nu t)P,$$

$$P_t = -\frac{m\omega^2}{\nu}e^{-\lambda t}\sinh(\nu t)Q + e^{-\lambda t}\Big(\cosh(\nu t) - \frac{\mu}{\nu}\sinh(\nu t)\Big)P. \tag{5}$$

Here ν is a complex parameter such that $\nu^2 = \mu^2 - \omega^2$.

Let us consider commutator $[Q_t, P_t]$ for the operators Q_t and P_t. Using Eq. (5), we get

$$[Q_t, P_t] = e^{-2\lambda t}\Big(\cosh^2(\nu t) - \frac{\mu^2}{\nu^2}\sinh^2(\nu t) + \frac{\omega^2}{\nu^2}\sinh^2(\nu t)\Big)[Q, P] = e^{-\lambda t}[Q, P].$$

As a result, we obtain

$$[Q_t, P_t] = e^{-2\lambda t}[Q, P] = i\hbar e^{-2\lambda t} I,$$

and the uncertainty relation of the Heisenberg type (1) has the form

$$\Delta Q_t\, \Delta P_t \geq \frac{\hbar}{2} e^{-2\lambda t}, \tag{6}$$

where $\Delta A_t = \sqrt{\langle (A_t)^2\rangle - \langle A_t\rangle^2}$. We see that for $t \to \infty$ the left hand side of the uncertainty relation vanishes.

However Eq. (4) are non-Hamiltonian, and then the time evolution of operators is non-unitary. Therefore, algebraic relations between operators are not preserved in general. It is connected with the fact that dynamics of non-Hamiltonian quantum systems is not an endomorphism of operator algebras of quantum observables.[14] In general, the time evolution of non-Hamiltonian systems cannot be considered as an endomorphism of algebraic structures on a set of quantum observables such as Lie and Jordan algebras.

III. PROPERTIES OF NON-HAMILTONIAN TIME EVOLUTION

On the set of quantum observables we can define mathematical structures such as linear operator space, Lie-Jordan algebra, associative operator algebra. Let us give the definitions of the Lie, Jordan, and Lie-Jordan algebras.[14]

A Lie algebra is a linear algebra $\mathcal{M}$ over some field F such that the multiplicative binary operation $*$ satisfies the following axioms:

The skew-symmetry condition

$$A * B = -B * A,$$

the Jacobi identity

$$((A * B) * C) + ((B * C) * A) + ((C * A) * B) = 0,$$

and the bilinear condition

$$(aA + bB) * C = a(A * C) + b(B * C)$$

for all $A, B, C \in \mathcal{M}$ and $a, b \in F$.

A Jordan algebra is a linear algebra $\mathcal{M}$ over some field F such that the multiplicative binary operation $\circ$ satisfies the following axioms:

The symmetry condition

$$(A \circ B) = (B \circ A),$$

the Jordan identity

$$(((A \circ A) \circ B) \circ A) - ((A \circ A) \circ (B \circ A)) = 0,$$

and the bilinear condition

$$(aA + bB) \circ C = a(A \circ C) + b(B \circ C)$$

for all $A, B, C \in \mathcal{M}$ and $a, b \in F$.

A Lie-Jordan algebra $\langle \mathcal{M}, *, \circ, \hbar \rangle$ is a linear space $\mathcal{M}$ together with two bilinear multiplicative operations $*$ and $\circ$, such that the following conditions are satisfied:

1. $\langle \mathcal{M}, * \rangle$ is a Lie algebra.
2. $\langle \mathcal{M}, \circ \rangle$ is a Jordan algebra.
3. The operations $\langle *, \circ \rangle$ are connected by the Leibnitz rule,

$$A * (B \circ C) = (A * B) \circ C + B \circ (A * C).$$

4. The associators of the operations $\langle *, \circ \rangle$ are connected by the equation

$$(A \circ B) \circ C - A \circ (B \circ C) = \frac{\hbar^2}{4}\Big((A * B) * C - A * (B * C)\Big), \qquad (7)$$

where $\hbar$ is a positive real number.

If $\hbar = 0$, then $\langle \mathcal{M}, \circ \rangle$ is an associative Jordan algebra and we have the algebra of classical observables, where the operation $\circ$ can be represented by multiplication of functions.

For wide class of operator algebras, we can define[14] the operations $\langle *, \circ \rangle$ by

$$A * B = \frac{1}{i\hbar}(AB - BA), \quad A \circ B = \frac{1}{2}(AB + BA). \qquad (8)$$

The representation of operations $\langle *, \circ \rangle$ by (8) are defined by the following relationship between the associative algebra and Lie, Jordan algebras. The replacement of the operation of multiplication AB in an associative algebra M by the operation of commutation $[A, B] = AB - BA$, makes it into a Lie algebra $M^{(-)}$. If M is an algebra over a commutative field F, then $M^{(-)}$ is also an algebra over the same field. For every Lie algebra L over an arbitrary commutative field F there exists an associative M over the same field such that L can be isomorphically embedded in the algebra $M^{(-)}$. This is the

Poincare-Birkhoff-Witt theorem (see, for example, Ref. 24). The replacement of the operation of multiplication AB in an associative algebra M by the operation $[A, B]_+ = (1/2)(AB + BA)$, makes it into a Jordan algebra $M^{(+)}$. If M is an algebra over a commutative field F, then $M^{(+)}$ is also an algebra over the same field. The algebra $M^{(+)}$ is called the special algebra. The Jordan algebras that are not special are called exceptional. The exceptional Jordan algebras are not considered in this paper.

The imaginary unit "i" is used in order to Lie multiplications of self-adjoint operators $A * B$ be self-adjoint. The existence of the parameter $\hbar$ in relation (7) it allows us to use the well-known form of the Weyl quantization (see Sec. 5.1. in Ref. 19 and Chap. 17.2 in Ref. 14), and to consider the classical limit.

The Lie-Jordan algebra is an algebraic structure that gives a uniform description of classical and quantum systems. The case $\hbar = 0$ corresponds to transition from a nonassociative Jordan algebra into associative. Common algebraic properties of classical and quantum systems are not depend on relation (7). Note that uncertainty relations for quantum non-Hamiltonian systems should be considered in the Schrödinger-Robertson form since it allows us to take into account Lie-Jordan algebra of quantum observables.

Let us consider a time evolution of quantum observables of non-Hamiltonian quantum systems. It can be described by the Heisenberg equation

$$\frac{dA_t}{dt} = \mathcal{L}(A_t),$$

where $\mathcal{L}$ is an infinitesimal generator of the quantum dynamical map. If we consider a Cauchy problem for this equation in which the initial condition is given by A at the time $t = 0$, then its solution can be written in the form $A_t = \Phi_t(A)$.

The quantum system is Hamiltonian if $\mathcal{L}$ is a differentiation on operator algebras with respect to Lie, Jordan, and associative multiplication operations, i.e., the conditions

$$J_{\mathcal{L}}(A, B) = \mathcal{L}(A * B) - \mathcal{L}(A) * B - A * \mathcal{L}(B) = 0, \tag{9}$$

$$K_{\mathcal{L}}(A, B) = \mathcal{L}(A \circ B) - \mathcal{L}(A) \circ B - A \circ \mathcal{L}(B) = 0, \tag{10}$$

$$Z_{\mathcal{L}}(A, B) = \mathcal{L}(AB) - \mathcal{L}(A)B - A\mathcal{L}(B) = 0 \tag{11}$$

hold for all $A, B \in D(\mathcal{L}) \subset \mathcal{M}$. In the case (8), we have

$$Z_{\mathcal{L}}(A, B) = K_{\mathcal{L}}(A, B) + \frac{i\hbar}{2} J_{\mathcal{L}}(A, B).$$

and condition (11) is equivalent to (9) and (10). A quantum system is called a non-Hamiltonian system, if there exist observables A and B, such that the inequality

$$J_{\mathcal{L}}(A, B) \neq 0 \tag{12}$$

is valid. As a result, we have

$$Z_{\mathcal{L}}(A, B) \neq 0. \tag{13}$$

If the time evolution, which is described by Φ_t, is not an endomorphism with respect to multiplication in operator algebra $\mathcal{M}$, then there exist observables A and B such that

$$\Delta_t(A, B) = \Phi_t(AB) - \Phi_t(A)\Phi_t(B) \neq 0. \tag{14}$$

The total time derivative of (14) gives

$$\left(\frac{d}{dt}\Delta_t(A, B)\right)_{t=0} = \mathcal{L}(AB) - \mathcal{L}(A)B - A\mathcal{L}(B).$$

It is easy to see that $\Delta_t(A, B) = 0$ for all $t > 0$ if condition (11) is satisfied. As a result, the infinitesimal generator is a derivative on the operator algebra, and the quantum system is Hamiltonian.

In uncertainty relations (1)–(3) the time dependence of quantum observables is not considered. In the general case, quantum observables depend on time. A time evolution Φ_t of observables of a non-Hamiltonian quantum system is not an endomorphism with respect to Lie, Jordan, and associative multiplications in general. In this case, there exist observables $A, B \in \mathcal{M}$ such that

$$\Phi_t(AB) \neq \Phi_t(A)\Phi_t(B),$$

$$\Phi_t(A * B) \neq \Phi_t(A) * \Phi_t(B),$$

$$\Phi_t(A \circ B) \neq \Phi_t(A) \circ \Phi_t(B).$$

The map Φ_t is an endomorphism $\mathcal{M}$ with respect to these multiplicative binary operations if and only if the system is locally Hamiltonian.[14]

Therefore, we should define four types of variance (second moments) for non-Hamiltonian dynamics.

(1) The average value of square deviation of the evolution of observable X,

$$D_t^{(1)}(X) = \left\langle \Big(\Phi_t(X - \langle X_t \rangle)\Big)^2 \right\rangle = \left\langle \Big(X_t - \langle X_t \rangle\Big)^2 \right\rangle. \tag{15}$$

(2) The average value of square of the evolution of deviation of observable

$$D_t^{(2)}(X) = \left\langle \Big(\Phi_t(X - \langle X \rangle)\Big)^2 \right\rangle. \tag{16}$$

(3) The average value of the evolution of square of deviation of observable

$$D_t^{(3)}(X) = \left\langle \Phi_t\Big(\Big(X - \langle X \rangle\Big)^2\Big) \right\rangle. \tag{17}$$

(4) The average (expected) value of the evolution of square of deviation of observable at different time moments

$$D_t^{(4)}(X) = \left\langle \Phi_t\Big(\Big(X - \langle X_t \rangle\Big)^2\Big) \right\rangle, \tag{18}$$

where $X_t = \Phi_t(X)$.

In addition, we should consider the same types of standard deviations $\Delta_t^{(k)} X = D_t^{(k)}(X)$ and the covariance between two quantum observables with finite second moments $Cov_t^{(k)}(X, Y)$, $k = 1, 2, 3, 4$.

In quantum non-Hamiltonian dynamics, there exists an effect of appearing noncommutativity.[14] Let A, B be commutative observables ($[A, B] = 0$). In general, the evolution gives

$$[A_t, B_t] = [\Phi_t(A), \Phi_t(B)] \neq 0.$$

This is the "environment-induced noncommutativity".

If the time evolution of the non-Hamiltonian system is an endomorphism of a linear operator space, then

$$\Phi_t(aA + bB) = a\Phi_t(A) + b\Phi_t(B), \quad \Phi_t(0) = 0 \tag{19}$$

hold for all $A, B \in \mathcal{M}$ and $a, b \in \mathbb{C}$. In general, the time evolution of non-Hamiltonian systems is not an endomorphism of a linear space structure, since these exist strange attractors[17] that cannot be considered as the linear spaces.

Note that it is possible to generalize Lie and Jordan operations such that it will be "invariant" with respect to time evolution. This generalization is realized as one-parameter operations by t-deformation of the underlying algebraic structure (see Secs. 19.1.–19.7. in Ref. 14).

IV. DERIVATION OF UNCERTAINTY RELATIONS

Let us consider the uncertainty relation for quantum observables X and Y. In general, the quantum observables depend on time $X_t = \Phi_t(X)$ and $Y_t = \Phi_t(Y)$ for $t > 0$. Then, we define

$$A = X_t - \langle X_t \rangle I, \quad B = Y_t - \langle Y_t \rangle I, \tag{20}$$

where I is an identity operator. The symbol $\langle\, \rangle$ denotes the average value by

$$\langle X_t \rangle = Tr[\rho X_t],$$

where ρ is a matrix density operator (statistical operator), which describes a quantum state. If X and Y are self-adjoint operators, then $A^\dagger = A$, $B^\dagger = B$. Note that $A = X_t - \langle X_t \rangle I$ cannot be considered as $\Phi_t(X - \langle X \rangle I)$ since

$$\Phi_t(X - \langle X \rangle I) = \Phi_t(X) - \langle X \rangle \Phi_t(I) = X_t - \langle X \rangle I \neq X_t - \langle X_t \rangle I. \tag{21}$$

Moreover to use equalities (21), we should assume that Φ_t is an endomorphism of a linear space structure on a set of quantum observables. In this section, we will not assume that Φ_t is an endomorphism of a linear space.

Let us consider the operator $C = zA + iB$, where z is a complex number. Using the non-negativity property of average values in the form $\langle C^\dagger C \rangle \geq 0$, we get the inequality

$$\Big\langle (z^* A - iB)(zA + iB) \Big\rangle \geq 0$$

for all $z \in \mathbb{C}$. Using the linear property for the average values

$$\langle aA + bB \rangle = a \langle A \rangle + b \langle B \rangle$$

for $a, b \in \mathbb{C}$, we obtain

$$z^* z \left\langle A^2 \right\rangle + i z^* \langle AB \rangle - iz \langle BA \rangle + \left\langle B^2 \right\rangle \geq 0.$$

This inequality can be rewritten in the form

$$(z_1^2 + z_2^2) \left\langle A^2 \right\rangle + i z_1 \langle AB - BA \rangle + z_2 \langle AB + BA \rangle + \left\langle B^2 \right\rangle \geq 0, \tag{22}$$

where z_1 and z_2 are real and imagine parts of $z = z_1 + iz_2$.

Using the Lie and Jordan operations (8) on the operator algebras of quantum observables, we rewrite inequality (22) in the form

$$(z_1^2 + z_2^2) \left\langle A^2 \right\rangle - \hbar z_1 \langle A * B \rangle + 2 z_2 \langle A \circ B \rangle + \left\langle B^2 \right\rangle \geq 0.$$

Using the Euler formula, we can represent z_1 and z_2 by the relations $z_1 = x \cos\varphi$, $z_2 = x \sin\varphi$. Then

$$\left\langle A^2 \right\rangle x^2 + \Big(2 \langle A \circ B \rangle \sin\varphi - \hbar \langle A * B \rangle \cos\varphi \Big) x + \left\langle B^2 \right\rangle \geq 0.$$

This inequality should be satisfied for all $\varphi \in \mathbb{R}$ and all $x \geq 0$. It is easy to see that the inequality $ax^2 + bx + x \geq 0$ holds for all $x \geq 0$ for two cases: (1) the discriminant $D = b^2 - 4ac$ is negative; (2) the conditions $D \geq 0$, $b \geq 0$, $c \geq 0$ hold. Using the phase shift method for linear combination of a cosine and a sine of equal angles, it is easy to prove that the condition

$$b = 2 \langle A \circ B \rangle \sin\varphi - \hbar \langle A * B \rangle \cos\varphi \geq 0$$

cannot be realized for all $\varphi \in \mathbb{R}$. Then the discriminant of this quadratic polynomial should be negative

$$D = \Big(2 \langle A \circ B \rangle \sin\varphi - \hbar \langle A * B \rangle \cos\varphi \Big)^2 - 4 \left\langle A^2 \right\rangle \left\langle B^2 \right\rangle \leq 0 \tag{23}$$

for all $\varphi \in \mathbb{R}$. We can be rewritten (23) in the form

$$\left\langle A^2 \right\rangle \left\langle B^2 \right\rangle \geq \frac{1}{4} \Big(2 \langle A \circ B \rangle \sin\varphi - \hbar \langle A * B \rangle \cos\varphi \Big)^2. \tag{24}$$

This inequality should satisfied for all $\varphi \in \mathbb{R}$. Using the relation of the phase shift method

$$a \sin\varphi - b\cos\varphi = \sqrt{a^2+b^2}\sin(\varphi-\alpha),$$

where $\sin\alpha = b/\sqrt{a^2+b^2}$, we obtain

$$\langle A^2\rangle \langle B^2\rangle \geq \left(\frac{\hbar^2}{4}\langle A * B\rangle^2 + \langle A\circ B\rangle^2\right)\sin^2(\varphi-\alpha) \tag{25}$$

for all $\varphi \in \mathbb{R}$, where

$$\sin\alpha = \frac{\hbar\langle A * B\rangle}{\sqrt{\hbar^2\langle A*B\rangle^2 + 4\langle A\circ B\rangle^2}}.$$

Then, we use $\sin^2(\varphi + \alpha) \leq 1$. As a result, we have the uncertainty relation

$$\langle A^2\rangle\langle B^2\rangle \geq \frac{\hbar^2}{4}\langle A * B\rangle^2 + \langle A\circ B\rangle^2 . \tag{26}$$

Using the definitions (20) of A and B, we get

$$(\Delta X_t)^2\,(\Delta Y_t)^2 \geq \frac{\hbar^2}{4}\langle X_t * Y_t\rangle^2 + \Big(\langle X_t\circ Y_t\rangle - \langle X_t\rangle\,\langle Y_t\rangle\Big)^2, \tag{27}$$

which holds for all $t \geq 0$. This is the uncertainty relation of Schrödinger-Robertson-type for quantum observables X_t and Y_t. The Poincare-Birkhoff-Witt theorem and the assumption that we consider only special Jordan algebras allows us to use the representation of operations $\langle *, \circ\rangle$ in the form (8). Note that relation (27) for non-Hamiltonian systems can be represented in the form (3) only if $\langle X_t\rangle = \langle Y_t\rangle = 0$ for all $t \geq 0$.

For non-Hamiltonian system (4) uncertainty relation (27) has the form

$$D_t^{(1)}(Q)\,D_t^{(1)}(P) \geq \frac{\hbar^2}{4}e^{-4\lambda t} + \Big(K_{qq}^{(1)}\langle Q^2\rangle + K_{pp}^{(1)}\langle P^2\rangle + K_{qp}^{(1)}\langle Q\circ P\rangle - $$
$$ - L_{qq}^{(1)}\langle Q\rangle^2 - L_{pp}^{(1)}\langle P\rangle^2 - L_{qp}^{(1)}\langle Q\rangle\,\langle P\rangle\Big)^2, \tag{28}$$

where

$$K_{qq}^{(1)} = L_{qq}^{(1)} = e^{-2\lambda t}\Big(-\frac{m\omega^2}{\nu}\Big)\sinh(\nu t)\Big(\cosh(\nu t) + \frac{\mu}{\nu}\sinh(\nu t)\Big), \tag{29}$$

$$K_{pp}^{(1)} = L_{pp}^{(1)} = e^{-2\lambda t} + \frac{1}{m\nu}\sinh(\nu t)\Big(\cosh(\nu t) - \frac{\mu}{\nu}\sinh(\nu t)\Big), \tag{30}$$

$$K_{qp}^{(1)} = L_{qp}^{(1)} = e^{-2\lambda t}\Big(1 - 2\frac{\omega^2}{\nu^2}\sinh^2(\nu t)\Big). \tag{31}$$

V. SPECIAL CASES OF UNCERTAINTY RELATION

(1) If X_t and Y_t are commutative operators, then $X_t * Y_t = 0$ for all $t > 0$ and inequality (27) gives

$$(\Delta X_t)^2\,(\Delta Y_t)^2 \geq \Big(\langle X_t\circ Y_t\rangle - \langle X_t\rangle\,\langle Y_t\rangle\Big)^2. \tag{32}$$

For example, $X * Y = 0$ if $X = Q^k$ and $Y = Q^l$ or $X = Q^k$ and $Y = P^l$, where $k \neq l$. Foe quantum non-Hamiltonian systems there exists an effect of appearing ("environment-induced") noncommutativity:[14] The Lie multiplication of quantum observables ($[Q_0^k, Q_0^l] = 0$) can evolve to the operator, which is not equal to Lie multiplication of the evolved observables ($[Q_t^k, Q_t^l] \neq 0$). Only if $\langle X_t\circ Y_t\rangle = \langle X_t\rangle\,\langle Y_t\rangle$, then we can have $(\Delta X_t)^2\,(\Delta Y_t)^2 = 0$.

In classical statistical mechanics, we have the same relation

$$(\Delta X_t)^2\,(\Delta Y_t)^2 \geq \Big(\langle X_t \circ Y_t\rangle_c - \langle X_t\rangle_c\,\langle Y_t\rangle_c\Big)^2,$$

where $X \circ Y = X(t, q, p)Y(t, q, p)$, and

$$\langle X\rangle_c = \int dqdp\,\rho(t, q, p)X(t, q, p).$$

Note that relation (32) can be obtained from (24) for $\varphi = \pi/2$.

(2) If X_t and Y_t are anticommutative operators, then $X_t \circ Y_t = 0$ for all $t > 0$, and inequality (27) gives

$$(\Delta X_t)^2\,(\Delta Y_t)^2 \geq \frac{\hbar^2}{4}\,\langle X_t * Y_t\rangle^2 + \langle X_t\rangle^2\,\langle Y_t\rangle^2\,. \tag{33}$$

Note that this relation can be derived from (24) for $\varphi = 0$.

(3) It is easy to see that we have the usual form of the Heisenberg uncertainty relation

$$\Big\langle (X_t - \langle X_t\rangle\, I)^2\Big\rangle \Big\langle (Y_t - \langle Y_t\rangle\, I)^2\Big\rangle \geq \frac{\hbar^2}{4}\,\langle X_t * Y_t\rangle^2 \tag{34}$$

only for the case

$$\langle X_t Y_t + Y_t X_t\rangle - 2\,\langle X_t\rangle\,\langle Y_t\rangle = 0. \tag{35}$$

Note the condition $\langle X_t Y_t + Y_t X_t\rangle = 0$,[9] cannot give the Heisenberg's uncertainty relation, since the average values can be nonzero, and

$$\Big\langle (X_t - \langle X_t\rangle\, I)^2\Big\rangle \Big\langle (Y_t - \langle Y_t\rangle\, I)^2\Big\rangle \geq \frac{\hbar^2}{4}\,\langle X_t * Y_t\rangle^2 + \langle X_t\rangle^2\,\langle Y_t\rangle^2\,. \tag{36}$$

In general, condition (35) is not held and uncertainty relation (27) should be used instead of relation (34) of the Heisenberg-type.

(4) If the average values of X_t and Y_t are equal to zero ($\langle X_t\rangle = \langle Y_t\rangle = 0$) for all $t > 0$, then the uncertainty relation

$$\Big\langle (\Phi_t(X))^2\Big\rangle \Big\langle (\Phi_t(Y))^2\Big\rangle \geq \frac{\hbar^2}{4}\,\langle \Phi_t(X) * \Phi_t(Y)\rangle^2 + \langle \Phi_t(X) \circ \Phi_t(Y)\rangle^2 \tag{37}$$

should be held for all time moments $t > 0$.

VI. UNCERTAINTY RELATION FOR COORDINATE AND MOMENTUM

In order to consider an uncertainty relation for operators of coordinate Q^k and momenta P^k, we should use the Heisenberg canonical commutation relations

$$[Q^k, P^l] = i\hbar\delta^{kl} I, \quad [Q^k, Q^l] = [P^k, P^l] = 0, \quad k, l = 1..n. \tag{38}$$

To consider uncertainty relation for $Q^k_t = \Phi_t(Q^k)$ and $P^l_t = \Phi_t(P^k)$, we should use the canonical commutation relations in the form

$$[Q^k_t, P^l_t] = i\hbar\delta^{kl} I, \quad [Q^k_t, Q^l_t] = [P^k_t, P^l_t] = 0, \quad k, l = 1..n, \tag{39}$$

for all $t \geq 0$.

There is the following statement.[14] If the rule of term-by-term differentiation (Leibnitz rule) with respect to time and the canonical commutation relations (39) are valid for all $t > 0$, then the conditions

$$J_{\mathcal{L}}(Q^k_t, P^l_t) = J_{\mathcal{L}}(Q^k_t, Q^l_t) = J_{\mathcal{L}}(P^k_t, P^l_t) = 0$$

are satisfied for all $t > 0$, where $J_{\mathcal{L}}(A, B)$ is defined by (9). As a result, the quantum system is Hamiltonian id $K_{\mathcal{L}}(A, B) = 0$. To prove this statement, we consider differentiation of the first relation in (39) with respect to time t,

$$\frac{d}{dt}[Q_t^k, P_t^l] = 0.$$

The rule of term-by-term differentiation for the commutator $[Q_t^k, P_t^l]$ has the form

$$\frac{d}{dt}[Q_t^k, P_t^l] = [\frac{d}{dt}Q_t^k, P_t^l] + [Q_t^k, \frac{d}{dt}P_t^l].$$

Consequently, we have

$$[\frac{d}{dt}Q_t^k, P_t^l] + [Q_t^k, \frac{d}{dt}P_t^l] = 0.$$

Using the equations of motion

$$\frac{dQ_t^k}{dt} = \mathcal{L}(Q_t^k), \quad \frac{dP_t^k}{dt} = \mathcal{L}(P_t^k),$$

we obtain

$$[\mathcal{L}(Q_t^k), P_t^l] + [Q_t^k, \mathcal{L}(P_t^l)] = 0.$$

Using the usual condition $\mathcal{L}(I) = 0$,[14] we get

$$\mathcal{L}([Q_t^k, P_t^l]) - [\mathcal{L}(Q_t^k), P_t^l] - [Q_t^k, \mathcal{L}(P_t^l)] = 0.$$

As a result, we have

$$J_{\mathcal{L}}(Q_t^k, P_t^l) = 0$$

for Lie multiplication. Similarly, considering other canonical commutation relations (39), we obtain all of the identities

$$J_{\mathcal{L}}(X_t^k, X_t^l) = 0,$$

where $X_t^k = Q_t^k, P_t^k$. Then, the quantum system is Hamiltonian if $K_{\mathcal{L}}(X_t^k, X_t^l) = 0$.

As a result, if the rule of term-by-term differentiations and the canonical commutation relations are valid for all $t \geq 0$, then the quantum system is Hamiltonian for the case $K_{\mathcal{L}}(X_t^k, X_t^l) = 0$. For quantum non-Hamiltonian systems, either the canonical commutation relations or Leibnitz rule for multiplication is not valid. Note that a generalization $[\,,\,]_t$ (t-invariant commutator) of the commutator $[\,,\,]$, such that the commutation relations

$$[Q_t^k, Q_t^l]_t = 0, \quad [P_t^k, P_t^l]_t = 0, \quad [Q_t^k, P_t^l]_t = i\hbar\delta^{kl},$$

are satisfied for all $t \geq 0$ for quantum non-Hamiltonian system are discussed in Ref. 14.

It was proved that we cannot use the commutation relations

$$Q_t * P_t = I, \quad Q_t * I = P_t * I = 0$$

for operators of coordinate and momentum ($X = Q$ and $Y = P$) of non-Hamiltonian quantum system. As a result, inequality (25) cannot be represented in the form

$$\left\langle (Q_t - \langle Q_t \rangle I)^2 \right\rangle \left\langle (P_t - \langle P_t \rangle I)^2 \right\rangle \geq \frac{\hbar^2}{4} \tag{40}$$

or

$$\left\langle (Q_t - \langle Q_t \rangle I)^2 \right\rangle \left\langle (P_t - \langle P_t \rangle I)^2 \right\rangle \geq \frac{\hbar^2}{4} + (\langle Q_t \circ P_t \rangle - \langle Q_t \rangle \langle P_t \rangle)^2 \tag{41}$$

for non-Hamiltonian quantum systems. In general, we should use the inequality

$$\left\langle (Q_t - \langle Q_t \rangle I)^2 \right\rangle \left\langle (P_t - \langle P_t \rangle I)^2 \right\rangle \geq \frac{\hbar^2}{4} \langle Q_t * P_t \rangle^2 + (\langle Q_t \circ P_t \rangle - \langle Q_t \rangle \langle P_t \rangle)^2 \tag{42}$$

for coordinate and momentum of non-Hamiltonian system.

If the time evolution Φ_t of a non-Hamiltonian quantum system is an endomorphism of a linear operator space, then we have additional uncertainty relations that will be considered in Sec. VII.

VII. FEATURES OF THE UNCERTAINTY RELATION FOR NON-HAMILTONIAN SYSTEMS

The linear property for Φ_t is not used to derive inequalities (27). If the time evolution of the non-Hamiltonian system is an endomorphism of a linear operator space, then

$$\Phi_t(aA + bB) = a\Phi_t(A) + b\Phi_t(B), \quad \Phi_t(0) = 0, \quad \Phi_t(I) = I \tag{43}$$

hold for all $A, B \in \mathcal{M}$ and $a, b \in \mathbb{C}$. In general, Φ_t is not an endomorphism of the linear space. It is well-known that strange attractors of classical non-Hamiltonian systems are not linear spaces. In quantum theory, there are analogous situations for quantum analogs of (regular or strange) attractors. If linear condition (43) holds, then we have additional uncertainty relations. To derive these inequalities, we define

$$A_0 = X - \langle X\rangle I, \quad B_0 = Y - \langle Y\rangle I.$$

We can consider the operator $C_0 = zA_0 + iB_0$, where z is a complex number, and we can use the non-negativity property of average values in two following different forms.

(1) Using the non-negativity property of average values in the form

$$\left\langle (\Phi_t(C_0))^{\dagger}\Phi_t(C_0)\right\rangle \geq 0,$$

we get that the inequality

$$\left\langle(\Phi_t(A_0))^2\right\rangle\left\langle(\Phi_t(B_0))^2\right\rangle \geq \frac{\hbar^2}{4}\langle\Phi_t(A_0) * \Phi_t(B_0)\rangle^2 + \langle\Phi_t(A_0)\circ\Phi_t(B_0)\rangle^2 \tag{44}$$

holds for all $t > 0$. Here $\left\langle(\Phi_t(A_0))^2\right\rangle = D_t^{(2)}(X)$ and $\left\langle(\Phi_t(B_0))^2\right\rangle = D_t^{(2)}(Y)$. Note that

$$\Phi_t(A_0) = X_t - \langle X\rangle I \neq X_t - \langle X_t\rangle I.$$

As a result, (44) is not equivalent to (27).

For non-Hamiltonian quantum system (4) uncertainty relation (27) has the form

$$D_t^{(2)}(Q)\, D_t^{(2)}(P) \geq \frac{\hbar^2}{4}e^{-4\lambda t} + \Big(K_{qq}^{(2)}\left\langle Q^2\right\rangle + K_{pp}^{(2)}\left\langle P^2\right\rangle + K_{qp}^{(2)}\langle Q\circ P\rangle -$$
$$- L_{qq}^{(2)}\langle Q\rangle^2 - L_{pp}^{(2)}\langle P\rangle^2 - L_{qp}^{(2)}\langle Q\rangle\,\langle P\rangle\Big)^2, \tag{45}$$

where the coefficients are

$$K_{qq}^{(2)} = K_{qq}^{(1)}, \quad K_{pp}^{(2)} = K_{pp}^{(1)}, \quad K_{qp}^{(2)} = K_{qp}^{(1)}, \tag{46}$$

$$L_{qq}^{(2)} = -\frac{m\omega^2}{\nu}e^{-\lambda t}\sinh(\nu t), \tag{47}$$

$$L_{pp}^{(2)} = \frac{1}{m\nu}e^{-\lambda t}\sinh(\nu t), \tag{48}$$

$$L_{qp}^{(2)} = e^{-\lambda t}\Big(\cosh(\nu t) + \frac{\mu}{\nu}\sinh(\nu t)\Big) + e^{-\lambda t}\Big(\cosh(\nu t) - \frac{\mu}{\nu}\sinh(\nu t)\Big) - 1. \tag{49}$$

(2) The quantum dynamical map Φ_t satisfies the condition $\Phi_t(A) \geq 0$ if $A \geq 0$ for all $t \geq 0$.[14] Using the non-negativity property of average values in the form

$$\left\langle \Phi_t(C_0^{\dagger}C_0)\right\rangle \geq 0,$$

we get the uncertainty relation

$$\left\langle\Phi_t(A_0^2)\right\rangle\left\langle\Phi_t(B_0^2)\right\rangle \geq \frac{\hbar^2}{4}\langle\Phi_t(A_0 * B_0)\rangle^2 + \langle\Phi_t(A_0\circ B_0)\rangle^2 . \tag{50}$$

Here $\langle \Phi_t(A_0^2) \rangle = D_t^{(3)}(X)$ and $\langle \Phi_t(B_0^2) \rangle = D_t^{(3)}(Y)$. Note that there is no the factor $e^{-4\lambda t}$ in relation (51) since $\langle \Phi_t(A_0 * B_0) \rangle = 1$.

For non-Hamiltonian system (4) uncertainty relation (27) has the form

$$D_t^{(3)}(Q)\, D_t^{(3)}(P) \geq \frac{\hbar^2}{4} + \Big(K_{qq}^{(3)} \langle Q^2 \rangle + K_{pp}^{(3)} \langle P^2 \rangle + K_{qp}^{(3)} \langle Q \circ P \rangle - - L_{qq}^{(3)} \langle Q \rangle^2 - L_{pp}^{(3)} \langle P \rangle^2 - L_{qp}^{(3)} \langle Q \rangle \, \langle P \rangle \Big)^2, \quad (51)$$

where the coefficients $K^{(3)}$ and $L^{(3)}$ are expressed in terms of the coefficients that are defined by Eqs. (3.77) and (3.78) in the paper.[12]

It is well-known that the relations

$$\Phi_t(AB) = \Phi_t(A)\Phi_t(B), \quad \Phi_t(A^2) = (\Phi_t(A))^2, \quad (52)$$

$$\Phi_t(A * B) = \Phi_t(A) * \Phi_t(B), \quad \Phi_t(A \circ B) = \Phi_t(A) \circ \Phi_t(B) \quad (53)$$

hold for Hamiltonian quantum systems. As a result, inequalities (44) and (50) are equivalent. In general, relations (52) and (53) are not realized for non-Hamiltonian quantum systems, and the systems have the following unusual properties:

(1) A time evolution Φ_t of observables of a quantum non-Hamiltonian system is not an endomorphism[14] with respect to Lie, Jordan, and associative multiplications. The multiplication of quantum observables evolves to the operator, which is not equal to multiplication of the evolved observables

$$\Phi_t(AB) \neq \Phi_t(A)\Phi_t(B), \quad \Phi_t(A^2) \neq (\Phi_t(A))^2. \quad (54)$$

As a result, inequalities (44) and (50) are not equivalent.

(2) In non-Hamiltonian quantum dynamics, there exists an effect of appearing noncommutativity. In general, the commutative observables A and B ($A * B = 0$) can evolve into noncommutative observables $A_t = \Phi_t(A)$ and $B_t = \Phi_t(B)$,

$$[A_t, B_t] = [\Phi_t(A), \Phi_t(B)] \neq 0$$

for $t > 0$, i.e., $A_t * B_t \neq 0$.

(3) In general, the same effect can exist for Jordan's multiplication of quantum observables. The anticommutative observables A and B ($A \circ B = 0$) can evolve into non-anticommutative observables $A_t = \Phi_t(A)$ and $B_t = \Phi_t(B)$: $A_t \circ B_t \neq 0$ for $t > 0$.

All these effects and properties affect on the right hand side of inequalities (44) and (50). As a result, it is necessary to use both uncertainty relations (44) and (50) in addition to relation (27).

VIII. CONCLUSION

In well-known uncertainty relations (1)–(3), the time dependence of quantum observables is not considered. In the general case, quantum observables depend on time. A time evolution Φ_t of observables of a non-Hamiltonian quantum system is not an endomorphism with respect to Lie, Jordan, and associative multiplications in general. The evolution Φ_t is an endomorphism $\mathcal{M}$ with respect to these multiplicative binary operations if and only if the system is locally Hamiltonian.[14] We consider the uncertainty relations for non-Hamiltonian quantum systems in the Schrödinger-Robertson form by using the Lie-Jordan algebra for a set of quantum observables. We take into account that a time evolution of observables of a non-Hamiltonian quantum system is not an endomorphism with respect to Lie, Jordan, and associative multiplications. Therefore, we define four types of variance (second moments) for non-Hamiltonian dynamics: (1) The average value (15) of square deviation of the evolution of observable; (2) The average value (16) of square of the evolution of deviation of observable; (3) The average value (17) of the evolution of square of deviation of observable. In addition, it is possible to consider the average value (18) of the evolution of square of deviation of observable at different time moments. The same types of standard deviations and the covariance between two quantum observables with finite second moments.

Note that it is possible to generalize a kinematical structure such that it will be "invariant" with respect to time evolution. This generalization should be connected with a notion of one-parameter operations and t-deformation of the underlying algebraic structure (see Secs. 19.1.– 19.7. in Ref. 14).

In general, we have the following unusual properties for non-Hamiltonian quantum systems: (1) A time evolution Φ_t of observables of a quantum non-Hamiltonian system is not an endomorphism[14] with respect to Lie, Jordan, and associative multiplications. The multiplication of quantum observables evolves to the operator, which is not equal to multiplication of the evolved observables. This is so called the "environment-induced noncommutativity"; (2) In non-Hamiltonian quantum dynamics, there exists an effect of appearing noncommutativity. In general, the commutative observables A and B ($A * B = 0$) can evolve into noncommutative observables ($A_t * B_t = 0$); (3) The anticommutative observables A and B ($A \circ B = 0$) can evolve into non-anticommutative observables ($A_t \circ B_t = 0$).

All these effects and properties affect on the right hand side of uncertainty relations (44) and (50). As a result, it is necessary to use both inequalities (44) and (50) in addition to relation (27) in non-Hamiltonian quantum dynamics.

In general, the time evolution of non-Hamiltonian systems is not an endomorphism of a linear space structure, since these exist strange attractors[17] that cannot be considered as the linear spaces. In quantum theory, there are analogous situations for quantum analogs of (regular or strange) attractors.[15,16] The linear property for time evolution Φ_t is not used to derive inequalities (27). We have additional uncertainty relations (44) and (50) only if linear property (43) holds.

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