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Anisotropic fractal media by vector calculus in non-integer dimensional space

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A review of different approaches to describe anisotropic fractal media is proposed. In this paper, differentiation and integration non-integer dimensional and multi-fractional spaces are considered as tools to describe anisotropic fractal materials and media. We suggest a generalization of vector calculus for non-integer dimensional space by using a product measure method. The product of fractional and non-integer dimensional spaces allows us to take into account the anisotropy of the fractal media in the framework of continuum models. The integration over non-integer-dimensional spaces is considered. In this paper differential operators of first and second orders for fractional space and non-integer dimensional space are suggested. The differential operators are defined as inverse operations to integration in spaces with non-integer dimensions. Non-integer dimensional space that is product of spaces with different dimensions allows us to give continuum models for anisotropic type of the media. The Poisson's equation for fractal medium, the Euler-Bernoulli fractal beam, and the Timoshenko beam equations for fractal material are considered as examples of application of suggested generalization of vector calculus for anisotropic fractal materials and media. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4892155]

I. INTRODUCTION

Fractals are measurable metric sets with non-integer dimensions.[1,2] The basic property of the fractal is non-integer Hausdorff dimension that should be observed at all scales. The definition of the Hausdorff dimension requires the diameter of the covering sets to vanish. The fractal structure of real materials cannot be observed on all scales. This structure exists only for scales $R > R_0$, where R_0 is the characteristic size of atoms or molecules of fractal media. Isotropic fractal materials can be characterized by the relation between the mass $M_D(W_B)$ of a ball region W_B of fractal medium, and the radius R of this ball in the form

$$M_D(W_B) = M_0 \left(\frac{R}{R_0}\right)^D, \quad R/R_0 \gg 1. \tag{1}$$

The parameter D is called the mass dimension of fractal medium. The parameter D, does not depend on the shape of the region W_B, or on whether the packing of sphere of radius R_0 is close packing, a random packing or a porous packing with a uniform distribution of holes. Anisotropic fractal materials can be characterized by the power-law relation for the mass of the parallelepiped region W_P in the form

$$M_D(W_P) = M_0 \left(\frac{L_x}{R_0}\right)^{\alpha_1} \left(\frac{L_y}{R_0}\right)^{\alpha_2} \left(\frac{L_z}{R_0}\right)^{\alpha_3}, \quad \min\{L_x, L_y, L_z\} \gg R_0, \tag{2}$$

where the parameter α_k is non-integer dimension along X_k-axis, $k = 1, 2, 3$, and L_x, L_y, L_z represent three edges that meet at one vertex. The parameter α_k describes how to increase the medium mass

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in the case of increasing the size of the parallelepiped along one axis, when the parallelepiped sizes along other axes do not change. The sum $D = \alpha_1 + \alpha_2 + \alpha_3$ is called the fractal mass dimension of the anisotropic fractal medium.

Using (1) and (2), we can define a fractal material as a medium with non-integer mass dimension. The non-integer dimension does not reflect completely all specific properties of the fractal media, but it is an main characteristic of fractal media and materials. For this reason, we assume that continuum models with non-integer dimensional spaces allow us to derive important conclusions about the behavior of the fractal media.

The main ways of describing of fractal media can be conventionally divided into the following five approaches.

(1) *Analysis on fractal approach*: The first approach is based on the use of methods of "Analysis on fractals."[3–8] Unfortunately, a possibility of application of the "Analysis on fractals" to solve differential equations on fractals[4] for real problems of fractal materials is very limited due to weak development of this area of mathematics to the present time.

(2) *Fractional-differential continuum model approach*: The second approach is based on the use of the fractional derivatives of non-integer orders with respect to space coordinates to describe some properties of fractal materials.[9] Therefore, the correspondent models can be called the fractional-differential models. It has been suggested by Carpinteri and co-workers in Refs. 10,11, and 12, where so-called local fractional derivatives are used, and then developed in Refs. 13–19. Unfortunately, there are not enough differential equations with these fractional derivatives that are solved for various problems of fractal materials. It should be noted that the usual Leibniz rule does not hold[20] for derivatives of non-integer orders (and integer orders $n \neq 1$). It is a characteristic property of fractional derivatives.

(3) *Fractional-integral continuum model approach*: The third approach has been suggested in Refs. 21–27 and it is based on application of continuum models of fractal media. These models can be called fractional-integral continuum model because the integrations of non-integer orders are used. The kernels of fractional integrals are defined by power-law density of states.[27] The orders of fractional integrals are equal to the mass (charge or other physical) dimensions of media. In these models, the density of states is applied in addition to the notion of distribution functions such as density of mass, density of charge. There are a lot of applications of these continuum models in different fields of mechanics and physics (see Refs. 27 and references cited therein). These models have been applied by Ostoja-Starzewski in Refs. 28–32. A generalization of fractional-integral continuum models for anisotropic fractal media has been suggested by Ostoja-Starzewski and co-workers in Refs. 33–39. In these models, the differential operators are modified by the density of states. However, these operators have integer-order differential operators.

(4) *Fractional space approach*: The fourth approach uses the concept of a fractional space, which is characterized by non-integer (fractional) powers of coordinates. The fractional space approach has been suggested in Refs. 40–42 and then it is used for applications in different areas[27,43–46] (see also Refs. 47–50). This approach has been developed by Calcagni in Refs. 51–54 (see also Refs. 55 and 56), and then it was generalized for anisotropic case by using a multi-fractional space ("multi-scale space") in Refs. 57–59. The first interpretation of the fractional phase space is connected with fractional dimension space. The fractional dimension interpretation follows from the formulas for dimensional regularizations and it was suggested in Ref. 40. In Refs. 41 and 42 the second interpretation of the fractional phase space is considered. This interpretation follows from the fractional measure[40] of phase space that is used in the fractional integrals, i.e., the integrals of non-integer orders. In the third interpretation, the fractional phase space is considered as a phase space that is described by the fractional powers of coordinates and momenta. In addition, almost all Hamiltonian systems with fractional phase space are non-Hamiltonian dissipative systems in the usual phase space. It allows us to have the fourth interpretation of the fractional phase space as a phase space of power-law type of non-Hamiltonian systems. Using fractional space approach we can consider wide class of non-Hamiltonian systems as generalized Hamiltonian systems. Differentiation in fractional space approach can be used in two forms: (a) the usual derivatives with respect to fractional powers of coordinates;[40–42] (b) the fractional derivatives of non-integer orders (fractional derivatives) with respect to coordinates.[51–54] The term "fractional space" is sometimes used for

non-integer-dimensional space. This leads to confusion and misunderstanding. We use the term "fractional space" for effective space $\mathbb{R}^n$ with coordinates that are non-integer powers of coordinates of physical space. In the fractional space approach, the integer-dimensional spaces $\mathbb{R}^n$, the integration and differentiation of integer-orders for these spaces are used. The coordinates of fractional space are considered as effective coordinates that are fractional powers of real space coordinates of physical system. Note that we also can use effective spaces for non-integer-dimensional physical spaces. Expressions for the effective coordinates for fractional and non-integer-dimensional spaces differ by factors in the density of states.

(5) *Non-integer-dimensional space approach*: The fifth approach is based on application of integration and differentiation for non-integer-dimensional spaces. The integration in non-integer dimensional space is well developed,[60–62] and it has a wide application in quantum field theory. The axioms for integrals in D-dimensional space is suggested in Ref. 60. This properties are natural and necessary in applications.[62] Integration in D-dimensional spaces with non-integer D is used for dimensional regularization in quantum field theory[62–64] and in physical kinetics.[65,66] Dimensional regularization is a way to get infinities that occur when one evaluates Feynman diagrams in quantum theory. Differentiation in non-integer dimensional space is considered in Refs. 61, 67, and 68. In Refs. 61 and 67 it was offered only a scalar Laplacian for non-integer dimensional space. Unfortunately, the gradient, divergence, curl operator, and the vector Laplacian[104] are not considered in Refs. 61 and 67. The scalar Laplace operators, which are suggested by Stillinger in Ref. 61 and Palmer, Stavrinou in Ref. 67 for non-integer dimensional spaces, have successfully been used for effective descriptions in different areas of physics and mechanics such as quantum mechanics (see Refs. 61, 67, 70–91), the diffusion processes,[92] the general relativity,[93,94] and the electrodynamics.[95–101] All these applications are based only on two generalization of the scalar Laplacian that are suggested in Refs. 61 and 67. To expand the range of possible applications of models with non-integer dimensional spaces it is important to have generalization of differential operators of first orders (gradient, divergence, curl operators) and the vector Laplacian. The continuation in dimension is recently suggested in Refs. 68 and 69 to define the gradient, divergence, curl operator, and the vector Laplacian for non-integer dimensional space. It allows us to describe isotropic fractal media in the framework of continuum models with non-integer dimensional spaces. To generalize non-integer dimensional space approach for anisotropic fractal media we can suggest to use the product measure approach suggested in Refs. 40–42 and 67. Generalizations of the gradient, divergence, curl operators, and the vector Laplace operator for non-integer dimensional and fractional spaces to describe anisotropic fractal media are not considered by the product measure approach in Refs. 61 and 67 and other papers. These generalizations are suggested in this paper as an extension of approach proposed in Refs. 68 and 69.

To present more clearly some of the differences between these five approaches to describe fractal media distributed in the space $\mathbb{R}^n$, we present the following table.

Approach	Set/space	Integration	Differentiation
Analysis on fractal	Fractal set	Integration for fractal set	Differentiation for fractal set
Fractional-differential continuum model	Integer-dimensional space $\mathbb{R}^n$	Integrals for $\mathbb{R}^n$	Fractional-order derivatives for $\mathbb{R}^n$
Fractional-integral continuum model	Integer-dimensional space $\mathbb{R}^n$	Fractional-order integrals for $\mathbb{R}^n$	Integer-order derivatives for $\mathbb{R}^n$
Fractional space	Integer-dimensional (effective) space $\mathbb{R}^n$	Integer-order integrals for $\mathbb{R}^n$	Integer-order derivatives for $\mathbb{R}^n$
Non-integer-dimensional space	Non-integer-dimensional space	Integrals for non-integer-dimensional space	Derivatives for non-integer-dimensional space

Let us note some advantages and disadvantages of the third, fourth, and fifth approaches. The forms of the functions that define the density of states in the third and fourth approaches have arbitrariness due to the existence of different types of fractional integrals. In the fifth approach, the form of density of states is uniquely fixed by the expression of volume of the region in the non-integer-dimensional space. In the third approach practically all properties of fractal media are reduced to coordinate transformations and to space curvature of power type. The transition by transformations to an effective Euclidean space cannot be considered as a consistent approach to describe fractal media. These transformations can be considered only as a part of mathematical method to solve some equations in the framework of continuum models with non-integer-dimensional space, i.e., in the fifth approach. Connections between the effective coordinates and physical coordinates for the non-integer-dimensional space approach are uniquely defined. In addition, the fifth approach allows us to use fractional derivatives and integrals to describe nonlocal type of fractal media that cannot be described by other approaches.

In this paper, we consider the two last approaches based on the fractional space and the non-integer dimensional space to describe anisotropic fractal materials and media. We suggest a generalization of vector calculus for non-integer dimensional space that is product of spaces with different dimensions. In Sec. II, we discuss the product measure method to describe anisotropic fractal media. In Sec. III, the integration for fractional and non-integer-dimensional spaces are considered. In Sec. IV, differential operations of first and second orders for fractional space and non-integer-dimensional space are suggested. In Sec. V, we give some examples of application of suggested generalization of vector calculus for anisotropic fractal materials and media.

II. PRODUCT MEASURE METHOD

A. Product measure for fractional and non-integer-dimensional spaces

The product measure method can be applied to the non-integer dimensional spaces and to the fractional spaces:

(I) *Product measure for the fractional spaces*: The product measure approach for the fractional spaces has been suggested in Refs. 41 and 42, where fractional phase space is considered with its interpretation as a non-integer (fractional) dimensional space. In Refs. 41 and 42, the following measure is used for generalized coordinates and momenta

$$d\mu_\alpha(x_k) = c(\alpha)\,|x_k|^{\alpha-1}\,dx_k, \tag{3}$$

where the numerical factor $c(\alpha)$ is

$$c(\alpha) = 1/\Gamma(\alpha). \tag{4}$$

We use the factor $c(\alpha)$ in the form (4) to get a relation with the Riemann-Liouville fractional integrations of non-integer order α. In Refs. 40–42, it has been shown that the integration for the suggested fractional space is directly connected with integration for non-integer dimensional space up to numerical factor. Therefore, the fractional space has been interpreted as a non-integer dimensional space. The suggested fractional space approach has been used in Refs. 43–46 and 27 for configuration and phase spaces, and then it has been applied by Calcagni in Refs. 51–54 for space-time. The differentiation and integration in fractional space are considered as differentiation and integration with respect to non-integer (fractional) powers of coordinates. The differential operator of the first order is defined[40–45] by

$$D_{\alpha,k} = \frac{\partial}{\partial Q_k} = \frac{1}{\alpha\,|x_k|^{\alpha-1}}\frac{\partial}{\partial x_k}, \tag{5}$$

where $Q_k = \mathrm{sgn}(x)\,|x|^\alpha$. The product measure approach also used in Refs. 33–36, and 39 to describe fractal media, but the fractional space as space with non-integer powers of coordinates has not been considered. Instead of the Riemann-Liouville fractional integration, which is used in Refs. 40–46, the product measure approach is used in Refs. 33–36, and 39 for so-called modified Riemann-Liouville integrations in the integer dimensional spaces.

(II) *Product measure for the non-integer dimensional spaces*: The product measure approach has been suggested by Palmer and Stavrinou in Ref. 67 for non-integer dimensional spaces, where each orthogonal coordinates has own dimension. The methods of this paper can be considered as a modification of Stillinger's[61] and Svozils's[103] methods for the product measure approach. In Ref. 67, the product of the following single-variable measures is used

$$d\mu_{\alpha_k}(x_k) = c(\alpha_k)\, |x_k|^{\alpha_k - 1}\, dx_k \tag{6}$$

with the numerical factor

$$c(\alpha) = \frac{2\,\pi^{\alpha/2}}{\Gamma(\alpha/2)}. \tag{7}$$

Note that $c(\alpha)$ is equal to the surface area $S_\alpha(R)$ of α-sphere with the radius $R = 1$, where

$$S_\alpha(R) = \frac{2\,\pi^{\alpha/2}}{\Gamma(\alpha/2)}\, R^\alpha. \tag{8}$$

The integration for non-integer dimensional space by product measure approach is described in Ref. 67. The scalar Laplacian operator for non-integer dimensional spaces is also suggested in Ref. 67 in the form

$${}^S\Delta^{(\alpha)} = \sum_{k=1}^{3} \Big(\frac{\partial^2}{\partial x_k^2} + \frac{\alpha_k - 1}{x_k} \frac{\partial}{\partial x_k} \Big), \tag{9}$$

where $(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$ is the multi-index. Unfortunately, definitions of the gradient, divergence, curl operations, and the vector Laplacian are not considered in Ref. 67.

B. Type of anisotropic fractal media described by non-integer dimensional spaces

In this section, we describe a type of anisotropic fractal media that can be considered by approach based on non-integer dimensional spaces.

Let us defined the parameter α_k ($k = 1; 2; 3$) that describes the scaling property along X_k-axis by

$$\alpha_1 = \alpha_x = D - d_{yz}, \tag{10}$$

where D is a non-integer mass dimension of the material. Here d_{yz} is the non-integer dimension (for example, the box-counting dimension) of the YZ-cross-section, which is perpendicular to the X-axis. We should assume that non-integer dimension d_{yz} of the YZ-cross-sections is the same for all points along the X-axis ($d_{yz}(x) = d_{yz} = \text{const}$). If this condition is not satisfied, then we have a variable order $\alpha_1(x)$ that depends on the coordinates.

In general, the parameters d_{xy}, d_{xz}, d_{yz} cannot be considered as dimensions of the boundaries of fractal media. The boundary of fractal medium can be fractal surface with dimension D_s. In the general case, D_s can be greater than two. The non-integer dimensions of the cross-section are always less than two

$$0 < d_{xy} < 2, \quad 0 < d_{xz} < 2, \quad 0 < d_{yz} < 2. \tag{11}$$

Similarly (10), we can define other two parameters $\alpha_2 = \alpha_y$ and $\alpha_3 = \alpha_z$.

Let $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ be the basis vectors that define a three-dimensional parallelepiped in $\mathbb{R}^3$. The parallelepiped region is the convex hull for these vectors

$$W_P := \left\{ \sum_{k=1}^{n=3} a_k\, \mathbf{L}_k : \quad 0 \le a_k \le 1 \right\}. \tag{12}$$

For simplification we will consider the rectangular parallelepiped only. Anisotropic fractal material can be characterized by the power-law relation for the mass $M_D(W_P)$ of the parallelepiped region (12) in the form

$$M_D(W_P) = M_0 \left(\frac{L_x}{R_0}\right)^{\alpha_1} \left(\frac{L_y}{R_0}\right)^{\alpha_2} \left(\frac{L_z}{R_0}\right)^{\alpha_3}, \tag{13}$$

where the parameter α_k is non-integer dimension along X_k-axis, $k = 1, 2, 3$, and $L_k = |\mathbf{L}_k|$ are the magnitudes of vectors $\mathbf{L}_k$, $k = 1, 2, 3$. The values L_k can be considered as three edges that meet at one vertex, and R_0 is a characteristic size of particles (atoms or molecules) of fractal medium. The parameter α_k describes how to increase the mass medium in the case of increasing the size of the parallelepiped region along one axis, when the parallelepiped sizes along other axes do not change. The sum $D = \alpha_1 + \alpha_2 + \alpha_3$ is called the dimension of the anisotropic fractal medium.

In general, the parameters $\alpha_k > 0$ ($k = 1, 2, 3$) can be either less than unity or greater than unity. In all cases, the following conditions should be satisfied

$$0 < \alpha_1 + \alpha_2 + \alpha_3 = D \leq 3. \tag{14}$$

For example, the conditions $0 < \alpha_k < 1$ for all $k \in \{1, 2, 3\}$ hold for fractal media similar to 3D Cantor dust. We assume that the parameter $\alpha_k > 1$ describes a fractal flow and motion of medium in X_k-direction. This possibility is based on an assumption that trajectories of the medium particles in the X_k-direction are fractal curve with the dimension $\alpha_k > 1$. For example, the Koch curve has $\alpha_k = \ln(4)/\ln(3) \approx 1.262$. We also assume that $\alpha_k > 1$ can be used for materials consisting of fractal molecular curves or fractal chains.[27]

C. Single-variable measure

For the fractional space approach and the non-integer dimensional space approach, we can use the single-variable measure

$$d\mu(\alpha, x) = c(\alpha)\, |x|^{\alpha - 1}\, dx, \tag{15}$$

where $\alpha > 0$ is a parameter that will be considered as a non-integer (fractional) dimension of the line, and $c(\alpha)$ is a function of α. Here, we take the absolute value of x in $|x|^{\alpha-1}$ to consider positive and negative values of x. For $\alpha = 1$, the numerical factor $c(\alpha)$ must be equal to 1 in order to have

$$d\mu(1, x) = dx.$$

Using the product measure approach for $\mathbb{R}^3$ with point coordinates x_1, x_2, x_3, the single-variable measures are

$$d\mu(\alpha_k, x_k) = c(\alpha_k)\, |x_k|^{\alpha_k - 1}\, dx_k. \tag{16}$$

In the product measure method we can use the following two ways to define a numerical factor $c(\alpha_k)$. In the first way of description, the factor is defined by a connection with integrals of non-integer orders α_k in the integer dimensional space. In the first way, the factor is defined by a connection with integrals in spaces with non-integer dimensions α_k along the X_k-axis.

Let us give the effective coordinates for these two cases.

(1) The fractional space approach is based on the use of the following new (effective) coordinates

$$Q_k = Q_k(\alpha_k, x_k) = \frac{1}{\Gamma(\alpha_k + 1)}\, \mathrm{sgn}(x_k)\, |x_k|^{\alpha}, \tag{17}$$

that is connected with the single-variable measure of the form

$$d\mu(\alpha_k, x_k) = dQ_k = \frac{1}{\Gamma(\alpha_k)}\, |x_k|^{\alpha_k - 1}\, dx_k, \tag{18}$$

where the numerical factor in the density of states is

$$c(\alpha) = \frac{1}{\Gamma(\alpha)}. \tag{19}$$

This form of $c(\alpha_k)$ is based on the connection of the Riemann-Liouville integrals of non-integer orders α_k.

(2) For non-integer dimensional space approach, we can use the effective coordinates

$$X_k = X_k(\alpha_k, x_k) = \frac{\pi^{\alpha_k/2}}{2\,\Gamma(\alpha_k/2 + 1)}\, \mathrm{sgn}(x_k)\, |x_k|^{\alpha_k}, \tag{20}$$

that is connected with the single-variable measure[61,67] of the form

$$d\mu(\alpha_k, x_k) = dX_k = \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)} |x_k|^{\alpha_k - 1} dx_k. \tag{21}$$

Here we take the density of states in the form

$$c(\alpha) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}, \tag{22}$$

such that the area of the sphere with radius r is equal to

$$S_{\alpha-1}(r) = \frac{2\,\pi^{\alpha/2}}{\Gamma(\alpha/2)} r^{\alpha-1}. \tag{23}$$

The absolute values $|x_k|$ can be interpreted as radii $r_k = |x_k|$ of sphere with non-integer dimension α_k. The presence of a factor of 2 for $S_{\alpha-1}$ in (23) is due to the fact that for $\alpha = 1$, the variable r is integrated from $-R$ to R, and when the limits are taken as 0 and R, one gets a factor of 2.

The space with coordinates (20) and the product of single-variable measures (21) can be considered as a non-integer dimensional space. The parameter $D = \alpha_1 + \alpha_2 + \alpha_3$ can be interpreted as a dimension of the space. For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we get $D = 3$, i.e., the dimension of space is the usual integer dimension. If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, where $0 < \alpha \leq 1$, we have a non-integer dimensional space for isotropic fractal materials. Regardless of the isotropic or the anisotropic case, the dimension of the space is given by

$$D = \alpha_1 + \alpha_2 + \alpha_3.$$

We have a non-integer dimensional space if at least one of the parameters α_k is not equal to 1.

D. Density of states

A connection between the single-variable measure $d\mu(\alpha_k, x_k)$ of non-integer dimensional space and the measure $d\mu(1, x_k)$ of integer dimensional space is

$$d\mu(\alpha_k, x_k) = c_1(\alpha_k, x_k)\, d\mu(1, x_k). \tag{24}$$

The functions $c_1(\alpha_k, x_k)$ should be considered as a density of states of fractal material[27] along the X_k-axis. We can define the density of states along the X_k-axis by the equation

$$c_1(\alpha_x, x) = \frac{c_3(D, x, y, z)}{c_2(d_{yz}, y, z)}, \tag{25}$$

where $c_3(D, x, y, z)$ is the density of state in the volume of material, and $c_2(d_{xy}, x, y)$, $c_2(d_{xz}, x, z)$, $c_2(d_{yz}, y, z)$ are density of states of the XY, XZ, YZ-cross-sections, respectively.

The interpretation of the functions $c_3(D, x, y, z)$,$c_2(d_{xy}, x, y)$, $c_2(d_{xz}, x, z)$, $c_2(d_{yz}, y, z)$, and $c_1(\alpha_k, x_k)$, $(k = 1, 2, 3)$, as densities of states has been suggested in Ref. 27. The density of states c_n describes how closely packed permitted states of particles in the space $\mathbb{R}^n$. In Ref. 27 the density of states is defined by integrations of non-integer orders that is also called the fractional integration. For $x_k \in [a_k; b_k]$ the function $c_1(\alpha_k, x_k)$ is defined in Ref. 27 as

$$c_1(\alpha_k, x_k) = \frac{1}{\Gamma(\alpha_k)} |x_k - a_k|^{\alpha_k - 1}. \tag{26}$$

This form is connected with the Riemann-Liouville fractional integral. In Ref. 33, an expression for $c_1(\alpha_k, x_k)$ has been suggested in the form

$$c_1(\alpha_k, x_k) = \alpha_k\, |b_k - x_k|^{\alpha_k - 1} \tag{27}$$

that does not contain the gamma function. This form is connected with the modified Riemann-Liouville fractional integral.

In this paper, we use integration in non-integer dimensional space instead of fractional integration. Then we should use the density of states in the form

$$c_1(\alpha_k, x_k) = \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)} |x_k|^{\alpha_k-1} \tag{28}$$

that is defined by the measure for integration in non-integer dimensional space. An application of density of states in the form (28) allows us to get the expression for lengths

$$\int_{-R}^{R} d\mu(\alpha, x) = 2\int_0^R d\mu(\alpha, x) = \frac{2\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)} R^\alpha = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2+1)} R^\alpha = V_\alpha(R) \tag{29}$$

that coincides with the well-known value for non-integer dimensional volume.

III. INTEGRATION IN FRACTIONAL AND NON-INTEGER DIMENSIONAL SPACES

A. Product spaces and product measures

The integral for non-integer dimensional space is defined for a single-variable in Ref. 103. It is useful for integrating spherically symmetric functions only. We can consider multiple variables by using the product spaces and product measures.[67]

Using a collection of $n = 3$ measurable sets (W_k, μ_k, D) with $k = 1, 2, 3$, we form a Cartesian product $W = W_1 \times W_2 \times W_3$ of the sets W_k. The definition of product measures and an application of Fubini's theorem gives a measure for the product set $W = W_1 \times W_2 \times W_3$ as

$$\mu_B(W) = (\mu_{\alpha_1} \times \mu_{\alpha_2} \times \mu_{\alpha_3})(W) = \prod_{k=1}^{n=3} \mu(\alpha_k, W_k). \tag{30}$$

Then integration over a function f on W is

$$\int_W f(x_1, x_2, x_3)\, d\mu_B = \int_{W_1}\int_{W_2}\int_{W_3} f(x_1, x_2, x_3) \prod_{k=1}^{n=3} d\mu(\alpha_k, x_k). \tag{31}$$

In this form, the single-variable measure may be used for each coordinate x_k, which has an associated non-integer dimension α_k, by the equation

$$d\mu(\alpha_k, x_k) = c_1(\alpha_k, x_k)\, dx_k, \quad (k = 1, 2, 3), \tag{32}$$

where $c_1(\alpha_k, x_k)$ is the density of states of the form

$$c_1(\alpha_k, x_k) = \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)} |x_k|^{\alpha_k-1}. \tag{33}$$

Note that we use $c_1(\alpha_k, x_k)$ without the factor 2.

Then the total dimension of $W = W_1 \times W_2 \times W_3$ is $D = \alpha_1 + \alpha_2 + \alpha_3$.

B. Reproduce the single-variable integration

Let us reproduce the result for the single-variable integration in the form

$$\int_W f(x_1, x_2, x_3)\, d\mu_B = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty f(r)\, r^{D-1}\, dr \tag{34}$$

for spherically symmetric function $f(x_1, x_2, x_3) = f(r)$ in $W_1 \times W_2 \times W_3$, where $r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. For this function, we can perform the integration in spherical coordinates (r, ϕ, θ). The Cartesian coordinates (x_1, x_2, x_3) can be expressed by the spherical coordinates (r, φ, θ), where $r \in [0, \infty)$, $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$, by:

$$x_1 = r\, \sin\theta\, \cos\varphi, \tag{35}$$

$$x_2 = r\ \sin\theta\ \sin\varphi, \tag{36}$$

$$x_3 = r\ \cos\theta. \tag{37}$$

In this case, Eq. (31) becomes

$$\int_W d\mu_B\ f(x_1, x_2, x_3) =$$

$$= A(\alpha_1, \alpha_2, \alpha_3) \int_{W_1} dx_1 \int_{W_2} dx_2 \int_{W_3} dx_3\, |x_1|^{\alpha_1 - 1} |x_2|^{\alpha_2 - 1} |x_3|^{\alpha_3 - 1}\, f(x_1, x_2, x_3) =$$

$$= A(\alpha_1, \alpha_2, \alpha_3) \int_0^\infty dr \int_0^{2\pi} d\varphi \int_0^\pi d\theta\ J(r, \theta)\ r^{\alpha_1+\alpha_2+\alpha_3-3}\, \cdot$$

$$\cdot\, |\cos\varphi|^{\alpha_1-1}\, |\sin\varphi|^{\alpha_2-1} |\sin\theta|^{\alpha_1+\alpha_2-1} |\cos\theta|^{\alpha_3-1}\ f(r),$$

where

$$A(\alpha_1, \alpha_2, \alpha_3) = \frac{\pi^{\alpha_1/2}}{\Gamma(\alpha_1/2)} \frac{\pi^{\alpha_2/2}}{\Gamma(\alpha_2/2)} \frac{\pi^{\alpha_3/2}}{\Gamma(\alpha_3/2)}, \tag{38}$$

and $J(r, \phi) = r^2 \sin\theta$ is the Jacobian of the coordinate change.

Since the function is only dependent on the radial variable and not the angular variables, we get the product of three integrals

$$\int_W d\mu_B\ f(x_1, x_2, x_3) = A(\alpha_1, \alpha_2, \alpha_3) \int_0^\infty\ f(r)\, r^{\alpha_1+\alpha_2+\alpha_3-1}\, \cdot$$

$$\cdot \int_0^{2\pi} d\varphi\, |\cos\varphi|^{\alpha_1-1}\, |\sin\varphi|^{\alpha_2-1} \int_0^\pi d\theta\, |\sin\theta|^{\alpha_1+\alpha_2-1}\, |\cos\theta|^{\alpha_3-1}. \tag{39}$$

Using Eq. (26) from Section 2.5.12 of Ref. 105, in the form

$$\int_0^{\pi/2} (\sin x)^{\mu-1} (\cos x)^{\nu-1}\, dx = \frac{\Gamma(\mu/2)\,\Gamma(\nu/2)}{2\,\Gamma((\mu+\nu)/2)}, \tag{40}$$

where $\mu > 0$, $\nu > 0$, we have

$$\int_0^{2\pi} d\varphi\, |\cos\varphi|^{\alpha_1-1}\, |\sin\varphi|^{\alpha_2-1} =$$

$$= 4 \int_0^{\pi/4} d\varphi\, (\cos\varphi)^{\alpha_1-1} (\sin\varphi)^{\alpha_2-1} = \frac{2\,\Gamma(\alpha_1/2)\,\Gamma(\alpha_2/2)}{\Gamma((\alpha_1+\alpha_2)/2)}, \tag{41}$$

$$\int_0^\pi d\theta\, |\sin\theta|^{\alpha_1+\alpha_2-1}\, |\cos\theta|^{\alpha_3-1} =$$

$$= 2 \int_0^{\pi/2} d\theta\, (\sin\theta)^{\alpha_1+\alpha_2-1} (\cos\theta)^{\alpha_3-1} = \frac{\Gamma((\alpha_1+\alpha_2)/2)\,\Gamma(\alpha_3/2)}{\Gamma((\alpha_1+\alpha_2+\alpha_3)/2)}. \tag{42}$$

Using $D = \alpha_1 + \alpha_2 + \alpha_3$, we obtain for $f(x_1, x_2, x_3) = f(r)$ the ralation

$$\int_W f(x_1, x_2, x_3)\, d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3) = \frac{2\,\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty f(r)\, r^{D-1}\, dr. \tag{43}$$

This equation describes the D-dimensional integration of a spherically symmetric function, and reproduces the result (34).

It is important to note that relation (43) holds only for density of states $c(\alpha_k, x_k)$ that corresponds to the non-integer dimensional space (33). Equation (43) cannot hold only for density of states $c_1(\alpha_k, x_k)$ suggested in Refs. 22 and 33.

IV. DIFFERENTIAL OPERATORS FOR FRACTIONAL AND NON-INTEGER DIMENSIONAL SPACES

A. Laplace operator for non-integer dimensional and fractional spaces

Let us consider possible definitions of the scalar Laplace operators for non-integer dimensional and fractional spaces.

(I) *Scalar Laplace operators for non-integer dimensional space*: The Laplacian operator for non-integer dimensional space has been suggested in Ref. 67 in the form

$$ {}^{S}\Delta^{(\alpha)} = \sum_{k=1}^{3} \Big(\frac{\partial^2}{\partial x_k^2} + \frac{\alpha_k - 1}{x_k} \frac{\partial}{\partial x_k} \Big), \tag{44}$$

where $(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$. Unfortunately, definitions of differential operators of first order such as gradient, divergence, curl operations are not considered in Ref. 67. The Laplacian operator (44) is used in Refs. 98–102. In Ref. 102, the gradient, divergence, and curl operators are suggested only as approximations of the square of the Laplace operator (9). The first order differential operator proposed in Ref. 102 is considered as an approximation of (44) in the form

$$ D_{\alpha_k,k} \approx \Big(\frac{\partial}{\partial x_k} + \frac{\alpha_k - 1}{2\, x_k} \Big). \tag{45}$$

For example, the gradient is

$$ \mathrm{grad}_D\, \varphi \approx \sum_{k=1}^{3} \Big(\frac{\partial \varphi}{\partial x_k} + \frac{\alpha_k - 1}{2\, x_k} \varphi \Big)\, \mathbf{e}_k, \tag{46}$$

and the divergence for vector field $\mathbf{u} = u_k\, \mathbf{e}_k$ is defined by

$$ \mathrm{div}_D\, \mathbf{u} \approx \sum_{k=1}^{3} \Big(\frac{\partial u_k}{\partial x_k} + \frac{\alpha_k - 1}{2\, x_k} u_k \Big). \tag{47}$$

Obviously, the corresponding scalar Laplacian

$$ \mathrm{div}_D\, \mathrm{grad}_D\, \varphi \approx \sum_{k=1}^{3} \left(\frac{\partial^2 \varphi}{\partial x_k^2} + \frac{\alpha_k - 1}{x_k} \frac{\partial \varphi}{\partial x_k} + \frac{(\alpha_k - 1)(\alpha_x - 3)}{4\, x_k^2} \varphi \right) \tag{48}$$

does not coincide with the operator (44).

(II) *Scalar Laplace operators for fractional space*: The scalar Laplace operators have been suggested by Calcagni and Nardelli[54] for fractional space-time. We can represent the suggested equations for fractional space in the following forms

$$ \mathcal{K}_1 \varphi = \sum_{k=1}^{3} \frac{1}{v(\alpha, x)} \frac{\partial}{\partial x_k} \left(v(\alpha, x) \frac{\partial \varphi}{\partial x_k} \right), \tag{49}$$

$$ \mathcal{K}_2 \varphi = \sum_{k=1}^{3} \frac{1}{\sqrt{v(\alpha, x)}} \frac{\partial^2}{\partial x_k^2} \Big(\sqrt{v(\alpha, x)}\, \varphi \Big), \tag{50}$$

and

$$ \mathcal{K}_{\alpha,l} \varphi = \sum_{k=1}^{3} \frac{(x_k)^{l-1/2}}{\sqrt{v(\alpha, x)}} \frac{\partial}{\partial x_k} \left((x_k)^{l-1/2} \frac{\partial}{\partial x_k} \Big((x_k)^{l-1/2} \sqrt{v(\alpha, x)}\, \varphi \Big) \right), \tag{51}$$

where $x_k > 0$, and $v(\alpha, x)$ is the isotropic measure weight

$$ v(\alpha, x) = \prod_{k=1}^{3} c_1(\alpha, x_k) = \prod_{k=1}^{3} \frac{(x_k)^{\alpha - 1}}{\Gamma(\alpha)}, \quad (x_k \geq 0). \tag{52}$$

Here $c_1(\alpha,\, x_k)$ is the density of states for fractional space.[27] We should note that Calcagni and Nardelli[54] consider the Laplace operators for Minkowski space-time $\mathbb{R}^4_{1,3}$. Equations (49)–(51) are given for the Euclidean space $\mathbb{R}^3$ to describe fractal material. Note that the expression (51) contains (49) and (50) as particular cases. The operators (49) and (50) can be represented as

$$\mathcal{K}_1 = \mathcal{K}_{\alpha, l=1-\alpha/2}, \quad \mathcal{K}_2 = \mathcal{K}_{\alpha, l=1/2}. \tag{53}$$

Substitution of (52) into (49)–(51) gives the Laplace operators in the forms

$$\mathcal{K}_1 \varphi = \sum_{k=1}^{3} \left(\frac{\partial^2 \varphi}{\partial x_k^2} + \frac{\alpha - 1}{x_k} \frac{\partial \varphi}{\partial x_k} \right), \tag{54}$$

$$\mathcal{K}_2 \varphi = \sum_{k=1}^{3} \left(\frac{\partial^2 \varphi}{\partial x_k^2} + \frac{\alpha - 1}{x_k} \frac{\partial \varphi}{\partial x_k} + \frac{(\alpha-1)(\alpha-3)}{4\, x_k^2} \varphi \right), \tag{55}$$

$$\mathcal{K}_{\alpha,l} \varphi = \sum_{k=1}^{3} \left(\frac{\partial^2 \varphi}{\partial x_k^2} + \frac{\alpha - 1}{x_k} \frac{\partial \varphi}{\partial x_k} + \frac{(\alpha-2)^2 - 4\, l^2}{4\, x_k^2} \varphi \right). \tag{56}$$

The Laplace operator (54) coincides with the expression (9) suggested in Ref. 67 The Laplace operator (55) coincides with the operator (48) that can be derived from the first order operators (46) and (46) suggested in Ref. 102. It was proved[54] that the case $l = 1/2$ is unique because it is only one, where the Laplace operator of the type (51) can be represented as the square of first order differential operator. This first order operator (see Eq. (3.43) in Ref. 54) is

$$\mathcal{D}_{\alpha,k} \varphi = \frac{1}{\sqrt{v(\alpha, x)}} \frac{\partial}{\partial x_k} \left(\sqrt{v(\alpha, x)}\, \varphi \right), \tag{57}$$

such that

$$\mathcal{K}_2 \varphi = \sum_{k=1}^{3} \left(\mathcal{D}_{\alpha,k} \right)^2 \varphi. \tag{58}$$

Substitution of (52) into (57) gives

$$\mathcal{D}_{\alpha,k} \varphi = \frac{\partial \varphi}{\partial x_k} + \frac{\alpha - 1}{2\, x_k} \varphi. \tag{59}$$

We can see that operator (59) can be considered as (45) with $\alpha_k = \alpha$ for all $k = 1, 2, 3$.

To anisotropic fractal media and materials in the framework of the non-integer dimensional space approach we should generalize the isotropic measure weight $v(\alpha, x)$ by using

$$v(\alpha, x) = \prod_{k=1}^{3} \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2 + 1)} |x_k|^{\alpha_k - 1} \tag{60}$$

instead of the expression (52) for fractional space. The derivative of the first order (59) should also be generalized for anisotropic fractal media as

$$\mathcal{D}_{\alpha,k} \varphi = \frac{\partial \varphi}{\partial x_k} + \frac{\alpha_k - 1}{2\, x_k} \varphi, \tag{61}$$

where we take into account different values of dimensions along the X_k-axis. Using the first order differential operator (61), it is easy to define the del operator, gradient, divergence, curl operators, and the vector Laplacian in order to describe fractal media in the framework of the non-integer dimensional space approach.

B. Approaches to formulation of the vector calculus in non-integer dimensional space

In the Stillinger's paper,[61] the scalar Laplace operator for non-integer dimensional space is suggested only. Generalizations of gradient, divergence, curl operators, and the vector Laplacian are not considered in Ref. 61. A generalization of the gradient, divergence, curl operators, the scalar and vector Laplace operators for non-integer dimensional space can be defined by the method of continuation in dimension Refs. 68 and 69. For simplification, only radial case is consider in Ref. 68, where the scalar and vector fields are independent of the angles, and the vector fields are directed along the radius vector. This simplification is analogous to consideration of the integration in non-integer dimensional space in Sec. 4 of Ref. 62.

The main advantage of the product measure approach to define the vector calculus for non-integer dimensional space is a possibility to describe anisotropic fractal materials. In this paper, we suggest differential vector operators for non-integer dimensional space by product measure method to describe anisotropic fractal media in the framework of continuum models. The differential operators are defined as an inverse operation to integration in non-integer dimensional space.

We can state that it is possible to reduce non-integer dimensional space to a power-law curved space by considering the density of states $c_1(\alpha_k, x_k)$ as the Lame coefficients

$$H_k = c_1(\alpha_k, x_k). \tag{62}$$

We consider Euclidean space for the effective coordinates $X_k(x_k)$, which are defined by (20), such that

$$d^2 s_X = \sum_{k=1}^{3} (dX_k)^2 = \sum_{k=1}^{3} c_1^2(\alpha_k, x_k)\,(dx_k)^2, \tag{63}$$

where $c_1(\alpha_k, x_k)$ are the density of states of the form

$$c_1(\alpha_k, x_k) = \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)}\, |x_k|^{\alpha_k - 1}. \tag{64}$$

Using (63), we can see that the densities of states (64) are the Lame coefficients

$$H_k = \sqrt{\sum_{k=1}^{n=3} \left(\frac{\partial X_i}{\partial x_k}\right)^2}. \tag{65}$$

It allows us to use the following well-known equations and definitions. The metric tensors of the Euclidean space in curvilinear coordinates x^k are

$$g_{kl}(x) = H_k^2\, \delta_{kl}, \quad g^{kl}(x) = \frac{1}{H_k^2}\, \delta_{kl}, \tag{66}$$

and indices can be raised and lowered by this metric

$$u^k = g^{kl}(x)\, u_l, \quad u_k = g_{kl}\, u^l. \tag{67}$$

For an orthogonal basis, we have

$$g = \det(g_{kl}) = \prod_{k=1}^{n=3} g_{kk} = \prod_{k=1}^{n=3} H_k^2, \tag{68}$$

and

$$J = \sqrt{|g(x)|} = \prod_{k=1}^{n=3} H_k = H_1\, H_2\, H_3. \tag{69}$$

Then the volume differential 3-form is given by

$$\mathrm{vol}_\alpha = \sqrt{|g(x)|}\, dx^1 \wedge dx^2 \wedge dx^3. \tag{70}$$

The correspondent volume for arbitrary set W in local coordinates is

$$\mathrm{vol}_\alpha(W) = \int_W \sqrt{|g(x)|}\, dx^1\, dx^2\, dx^3. \tag{71}$$

Using the metric tensor (66), the gradient of scalar function φ is the vector field

$$\mathrm{grad}_{(\alpha)}\, \varphi = \sum_{k,l=1}^{n=3} \mathbf{e}_k\, g^{kl}(x)\, \frac{\partial \varphi}{\partial x_l}, \tag{72}$$

and the divergence of vector field $\mathbf{u} = u^k\, \mathbf{e}_k$ is defined as the scalar function by

$$\mathrm{div}_{(\alpha)}\, \mathbf{u} = \sum_{k=1}^{n=3} \frac{1}{\sqrt{|g(x)|}} \frac{\partial(\sqrt{|g(x)|}\, u_k)}{\partial x^k}. \tag{73}$$

T the Laplace-Beltrami operator is defined as the divergence of the gradient. Using the definitions of the gradient and divergence, the Laplace-Beltrami operator applied to a scalar function φ is given in local coordinates by

$$\Delta_{(\alpha)}\varphi = \mathrm{div}_{(\alpha)}\, \mathrm{grad}_{(\alpha)}\, \varphi = \sum_{k=1}^{n=3} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g(x)|}\, g^{kl}(x) \frac{\partial \varphi}{\partial x^l} \right). \tag{74}$$

Using the effective coordinates $X_k = X_k(\alpha_k, x_k)$, which are defined in (20), we can define the nabla operator (the del operator) by

$$\nabla_{(\alpha)} = \sum_{k=1}^{3} \mathbf{e}_k\, \frac{\partial}{\partial X_k} = \sum_{k=1}^{3} \mathbf{e}_k\, \frac{1}{c_1(\alpha_k, x_k)}\, \frac{\partial}{\partial x_k}, \tag{75}$$

where $(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$ is the multi-index, and $c_1(\alpha_k, x_k)$ is defined by (64). We can use the well-known relations for the gradient, divergence, the curl operator, the scalar and vector Laplace operators through the Lame coefficients. The gradient for scalar field

$$\mathrm{grad}_{(\alpha)}\, \varphi = \sum_{k=1}^{3} \frac{1}{H_k^2} \frac{\partial \varphi}{\partial x_k}\, \mathbf{e}_k. \tag{76}$$

The divergence for vector field

$$\mathrm{div}_{(\alpha)}\, \mathbf{u} = \sum_{k=1}^{3} \frac{1}{H_1\, H_2\, H_3} \frac{\partial}{\partial x_k} \left(\frac{H_1\, H_2\, H_3}{H_k} u_k \right). \tag{77}$$

The curl operator for vector field

$$\mathrm{curl}_{(\alpha)}\, \mathbf{u} = \sum_{k,i,j=1}^{3} \frac{1}{H_1\, H_2\, H_3}\, \mathbf{e}_i \epsilon_{ijk}\, H_i \frac{\partial(H_k\, u_k)}{\partial x_j}, \tag{78}$$

where ϵ_{ijk} is the Levi-Civita symbol. The scalar Laplacian

$$\Delta_{(\alpha)}\varphi = \sum_{k=1}^{3} \frac{1}{H_1\, H_2\, H_3} \frac{\partial}{\partial x_k} \left(\frac{H_1\, H_2\, H_3}{H_k^2} \frac{\partial \varphi}{\partial x_k} \right). \tag{79}$$

Using that $H_k = c_1(\alpha_k, x_k)$, we get $\partial H_k/\partial x_l = 0$ for $k \neq l$, and the divergence

$$\mathrm{div}_{(\alpha)}\, \mathbf{u} = \sum_{k=1}^{3} \frac{1}{H_k} \frac{\partial u_k}{\partial x_k}. \tag{80}$$

The curl operator for vector field

$$\mathrm{curl}_{(\alpha)}\, \mathbf{u} = \sum_{k,i,j=1}^{3} \frac{1}{H_j}\, \mathbf{e}_i \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}, \tag{81}$$

where ϵ_{ijk} is the Levi-Civita symbol. The scalar Laplacian

$$\Delta_{(\alpha)}\varphi = \sum_{k=1}^{3} \frac{1}{H_k} \frac{\partial}{\partial x_k} \left(\frac{1}{H_k} \frac{\partial \varphi}{\partial x_k} \right). \tag{82}$$

We can define a differential operator that takes into account the density of states $c_1(\alpha_k, x_k)$ by

$$\partial_{x_k,\alpha_k} = \frac{\partial}{\partial X_k} = \frac{1}{c_1(\alpha_k, x_k)} \frac{\partial}{\partial x_k}, \tag{83}$$

where $c_1(\alpha_k, x_k)$ corresponds to the non-integer dimensionality along the X_k-axis and it is defined by (64). These derivatives cannot be considered as derivatives of non-integer orders (as fractional derivatives) or as fractal derivatives (derivatives on fractal set). The operators ∂_{x_k,α_k} are usual differential operators of the first order that is defined for differentiable functions on $\mathbb{R}^3$.

The form of derivatives (83) is analogous to differential operator suggested in Ref. 39. The main difference is that we define the operators (83) for the non-integer dimensional spaces in the framework of the measure product approach that is described by Palmer and Stavrinou[67] (see also Refs. 27, 48, and 49). The differential operators suggested in Ref. 39 are defined for modified Riemann-Liouville fractional integral of orders α_k in the integer dimensional space.

Using the operators (83), we can introduce generalized vector differential operations. The gradient

$$\mathrm{grad}_{(\alpha)}\, \varphi(\mathbf{r}) = \sum_{k=1}^{3} \mathbf{e}_k\, \partial_{x_k,\alpha_k} \varphi(\mathbf{r}), \tag{84}$$

where $\mathbf{e}_k$ are unit base vector of the Cartesian coordinate system. The divergence of the vector field $\mathbf{u}(\mathbf{r}) = \mathbf{e}_k\, u_k(\mathbf{r})$ is

$$\mathrm{div}_{(\alpha)}\, \mathbf{u}(\mathbf{r}) = \sum_{k=1}^{3} \partial_{x_k,\alpha_k}\, u_k(\mathbf{r}). \tag{85}$$

The curl for the vector field $\mathbf{u}(\mathbf{r}) = \mathbf{e}_k\, u_k(\mathbf{r})$ is

$$\mathrm{curl}_{(\alpha)}\, \mathbf{u}(\mathbf{r}) = \sum_{k,i,l=1}^{3} \mathbf{e}_i\, \epsilon_{ikl}\, \partial_{x_k,\alpha_k}\, u_l(\mathbf{r}), \tag{86}$$

where ϵ_{ikl} is the Levi-Civita symbol (or alternating symbol).

Using (84) and (85) we can define the second order differential operators such as the scalar Laplacian and vector Laplacian. The scalar Laplacian has the from

$${}^S\Delta_{(\alpha)}\varphi(\mathbf{r}) = \mathrm{div}_{(\alpha)}\, \mathrm{grad}_{(\alpha)}\, \varphi(\mathbf{r}). \tag{87}$$

The vector Laplacian[104] has the from

$${}^V\Delta_{(\alpha)}\mathbf{u}(\mathbf{r}) = \mathrm{grad}_{(\alpha)}\, \mathrm{div}_{(\alpha)}\, \mathbf{u}(\mathbf{r}) - \mathrm{curl}_{(\alpha)}\, \mathrm{curl}_{(\alpha)}\, \mathbf{u}(\mathbf{r}). \tag{88}$$

The scalar Laplacian (87) can be represented by using the usual partial derivatives

$${}^S\Delta_{(\alpha)}\varphi(\mathbf{r}) = \sum_{k=1}^{3} \frac{1}{c_1^2(\alpha_k, x_k)} \left(\frac{\partial^2 \varphi}{\partial x_k^2} - \frac{\alpha_k - 1}{x_k} \frac{\partial \varphi}{\partial x_k} \right), \tag{89}$$

where $c_1(\alpha_k, x_k)$ is density of states (64) along the X_k-axis for model with the non-integer dimensional space. It is easy to see that this operator does not coincide with the Laplace operators proposed by Palmer and Stavrinou,[67] Stillinger,[61] and Calcagni.[54] The main advantage of the suggested Laplace operator (89) is that first, it is obtained as the action of the gradient and divergence, and second, it is adapted for models with non-integer spatial dimensions.

The differential operators of the first order such as the gradient (76), the divergence (77), the curl operator (78), and the second order differential operators such as the scalar Laplacian (87), and the vector Laplacian (88), allow us to describe anisotropic fractal media and materials in the framework of continuum models with non-integer spatial dimensions.

V. EXAMPLES OF APPLICATION

A. Poisson's equation

Let us consider the Poisson's equation for a fractal medium that is distributed along the positive half-X-axis

$$ {}^{S}\Delta_{(\alpha)}\varphi(x) = f(x). \tag{90} $$

Here we use the Laplace operator suggested in this paper. Equation (90) for single-variable case can be written as

$$ \frac{1}{c_1^2(\alpha, x)}\left(\frac{\partial^2\varphi(x)}{\partial x^2} - \frac{\alpha-1}{x}\varphi(x)\right) = f(x) \quad (x > 0). \tag{91} $$

The general solution of this equation is

$$ \varphi(x) = C_1 + C_2\, x^{\alpha} - \frac{\pi^{\alpha}}{\alpha(\Gamma(\alpha/2))^2}\left(\int f(x)\, x^{2\alpha-1}\, dx - x^{\alpha}\int f(x)\, x^{\alpha-1}\, dx\right). \tag{92} $$

Let us compare this result with the solution of Poisson's equation Laplace operator $\mathcal{K}_2$. The Poisson's equation of the form

$$ \mathcal{K}_2\varphi(x) = f(x) \tag{93} $$

has the general solution

$$ \varphi(x) = C_1\, x^{(3-\alpha)/2} + C_2\, x^{(1-\alpha)/2} + x^{(3-\alpha)/2}\int f(x)\, x^{(1-\alpha)/2}\, dx - x^{(1-\alpha)/2}\int f(x)\, x^{(1+\alpha)/2}\, dx. \tag{94} $$

In Eq. (93), we use the Laplace operator (89) for single-variable case.

B. Timoshenko beam equations for fractal material

The Euler-Bernoulli beam theory gives a simplification of the linear theory of elasticity, which provides a means of calculating the load-carrying and deflection characteristics of beams and it covers the case for small deflections of a beam, which is subjected to lateral loads only. Note that the Timoshenko beam equation for fractal materials is discussed in Refs. 34 and 39.

In the Timoshenko beam theory without axial effects, the displacement vector $\mathbf{u}(x, y, z, t)$ of the beam are assumed to be given by

$$ u_x(x, y, z, t) = -z\,\varphi(x, t) \quad u_y(x, y, z, t) = 0, \quad u_z(x, y, t) = w(x, t), \tag{95} $$

where (x, y, z) are the coordinates of a point in the beam, u_x, u_y, u_z are the components of the displacement vector $\mathbf{u}$, $\varphi = \varphi(x, t)$ is the angle of rotation of the normal to the mid-surface of the beam, and $w = w(x, t)$ is the displacement of the mid-surface in the z-direction.

Using a model with non-integer dimensional space for fractal media, we can use the derivatives

$$ \partial_{x,\alpha} = c_1^{-1}(\alpha_x, x)\, D_x^1, \quad \partial_{x,\alpha}^n = (\partial_{x,\alpha})^n \quad (n \in \mathbb{N}), \tag{96} $$

where $c_1(\alpha_x, x)$ is defined for non-integer dimensional space by

$$ c_1(\alpha_x, x) = \frac{\pi^{\alpha_x/2}}{\Gamma(\alpha_x/2)}\, |x|^{\alpha_x-1} \tag{97} $$

instead of the usual derivatives D_x^1 and D_x^n for fractal materials. If we use the derivatives (96), then the Timoshenko equation for fractal beam can be derived from the force and moment balance equations

$$ \rho\, A\, D_t^2 w = \partial_{x,\alpha} Q, \quad \rho\, I^{(d)}\, D_t^2\varphi = Q - \partial_{x,\alpha} M, \tag{98} $$

with the bending moment

$$M = -E\, I^{(d)}\, \partial_{x,\alpha}\varphi, \tag{99}$$

and the shear force

$$Q = k\, G\, A \left(\partial_{x,\alpha} w - \varphi\right). \tag{100}$$

Here $I^{(d)}$ is the second moment of the fractal beam's cross-section.

The Timoshenko equations for homogeneous fractal beam has the form

$$\rho\, A\, D_t^2 w = k\, G\, A\, \partial_{x,\alpha}(\partial_{x,\alpha} w - \varphi), \tag{101}$$

$$\rho\, I^{(d)}\, D_t^2 \varphi = k\, G\, A\, (\partial_{x,\alpha} w - \varphi) + E\, I^{(d)}\, \partial^2_{x,\alpha}\varphi. \tag{102}$$

The Timoshenko fractal beam equations (101) and (102) can also be derived from the variational principle. The Lagrangian of the Timoshenko fractal beam has the form

$$\mathcal{L}_{GTFB} = \frac{1}{2}\rho\, I^{(d)}\, \left(D_t^1\varphi(x,t)\right)^2 + \frac{1}{2}\rho\, A\, \left(D_t^1 w(x,t)\right)^2 -$$

$$-\frac{1}{2}(kGA)\, \left(\partial_{x,\alpha} w(x,t) - \varphi(x,t)\right)^2 - \frac{1}{2}(EI^{(d)})\, \left(\partial_{x,\alpha}\varphi(x,t)\right)^2. \tag{103}$$

The stationary action principle gives the equations

$$\frac{\partial \mathcal{L}}{\partial w} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 w}\right) - D_x^1 \left(\frac{\partial \mathcal{L}}{\partial D_x^1 w}\right) = 0, \tag{104}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - D_t^1 \left(\frac{\partial \mathcal{L}}{\partial D_t^1 \varphi}\right) - D_x^1 \left(\frac{\partial \mathcal{L}}{\partial D_x^1 \varphi}\right) = 0. \tag{105}$$

Equations (104) and (105) are the Euler-Lagrange equation for the model of fractal material described by the Lagrangian (103). Substitution of (103) into Eqs. (104) and (105) gives the Timoshenko fractal beam equations (101) and (102) that can be presented as

$$\rho\, A\, D_t^2 w = k\, G\, A\, \partial_{x,\alpha} \left(\partial_{x,\alpha} w - \varphi\right), \tag{106}$$

$$\rho\, I^{(d)}\, D_t^2 \varphi = k\, G\, A \left(\partial_{x,\alpha} w - \varphi\right) + E\, I^{(d)}\, \partial^2_{x,\alpha}\varphi. \tag{107}$$

If $\alpha = 1$ then Eqs. (106) and (107) are the Timoshenko equations for beam with homogeneous non-fractal material.

For models with non-integer dimensional spaces, solutions of equations for fractal materials can be obtained from solutions of equations for non-fractal materials. Let $w_c(x, t)$ and $\varphi_c(x, t)$ be solutions of (106) and (107) with $\alpha = 1$ and $x > 0$, i.e., the Timoshenko equations for homogeneous non-fractal beam. Then solutions $w_F(x, t)$ and $\varphi_F(x, t)$ of equations (106) and (107) for fractal beam with $0 < \alpha < 1$ can be represented by

$$w_F(x,t) = w_c \left(\frac{\pi^{\alpha_x/2}}{\Gamma(\alpha_x/2)}\, |x|^{\alpha_x - 1}, t\right), \quad \varphi_F(x,t) = \varphi_c \left(\frac{\pi^{\alpha_x/2}}{\Gamma(\alpha_x/2)}\, |x|^{\alpha_x - 1}, t\right). \tag{108}$$

C. Euler-Bernoulli fractal beam

As an example, we consider the equation for the Euler-Bernoulli homogeneous fractal beam in the absence of a transverse load ($q(x) = 0$),

$$\rho\, A\, D_t^2 w(x,t) + E\, I^{(d)}\, \partial^4_{x,\alpha} w(x,t) = 0. \tag{109}$$

This equation can be solved using the Fourier decomposition of the displacement into the sum of harmonic vibrations of the form $w(x,t) = \mathrm{Re}[w(x)\exp(-i\omega t)]$, where ω is the frequency of vibration. Then, Eq. (109) gives the ordinary differential equation

$$-\rho\, A\, \omega^2 w(x) + E\, I^{(d)}\, \partial^4_{x,\alpha} w(x) = 0. \tag{110}$$

The boundary conditions for fractal beam of length L fixed at $x = 0$ are

$$w(0) = 0, \quad (\partial^1_{x,\alpha} w)(0) = 0, \tag{111}$$

$$(\partial^2_{x,\alpha} w)(L) = 0, \quad (\partial^3_{x,\alpha} w)(L) = 0. \tag{112}$$

The solution for the Euler-Bernoulli homogeneous fractal beam is defined by

$$w_{F,n}(x) = w_0 \left(\cosh(k_n x^\alpha) - \cos(k_n x^\alpha) + C_n(\alpha)\left(\sin(k_n x^\alpha) - \sinh(k_n x^\alpha)\right)\right), \quad x \in [0; L], \tag{113}$$

where w_0 is a constant, and

$$C_n(\alpha) = \frac{\cos(k_n L^\alpha) + \cosh(k_n L^\alpha)}{\sin(k_n L^\alpha) + \sinh(k_n L^\alpha)}, \qquad k_n = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \left(\frac{\rho\, A\, \omega_n^2}{E\, I^{(d)}}\right)^{1/4}. \tag{114}$$

For boundary conditions (111) and (112), the solution (113) exist only if k_n are defined by

$$\cosh(k_n L)\, \cos(k_n L) + 1 = 0. \tag{115}$$

This trigonometric equation is solved numerically. The corresponding natural frequencies of vibration are $\omega_n = k_n^2 \sqrt{(E\, I^{(d)})/\rho\, A}$. For a non-trivial value of the displacement, w_0 has to remain arbitrary, and the magnitude of the displacement is unknown for free vibrations. Usually $w_0 = 1$ is used when plotting mode shapes.

VI. CONCLUSION

We give a review of possible approaches to describe anisotropic fractal media. We focused on two approaches based on the fractional space and the non-integer dimensional space. There approaches allow us to describe anisotropic fractal media and materials in the framework of continuum models by using the concept of density of states[27] and the product measure method. Fractal medium is considered as a medium with non-integer mass dimension. The non-integer dimensionality is a main characteristic property of fractal materials. Therefore we suggest an application of differentiation and integration over non-integer dimensional spaces as natural way to describe fractal media.[68,69] Although, the non-integer dimension does not reflect all specific properties of real fractal materials, it allows us to formulate continuum models to derive important conclusions about the behavior of the media. In this paper a generalization of the vector calculus for multi-fractional and non-integer dimensional spaces is proposed as tools to describe anisotropic fractal media and materials in the framework of continuum models. We suggest a generalization of vector calculus for non-integer dimensional space that is product of spaces with different dimensions. The product measure method allows us to describe anisotropic fractal materials by taking into account various non-integer dimensions in different directions. The differential operators of the first order such as the gradient (76), the divergence (77), the curl operator (78), and the second order differential operators such as the scalar Laplacian (87), and the vector Laplacian (88), are suggested to describe anisotropic fractal media and materials by continuum models with non-integer dimensional spaces. To demonstrate some simple applications of proposed approach to the description of fractal materials, we consider the Poisson's equation for fractal medium, the Euler-Bernoulli fractal beam and the Timoshenko beam equations for fractal material.

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