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Fractional-order difference equations for physical lattices and some applications

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Fractional-order operators for physical lattice models based on the Grünwald-Letnikov fractional differences are suggested. We use an approach based on the models of lattices with long-range particle interactions. The fractional-order operators of differentiation and integration on physical lattices are represented by kernels of lattice long-range interactions. In continuum limit, these discrete operators of non-integer orders give the fractional-order derivatives and integrals with respect to coordinates of the Grünwald-Letnikov types. As examples of the fractional-order difference equations for physical lattices, we give difference analogs of the fractional nonlocal Navier-Stokes equations and the fractional nonlocal Maxwell equations for lattices with long-range interactions. Continuum limits of these fractional-order difference equations are also suggested. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4933028]

I. INTRODUCTION

Derivatives and integrals of non-integer order[1–5] have a long history of over 300 years that is connected with the names of famous scientists such as Leibniz, Liouville, Riemann, Grünwald, Letnikov, and Riesz. Fractional-order differential and integral equations have a lot of application in mechanics and physics.[6–11] Fractional derivatives and integrals are very important to describe processes in nonlocal media and nonlocal continuously distributed systems. As it was shown in Refs. 14–18, the continuum equations with fractional derivatives are directly connected to lattice models with long-range interactions. Therefore, fractional-order differential and integral operators can be used for different systems with long-range power-law interactions.[9] Linear and nonlinear systems with long-range interactions are well known in physics and mechanics.[12,13,9] The lattice equations for fractional nonlocal continuum and the corresponding continuum equations have been considered in Refs. 21–28 and Refs. 29–34.

Differences of non-integer orders and the correspondent fractional derivatives have been first proposed by Grünwald[19] in 1867 and independently by Letnikov[20] in 1868. These differences of fractional orders are infinite differences that are defined by infinite series (see Section 20 in Ref. 1) as a generalization of the usual finite difference of integer orders. Now, these differences and derivatives are called the Grünwald-Letnikov fractional differences and derivatives.[1–3] One-dimensional lattice models with long-range interactions of the Grünwald-Letnikov type and the correspondent fractional differential and integral continuum equations have been suggested in Ref. 30. The suggested form of long-range interaction is based on the form of the left-sided and right-sided Grünwald-Letnikov fractional differences.

In this paper, we use an approach that is based on the lattice models with long-range particle interactions and its continuum limit. We propose a generalization of the models considered in Ref. 30 to formulate a fractional difference calculus for physical lattices.[17,18] The continuum limits of the suggested fractional operators are described by the fractional derivatives of the

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Grünwald-Letnikov type. As examples of applications of the suggested approach, we consider the fractional-order difference equations for physical lattice models with long-range interactions and correspondent continuum limits for the fractional generalizations of the Navier-Stokes equations and the Maxwell equations of nonlocal continuous media.

II. GRÜNWALD-LETNIKOV FRACTIONAL-ORDER DIFFERENCES, DERIVATIVES, AND INTEGRALS

In this section, we briefly describe the Grünwald-Letnikov fractional differences and derivatives to fix notation for further consideration. For more details, can refer to Refs. 1–3.

The difference of a fractional order and the correspondent fractional derivatives has been introduced by Grünwald in 1867 and independently by Letnikov in 1868. Definitions of differences of non-integer orders are based on a generalization of the usual difference of integer orders. The difference of positive real order $\alpha \in \mathbb{R}_+$ is defined by the infinite series (see Section 20 in Ref. 1).

Definition 1. The Grünwald-Letnikov fractional-order differences $\nabla^{\alpha}_{a,\pm}$ of order $\alpha \in \mathbb{R}_+$ with step $a > 0$ are defined by the equation

$$\nabla^{\alpha}_{a,\pm} u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, \Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} \, u(x \mp na), \quad (a > 0). \tag{1}$$

The difference $\nabla^{\alpha}_{a,+}$ is called left-sided fractional difference, and $\nabla^{\alpha}_{a,-}$ is called a right-sided fractional difference.

We note that the series in (1) converges absolutely and uniformly for every bounded function $u(x)$ and $\alpha > 0$.

Let us give some basic properties of fractional-order differences $\nabla^{\alpha}_{a,\pm}$.

1. For the fractional difference, the semigroup property

$$\nabla^{\alpha}_{a,\pm} \nabla^{\beta}_{a,\pm} u(x) = \nabla^{\alpha+\beta}_{a,\pm} u(x), \quad (\alpha > 0, \quad \beta > 0) \tag{2}$$

is valid for any bounded function $u(x)$ (see Property 2.29 in Ref. 3).

2. The Fourier transform $\mathcal{F}$ of the fractional differences $\nabla^{\alpha}_{a,\pm}$ is

$$\mathcal{F}\{\nabla^{\alpha}_{a,\pm} u(x)\}(k) = (1 - \exp\{\mp i\, k\, a\})^{\alpha} \mathcal{F}\{u(x)\}(k) \tag{3}$$

for any function $u(x) \in L_1(\mathbb{R})$ (see Property 2.30 in Ref. 3) and the step $a > 0$. This expression will be used by us to prove that the proposed expression of long-range interactions of the Grünwald-Letnikov type are the power-law lattice interactions.

3. Using $\Gamma(n+1) = n!$ and $\lim_{x \to -k} 1/\Gamma(x+1) = 0$ for $k \in \mathbb{N}$, we get that Grünwald-Letnikov differences (1) with $\alpha = m \in \mathbb{N}$ are the well-known finite differences of integer-order m. The marked property of the gamma function ($|\Gamma(x+1)| \to \infty$ for $x \to -k$, $k \in \mathbb{N}$) leads us to the fact that all terms with $n \geq m+1$ vanish in infinite sum (1) if $\alpha = m \in \mathbb{N}$. For integer values of $\alpha = m \in \mathbb{N}$, the differences $\nabla^{\alpha}_{a,\pm}$ are represented by the finite series

$$\nabla^{m}_{a,\pm} u(x) = \sum_{n=0}^{m} \frac{(-1)^n \, m!}{n!\,(m-n)!} \, u(x \mp n\, a), \quad (a \in \mathbb{R}_+). \tag{4}$$

This property of fractional-order differences (1) was first proved by Letnikov[20] in 1868 (see also Section 2.8 of Ref. 3).

Definition 2. The left- and right-sided Grünwald-Letnikov derivatives of order $\alpha > 0$ are defined by the equation

$$^{GL}D^{\alpha}_{x,\pm} u(x) = \lim_{a \to 0+} \frac{\nabla^{\alpha}_{a,\pm} u(x)}{|a|^{\alpha}}. \tag{5}$$

Let us give some basic properties of the fractional-order derivatives $^{GL}D^{\alpha}_{x,\pm}$.

1. For integer values of $\alpha = m \in \mathbb{N}$, Grünwald-Letnikov derivatives (5) are equal to the usual derivatives of integer order m up to the sign in the form

$$^{GL}D^m_{x,\pm}u(x) = (\pm 1)^m \frac{d^m u(x)}{dx^m}, \quad (\alpha > 0). \tag{6}$$

2. The Grünwald-Letnikov fractional derivatives coincide with the Marchaud fractional derivatives for the functions $u(x) \in L_r(\mathbb{R})$, where $1 \leqslant r < \infty$ (see Theorem 20.4 in Ref. 1).

3. It is interesting that fractional-order differences (1) can be used not only to define fractional derivatives, but also to define the fractional-order integrals. Equation (5) with $\alpha < 0$ defines the Grünwald-Letnikov fractional integral (see Section 20 in Ref. 1 and Section 2.2 in Ref. 2) if the functions $u(x)$ satisfy the condition

$$|u(x)| < c(1 + |x|)^{-\mu}, \quad \mu > |\alpha|. \tag{7}$$

It allows us to have a common approach to define the differential and integral operators on the physical lattices.

III. FRACTIONAL-ORDER DIFFERENCE OPERATORS FOR UNBOUNDED PHYSICAL LATTICES WITH LONG-RANGE INTERACTION

As a model of physical lattices, we consider an unbounded physical lattice that is characterized by N non-coplanar vectors $\mathbf{a}_j$, where $j = 1, \ldots, N$. These vectors are the shortest vectors by which a lattice can be displaced such that this lattice is brought back into itself. For simplification, we will consider the mutually perpendicular primitive lattice vectors $\mathbf{a}_i$. This simplification means that the lattice is an N-dimensional analog of the primitive orthorhombic Bravais lattice. We choose the basis vectors $\mathbf{e}_j$ for the Cartesian coordinate system for $\mathbb{R}^N$ such that $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$, where $j = 1, \ldots, N$. The position of the lattice sites is defined by the vectors $\mathbf{r}(\mathbf{n}) = \sum_{j=1}^N n_j \mathbf{a}_j$, where n_j are integer numbers. The vector $\mathbf{n} = (n_1, \ldots, n_N)$ can be used for numbering the lattice sites and the corresponding lattice particles. In the lattice model, the equilibrium positions of these particles coincide with the lattice sites. In general case, the vectors $\mathbf{r}(\mathbf{n})$ of lattice sites differ from position of the corresponding particles, when the particles are displaced from the equilibrium positions. To define the positions of lattice particles, we define the displacement vector field $\mathbf{u}(\mathbf{n},t) = \sum_{j=1}^N u_j(\mathbf{n},t)\,\mathbf{e}_j$, or the scalar displacement field $u(\mathbf{n},t)$. The components $u_j(\mathbf{n},t) = u_j(n_1, \ldots, n_N, t)$ of the displacement vector $\mathbf{u}(\mathbf{n},t)$ are the functions of the vector $\mathbf{n} = (n_1, \ldots, n_N)$ and time t. For simplification, fractional-order difference operators for the physical lattices will be defined for scalar bounded functions $u = u(\mathbf{n},t) = u(n_1, \ldots, n_N, t)$ that are defined for all $n_j \in \mathbb{Z}$, where $j = 1, 2, \ldots, N$. All expressions can be easily rewritten for the case of the vector bounded functions $\mathbf{u}(\mathbf{n},t)$.

Let us define fractional difference operators of the Grünwald-Letnikov type for unbounded physical lattice in the direction $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$.

Definition 3. Fractional-order difference operators of the Grünwald-Letnikov type for unbounded lattice are the operators $^{GL}\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha \\ j \end{bmatrix}$ *and* $^{GL}\mathbb{I}^{\pm}_L \begin{bmatrix} \alpha \\ j \end{bmatrix}$ *that act on the function* $u(\mathbf{m})$ *as*

$$^{GL}\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{GL}K^{\pm}_{\alpha}(n_j - m_j)\, u(\mathbf{m}) \quad (\alpha > 0, \quad j = 1, \ldots, N), \tag{8}$$

$$^{GL}\mathbb{I}^{\pm}_L \begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{GL}L^{\pm}_{\alpha}(n_j - m_j)\, u(\mathbf{m}) \quad (\alpha > 0, \quad j = 1, \ldots, N), \tag{9}$$

where $u(\mathbf{m})$ *is a bounded function defined on the whole lattice,* $a_j = |\mathbf{a}_j|$*, and the interaction kernels* $^{GL}K^{\pm}_{\alpha}(n)$ *and* $^{GL}L^{\pm}_{\alpha}(n)$ *are defined by the equations*

$$^{GL}K^{\pm}_{\alpha}(n) = \frac{(-1)^n\, \Gamma(1+\alpha)\, (H[n] \pm H[-n])}{2\, \Gamma(|n|+1)\, \Gamma(1+\alpha-|n|)}, \quad (\alpha > 0), \tag{10}$$

$$^{GL}L^{\pm}_{\alpha}(n) = \frac{(-1)^n\, \Gamma(1-\alpha)\, (H[n] \pm H[-n])}{2\, \Gamma(|n|+1)\, \Gamma(1-\alpha-|n|)}, \quad (\alpha > 0), \tag{11}$$

and $\Gamma(z)$ is the gamma function, $H[n]$ is the Heaviside step function. The parameter α is called the order of these operators.

Remark 1. In expressions (10) and (11), we use the Heaviside step function $H[n]$ (also called the unit step function) of a discrete variable n that is defined by the equation

$$H[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} \tag{12}$$

where n is an integer. Note that the definition of $H[0] = 1$ for discrete variable Heaviside function is significant. This allows us to write the kernels ${}^{GL}K^{+}_{\alpha}(n)$ and ${}^{GL}L^{+}_{\alpha}(n)$ in the simple form without allocating repeated zero terms.

Remark 2. For integer values $\alpha = m \in \mathbb{N}$ the expression of kernel (11) corresponds to the indeterminate form ∞/∞ since the gamma function $\Gamma(z)$ has simple poles at $z = -n$ ($n \in \mathbb{N}$). In order to eliminate this uncertainty, we should use

$$\frac{\Gamma(1-m)}{\Gamma(1-m-|n|)} = (-m)(-m-1)\ldots(-m-|n|) = \frac{(-1)^{|n|+1}\Gamma(m+1+|n|)}{\Gamma(m)}. \tag{13}$$

Remark 3. To demonstrate the properties of (10), we visualize the functions

$$f_{\pm}(x,y) = {}^{GL}K^{\pm}_{y}(x) = \frac{Re[(-1)^{x}]\,\Gamma(1+y)\,(H[x] \pm H[-x])}{2\,\Gamma(|x|+1)\,\Gamma(1+y-|x|)} \tag{14}$$

with continuous variables x and y by Figures 1 and 2. The functions $f_{+}(x,y)$ and $f_{-}(x,y)$, which are defined by (14), are presented by Figures 1 and 2, respectively. To demonstrate the properties of (11), we visualize the functions

$$g_{\pm}(x,y) = {}^{GL}L^{\pm}_{y}(x) = \frac{Re[(-1)^{x}]\,\Gamma(1-y)\,(H[x] \pm H[-x])}{2\,\Gamma(|x|+1)\,\Gamma(1-y-|x|)} \tag{15}$$

with continuous variables x and y by Figures 3 and 4. The functions $g_{+}(x,y)$ and $g_{-}(x,y)$, which are defined by (15), are presented by Figures 3 and 4, respectively.

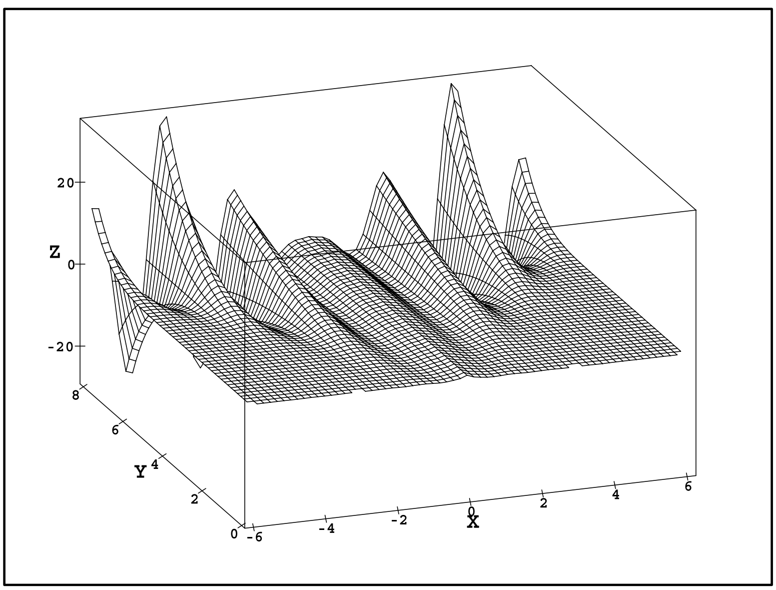

FIG. 1. Plot of the function $f_{+}(x,y)$ (14) for the range $x \in [-6;+6]$ and $y = \alpha \in [0;8]$ that represents the kernels ${}^{GL}K^{+}_{y}(x)$ of the fractional-order difference operators of the Grünwald-Letnikov type ${}^{GL}\mathbb{D}^{+}_{L}\left[{\alpha \atop j}\right]$ with $\alpha = y$.

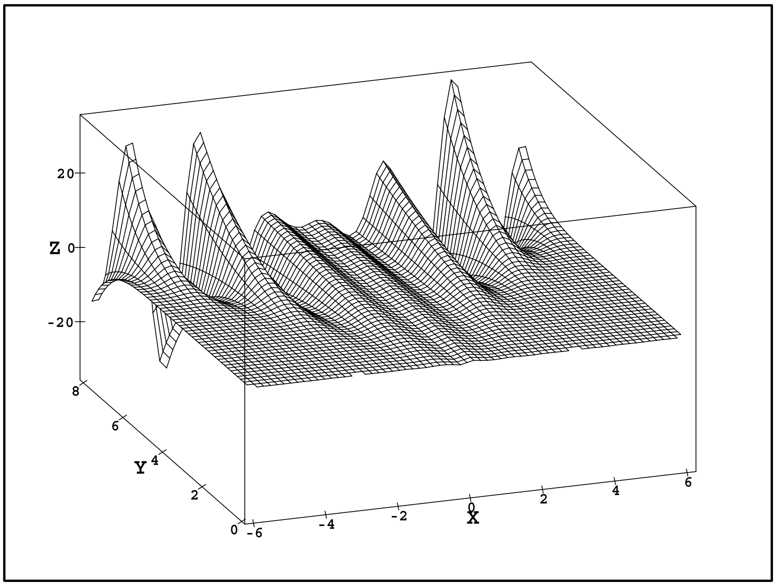

FIG. 2. Plot of the function $f_{-}(x,y)$ (14) for the range $x \in [-6;+6]$ and $y = \alpha \in [0;8]$ that represents the kernels ${}^{GL}K^{-}_{y}(x)$ of the fractional-order difference operators of the Grünwald-Letnikov type ${}^{GL}\mathbb{D}^{-}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ with $\alpha = y$.

Remark 4. Fractional-order difference operators (8) and (9) can be called a lattice fractional partial derivative and a lattice fractional integral in the direction $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$. Therefore, fractional-order difference operators (8) and (9) depend on the vector $\mathbf{n} = \sum_{j=1}^{N} n_j \mathbf{e}_j$ such that

$$ {}^{GL}\mathbb{D}^{+}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \left({}^{GL}\mathbb{D}^{+}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u\right)(\mathbf{n}), \tag{16} $$

$$ {}^{GL}\mathbb{I}^{+}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \left({}^{GL}\mathbb{I}^{+}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u\right)(\mathbf{n}), \tag{17} $$

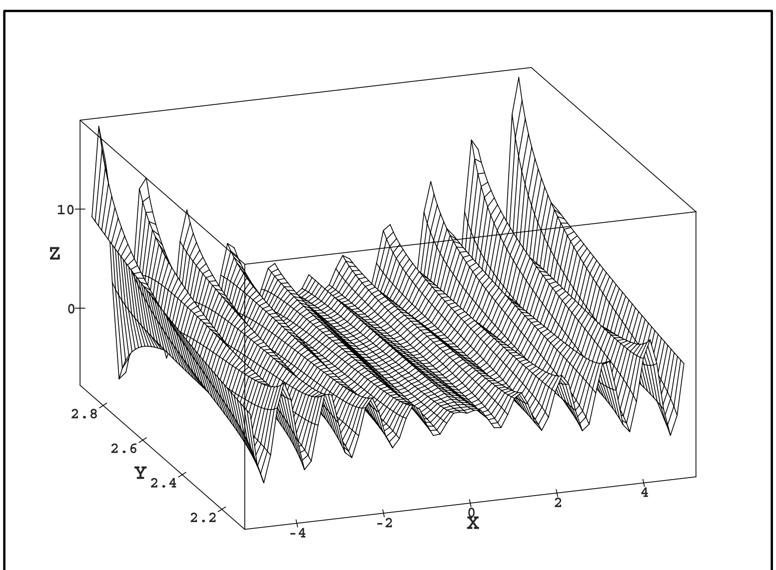

FIG. 3. Plot of the function $g_{+}(x,y)$ (15) for the range $x \in [-5;+5]$ and $y = \alpha \in [2.1, 2.9]$ that represents the kernels ${}^{GL}L^{+}_{y}(x)$ of the fractional-order difference operators of the Grünwald-Letnikov type ${}^{GL}\mathbb{I}^{+}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ with $\alpha = y$.

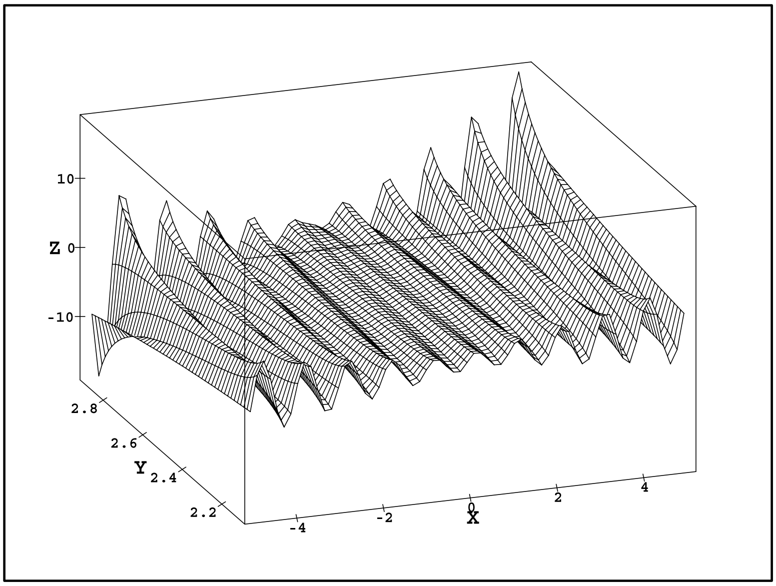

FIG. 4. Plot of the function $g_{-}(x,y)$ (15) for the range $x \in [-5;+5]$ and $y = \alpha \in [2.1;2.9]$ that represents the kernels ${}^{GL}L^{-}_{y}(x)$ of the fractional-order difference operators of the Grünwald-Letnikov type ${}^{GL}\mathbb{I}^{-}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ with $\alpha = y$.

where $n_i = m_i$ for $i \neq j$, i.e., $\mathbf{n} = \mathbf{m} + (n_j - m_j)\mathbf{e}_j$.

Remark 5. It should be noted that one-dimensional lattice models with the long-range interaction of the form ${}^{GL}K^{+}_{\alpha}(n)$ and correspondent fractional nonlocal continuum models have been suggested in Ref. 30, (see also Ref. 9).

Remark 6. It is easy to see that the kernels ${}^{GL}K^{\pm}_{\alpha}(n)$ and ${}^{GL}L^{\pm}_{\alpha}(n)$ are even and odd functions

$$ {}^{GL}K^{\pm}_{\alpha}(-n) = \pm\, {}^{GL}K^{\pm}_{\alpha}(n), \qquad {}^{GL}L^{\pm}_{\alpha}(-n) = \pm\, {}^{GL}L^{\pm}_{\alpha}(n). $$

The form of these fractional-order difference operators, which are defined by (8) with (10) and (9) with (11), can be represented as the addition and subtraction of the Grünwald-Letnikov fractional differences defined by (1). Let us prove the following proposition.

Proposition 1. The fractional-order difference operators of the Grünwald-Letnikov type, which are defined by (8) with (10) and (9) with (11), can be represented in the form

$$ {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{|a_j|^{\alpha}} \sum_{m_j=0}^{\infty} \frac{(-1)^{m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m_j+1)\Gamma(1+\alpha-m_j)} \Big(u(\mathbf{n}-m_j\mathbf{e}_j) \pm u(\mathbf{n}+m_j\mathbf{e}_j)\Big), \quad (18) $$

$$ {}^{GL}\mathbb{I}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{|a_j|^{\alpha}} \sum_{m_j=0}^{\infty} \frac{(-1)^{m_j}\,\Gamma(1-\alpha)}{2\,\Gamma(m_j+1)\Gamma(1-\alpha-m_j)} \Big(u(\mathbf{n}-m_j\mathbf{e}_j) \pm u(\mathbf{n}+m_j\mathbf{e}_j)\Big), \quad (19) $$

where $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$ *and* $\alpha > 0$.

Proof. Let us first prove this proposition for operator (8) with kernel (10).

Using (12), Equation (8) can be rewritten in the form

$$ {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{GL}K^{\pm}_{\alpha}(n_j-m_j)\, u(\mathbf{m}) = $$

$$ = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} \frac{(-1)^{n_j-m_j}\,\Gamma(1+\alpha)\,(H[n_j-m_j] \pm H[-(n_j-m_j)])}{2\,\Gamma(|n_j-m_j|+1)\,\Gamma(1+\alpha-|n_j-m_j|)}\, u(\mathbf{m}) = $$

$$= \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} \frac{(-1)^{n_j-m_j}\,\Gamma(1+\alpha)\,H[n_j-m_j]}{2\,\Gamma(|n_j-m_j|+1)\,\Gamma(1+\alpha-|n_j-m_j|)}\, u(\mathbf{m}) \pm$$

$$\pm \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} \frac{(-1)^{n_j-m_j}\,\Gamma(1+\alpha)\,H[m_j-n_j]}{2\,\Gamma(|n_j-m_j|+1)\,\Gamma(1+\alpha-|n_j-m_j|)}\, u(\mathbf{m}) =$$

$$= \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{n_j} \frac{(-1)^{n_j-m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(|n_j-m_j|+1)\,\Gamma(1+\alpha-|n_j-m_j|)}\, u(\mathbf{m}) \pm$$

$$\pm \frac{1}{a_j^{\alpha}} \sum_{m_j=n_j}^{+\infty} \frac{(-1)^{n_j-m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(|n_j-m_j|+1)\,\Gamma(1+\alpha-|n_j-m_j|)}\, u(\mathbf{m}).$$

Let us use new variables $m'_j = n_j - m_j$ and $m''_j = m_j - n_j$, and then introduce the redesignation $m'_j \to m_j$ and $m''_j \to m_j$. This leads to the following expressions:

$${}^{GL}\mathbb{D}^{\pm}_{L} \begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m'_j=0}^{+\infty} \frac{(-1)^{m'_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m'_j+1)\,\Gamma(1+\alpha-m'_j)}\, u(\mathbf{n}-m'_j\mathbf{e}_j) \pm$$

$$\pm \frac{1}{a_j^{\alpha}} \sum_{m''_j=0}^{+\infty} \frac{(-1)^{m''_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m''_j+1)\,\Gamma(1+\alpha-m''_j)}\, u(\mathbf{n}+m''_j\mathbf{e}_j) =$$

$$= \frac{1}{a_j^{\alpha}} \sum_{m_j=0}^{+\infty} \frac{(-1)^{m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m_j+1)\,\Gamma(1+\alpha-m_j)}\, u(\mathbf{n}-m_j\mathbf{e}_j) \pm$$

$$\pm \frac{1}{a_j^{\alpha}} \sum_{m_j=0}^{+\infty} \frac{(-1)^{m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m_j+1)\,\Gamma(1+\alpha-m_j)}\, u(\mathbf{n}+m_j\mathbf{e}_j) =$$

$$= \frac{1}{a_j^{\alpha}} \sum_{m_j=0}^{+\infty} \frac{(-1)^{m_j}\,\Gamma(1+\alpha)}{2\,\Gamma(m_j+1)\,\Gamma(1+\alpha-m_j)} \Big(u(\mathbf{n}-m_j\mathbf{e}_j) \pm u(\mathbf{n}+m_j\mathbf{e}_j) \Big).$$

As a result, we get representation (18). Representation (19) is proved similarly. □

Remark 7. Note that expressions (18) and (19) contain the summation over non-negative values m_j, in contrast to the sum over all integer values in Equations (8) and (9).

Definition 4. An interaction of lattice particles is called the interaction of power-law type if the kernel $K(n-m)$ of this interaction satisfies the conditions

$$\lim_{k\to 0+} \frac{\hat{K}_{\alpha}(k)}{k^{\alpha}} = A_{\alpha}, \quad \alpha > 0, \quad 0 < |A_{\alpha}| < \infty, \tag{20}$$

where $\hat{K}_{\alpha}(k)$ is the Fourier series transform of the kernel $K(n)$ such that

$$\hat{K}_{\alpha}(k\,a) = \sum_{n=-\infty}^{+\infty} K(n)\, e^{-i\,k\,n\,a}. \tag{21}$$

Let us prove that the suggested long-range interactions of the Grünwald-Letnikov type with kernels (10) and (11) are the interaction of power-law type.

Proposition 2. The long-range interactions of the Grünwald-Letnikov type, which are defined by Equations (8) and (9), are the interaction of power-law type.

Proof. Let us first prove this proposition for operator (8) with kernel (10). Using (3) and (18), we get

$${}^{GL}\hat{K}^{\pm}_{\alpha}(k\,a) = \sum_{n=-\infty}^{+\infty} {}^{GL}K^{\pm}_{\alpha}(n)\, e^{-ik\,n\,a} = \frac{1}{2}\Big((1-\exp\{i\,k\,a\})^{\alpha} \pm (1-\exp\{-i\,k\,a\})^{\alpha} \Big). \tag{22}$$

Then limit (20) gives

$$\lim_{k\to 0+} \frac{{}^{GL}\hat{K}^{\pm}_{\alpha}(a\,k)}{k^{\alpha}} = \frac{1}{2} a^{\alpha} \Big((-i)^{\alpha} \pm (i)^{\alpha} \Big). \tag{23}$$

Using the Euler's formula $\exp(\pm i\,x) = \cos(x) \pm i\, \sin(x)$, limits (23) have the forms

$$\lim_{k\to 0+} \frac{{}^{GL}\hat{K}^{+}_{\alpha}(k)}{k^{\alpha}} = \cos\left(\frac{\pi\,\alpha}{2}\right), \tag{24}$$

$$\lim_{k\to 0+} \frac{{}^{GL}\hat{K}^{-}_{\alpha}(k)}{k^{\alpha}} = -\,i\, \sin\left(\frac{\pi\,\alpha}{2}\right). \tag{25}$$

As a result, we get $A_{\alpha} = \cos(\pi\,\alpha/2)$ for the long-range interactions with the kernel ${}^{GL}K^{+}_{\alpha}(n)$, and $A_{\alpha} = -\,i\, \sin(\pi\,\alpha/2)$ for the long-range interactions with the kernel ${}^{GL}K^{-}_{\alpha}(n)$. Similarly, considering the negative values of α, we get that the interactions with the kernel ${}^{GL}L^{\pm}_{\alpha}(n)$ are power-law interactions. Therefore, the long-range interactions of the Grünwald-Letnikov type are power-law lattice interactions. □

The suggested fractional-order difference operators can be extended for bounded physical lattice models in the following form.

Definition 5. Fractional-order difference operators of the Grünwald-Letnikov type for bounded lattice with $m_j \in [m^1_j; m^2_j]$ ($m^1_j \le m_j \le m^2_j$) *are the operators* ${}^{GL}_{B}\mathbb{D}^{\pm}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ *and* ${}^{GL}_{B}\mathbb{I}^{\pm}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ *that act on the function* $u(\mathbf{m})$ *as*

$${}^{GL}_{B}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a^{\alpha}_j} \sum_{m_j=m^1_j}^{m^2_j} {}^{GL}K^{\pm}_{\alpha}(n_j - m_j)\, u(\mathbf{m}) \quad (j = 1, \ldots, N), \tag{26}$$

$${}^{GL}_{B}\mathbb{I}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a^{\alpha}_j} \sum_{m_j=m^1_j}^{m^2_j} {}^{GL}L^{\pm}_{\alpha}(n_j - m_j)\, u(\mathbf{m}) \quad (j = 1, \ldots, N), \tag{27}$$

where $u(\mathbf{m})$ *is a bounded function,* $a_j = |\mathbf{a}_j|$, *and the interaction kernels* ${}^{GL}K^{\pm}_{\alpha}(n)$ *and* ${}^{GL}L^{\pm}_{\alpha}(n)$ *are defined by Equations (10) and (11).*

Remark 8. The suggested forms of fractional difference operators for bounded physical lattice models are based on the Grünwald-Letnikov fractional differences on finite intervals (see Section 20.4 in Ref. 1). For the finite interval $[x^1_j, x^2_j]$, the integer values m^1_j, m^2_j, and m_j are defined by the equations

$$m^1_j = \left[\frac{x^1_j}{a_j}\right], \quad m^2_j = \left[\frac{x^2_j}{a_j}\right], \quad m_j = \left[\frac{x_j}{a_j}\right], \tag{28}$$

where the brackets [ ] of (28) mean the floor function that maps a real number to the largest previous integer number.

Let us give some properties of the suggested fractional-order difference operators.

Proposition 3. For fractional-order operators (8), the semi-group property

$${}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\beta\\ j\end{bmatrix} u(\mathbf{m}) = {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha+\beta\\ j\end{bmatrix} u(\mathbf{m}), \quad (\alpha > 0, \quad \beta > 0) \tag{29}$$

holds for any bounded functions $u(\mathbf{m})$.

Proof. Using semigroup property (2) for fractional differences (4) and equation (8), we obtain (29). □

Remark 9. Using Equation (29), it is easy to prove the commutativity of fractional operator (8) of the Grünwald-Letnikov type

$$^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}\,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\beta\\ j\end{bmatrix} = \,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\beta\\ j\end{bmatrix}\,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}, \quad (\alpha > 0, \quad \beta > 0), \tag{30}$$

and the associativity of the fractional operator

$$^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_1\\ j\end{bmatrix}\left(^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_3\\ j\end{bmatrix}\right) = \left(^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_1\\ j\end{bmatrix}\,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\right)\,^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_3\\ j\end{bmatrix}, \tag{31}$$

where the parameters α_1, α_2, α_3 are positive real. The commutativity of fractional operators (8) and (9) of the Grünwald-Letnikov type for different directions $\mathbf{e}_j$, $\mathbf{e}_k$, $\mathbf{e}_l$ are obvious.

Remark 10. Semigroup property (2) does not hold for fractional differences of negative orders, i.e., this property is not satisfied for operator (9), and we have the inequality

$$^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\beta\\ j\end{bmatrix} \neq \,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha+\beta\\ j\end{bmatrix}, \quad (\alpha > 0, \quad \beta > 0). \tag{32}$$

Therefore, the properties of commutative and associativity are not satisfied for the difference operators $^{GL}\mathbb{I}_L^{\pm}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ in general,

$$^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\beta\\ j\end{bmatrix} \neq \,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\beta\\ j\end{bmatrix}\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}, \quad (\alpha > 0, \quad \beta > 0), \tag{33}$$

$$^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_1\\ j\end{bmatrix}\left(^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_3\\ j\end{bmatrix}\right) \neq \left(^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_1\\ j\end{bmatrix}\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\right)\,^{GL}\mathbb{I}_L^{\pm}\begin{bmatrix}\alpha_3\\ j\end{bmatrix}. \tag{34}$$

Remark 11. In the general case, we can consider anisotropic physical lattices, where the properties are different for different directions $\mathbf{a}_j$. In this case, we should use the difference operators $^{GL}\mathbb{D}_L^{\pm}\left[\begin{smallmatrix}\alpha_j\\ j\end{smallmatrix}\right]$ and $^{GL}\mathbb{I}_L^{\pm}\left[\begin{smallmatrix}\alpha_j\\ j\end{smallmatrix}\right]$ with orders α_j that are different for different j.

IV. CONTINUUM LIMIT OF FRACTIONAL-ORDER DIFFERENCE OPERATORS

In continuum models with power-law nonlocality, the fractional-order derivatives with respect to space coordinates are used. Let us give a definition of the partial Grünwald-Letnikov fractional derivatives and fractional integrals of order $\alpha > 0$ in the direction $\mathbf{a}_j$.

Definition 6. The Grünwald-Letnikov fractional derivatives $^{GL}D^{\alpha}_{x_j,\pm}$ *of order* $\alpha > 0$ *with respect to* x_j *are defined by the equation*

$$^{GL}D^{\alpha}_{x_j,\pm}u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{|a_j|^{\alpha}} \sum_{m_j=0}^{\infty} \frac{(-1)^{m_j}\,\Gamma(\alpha+1)}{\Gamma(m_j+1)\Gamma(\alpha-m_j+1)}\, u(\mathbf{r} \mp m_j \mathbf{a}_j), \quad (\alpha > 0). \tag{35}$$

Definition 7. For functions $u(\mathbf{r})$ *that satisfy condition (7) with respect to* x_j*, the Grünwald-Letnikov fractional integration of order* α *in the direction* $\mathbf{a}_j$ *is given by the equation*

$$^{GL}I^{\alpha}_{x_j,\pm}u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{|a_j|^{\alpha}} \sum_{m_j=0}^{\infty} \frac{(-1)^{m_j}\,\Gamma(1-\alpha)}{\Gamma(m_j+1)\Gamma(1-\alpha-m_j)}\, u(\mathbf{r} \mp m_j \mathbf{a}_j), \quad (\alpha > 0). \tag{36}$$

Remark 12. The Grünwald-Letnikov fractional derivatives coincide with the Marchaud fractional derivatives (see Sections 20.3 in Ref. 1) for the functions from the space $L_r(\mathbb{R})$, where $1 \leqslant r < \infty$ (see Theorem 20.4 in Ref. 1). Moreover, both the Grünwald-Letnikov and Marchaud derivatives have the same domain of definition. The Marchaud fractional derivative is defined by the equation

$$ {}^{M}D^{\alpha,\pm}_{x_j} u(\mathbf{r}) = \frac{1}{a(\alpha,s)} \int_0^{\infty} \frac{\Delta^{s,\pm}_{z_j} u(\mathbf{r})}{z_j^{\alpha+1}} dz_j, \quad (0 < \alpha < s), \tag{37} $$

where $\Delta^{s,\pm}_{z_j}$ is the finite difference of integer order s,

$$ \Delta^{s,\pm}_{z_j} u(\mathbf{r}) = \sum_{k=0}^{s} \frac{(-1)^k \, s!}{(s-k)! \, k!} u(\mathbf{r} - k \, z_j \mathbf{e}_j), \tag{38} $$

and $a(\alpha,s)$ is

$$ a(\alpha,s) = \frac{s}{\alpha} \int_0^1 \frac{(1-\xi)^{s-1}}{(\ln(1/\xi))^{\alpha}} d\xi. \tag{39} $$

In addition, the expression $a(\alpha,s)$ for non-integer values of $\alpha \neq 1,2,\dots,s-1$ can defined (see Section 5.6 in Ref. 1) by

$$ a(\alpha,s) = -\Gamma(-\alpha)\, A_s(\alpha), \tag{40} $$

where the function $A_s(\alpha)$ is

$$ A_s(\alpha) = \sum_{k=0}^{s} \frac{(-1)^{k-1} \, s!}{(s-k)! \, k!} k^{\alpha}, \quad (\alpha > 0). \tag{41} $$

Note that this function has (see Section 5.6 in Ref. 1) the important property $A_s(\alpha) = 0$ for $\alpha = 1,2,\dots,s-1$.

Let us give the proposition that determines continuum analogs of the fractional-order difference operators defined on unbounded physical lattice.

Proposition 4. Fractional-order difference operators ${}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ *and* ${}^{GL}\mathbb{I}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ *defined by (8) and (9) are transformed by the continuous limit operation into the fractional derivative and fractional integral of Grünwald-Letnikov type with respect to coordinate* x_j *in the form*

$$ \lim_{a_j \to 0+} \left({}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) \right) = {}^{GL}\mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}), \tag{42} $$

$$ \lim_{a_j \to 0+} \left({}^{GL}\mathbb{I}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) \right) = {}^{GL}\mathbb{I}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}), \tag{43} $$

where ${}^{GL}\mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ *and* ${}^{GL}\mathbb{I}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ *are the continuum fractional derivative and integral of the Grünwald-Letnikov type, respectively, that are defined by*

$$ {}^{GL}\mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} = \frac{1}{2} \left({}^{GL}D^{\alpha}_{x_j,+} \pm {}^{GL}D^{\alpha}_{x_j,-} \right), \tag{44} $$

$$ {}^{GL}\mathbb{I}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} = \frac{1}{2} \left({}^{GL}I^{\alpha}_{x_j,+} \pm {}^{GL}I^{\alpha}_{x_j,-} \right), \tag{45} $$

which contain the Grünwald-Letnikov fractional derivatives and integrals ${}^{GL}D^{\alpha}_{x_j,\pm}$ *and* ${}^{GL}I^{\alpha}_{x_j,\pm}$ *with respect to space coordinate* x_j.

Proof. Using definitions 5 and 6, this proposition can be proved by analogy with the proof for lattice model with long-range interaction of the Grünwald-Letnikov type suggested in Ref. 30. □

Remark 13. Using (6), we can note that derivatives (44) for integer orders $\alpha = n \in \mathbb{N}$ have the forms

$$ {}^{GL}\mathbb{D}^{+}_{C}\begin{bmatrix}n\\ j\end{bmatrix} = \frac{1}{2}\left(\frac{\partial^n}{\partial x_j^n} + (-1)^n \frac{\partial^n}{\partial x_j^n} \right) = \begin{cases} 0, & n = 2m-1, \quad m \in \mathbb{N}, \\ \dfrac{\partial^n}{\partial x_j^n}, & n = 2m, \quad m \in \mathbb{N}, \end{cases} \tag{46} $$

$$ {}^{GL}\mathbb{D}^{-}_{C}\begin{bmatrix} n \\ j \end{bmatrix} = \frac{1}{2}\left(\frac{\partial^n}{\partial x_j^n} - (-1)^n \frac{\partial^n}{\partial x_j^n} \right) = \begin{cases} \dfrac{\partial^n}{\partial x_j^n}, & n = 2m-1, \quad m \in \mathbb{N}, \\ 0, & n = 2m, \quad m \in \mathbb{N}. \end{cases} \tag{47} $$

Therefore, the continuum fractional derivative and integral ${}^{GL}\mathbb{D}^{+}_{C}\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$, ${}^{GL}\mathbb{I}^{+}_{C}\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$ are the usual derivative and integral of integer order n for even values α only, and the continuum operators ${}^{GL}\mathbb{D}^{-}_{C}\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$, ${}^{GL}\mathbb{I}^{-}_{C}\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$ are the derivative and integral of integer order n for odd values α only.

Remark 14. In the general case, we can consider anisotropic nonlocal continua, where the properties are different for different directions. In this case, we can use the fractional-order differentiation and integration ${}^{GL}\mathbb{D}^{+}_{C}\left[\begin{smallmatrix} \alpha_j \\ j \end{smallmatrix}\right]$, ${}^{GL}\mathbb{I}^{+}_{C}\left[\begin{smallmatrix} \alpha_j \\ j \end{smallmatrix}\right]$ with orders $\alpha_j > 0$ that are different for different $j = 1, 2, \ldots, N$.

Let us give the proposition that defines continuum analogs of the fractional-order difference operators defined on bounded physical lattice.

Proposition 5. The fractional-order difference operators ${}^{GL}_{B}\mathbb{D}^{\pm}_{L}\left[\begin{smallmatrix} \alpha \\ j \end{smallmatrix}\right]$, ${}^{GL}_{B}\mathbb{I}^{\pm}_{L}\left[\begin{smallmatrix} \alpha \\ j \end{smallmatrix}\right]$ *defined by (26) and (27) are transformed by the continuous limit by the equations*

$$ \lim_{a_j \to 0+} \left({}^{GL}_{B}\mathbb{D}^{\pm}_{L}\begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{m}) \right) = {}^{GL}_{B}\mathbb{D}^{\pm}_{C}\begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{r}), \tag{48} $$

$$ \lim_{a_j \to 0+} \left({}^{GL}_{B}\mathbb{I}^{\pm}_{L}\begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{m}) \right) = {}^{GL}_{B}\mathbb{I}^{\pm}_{C}\begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{r}) \tag{49} $$

into the continuum fractional derivatives and integral of the Grünwald-Letnikov type with respect to space coordinate x_j,

$$ {}^{GL}_{B}\mathbb{D}^{\pm}_{C}\begin{bmatrix} \alpha \\ j \end{bmatrix} = \frac{1}{2}\left({}^{GL}_{x_j^1}D^{\alpha}_{x_j,+} \pm {}^{GL}_{x_j^2}D^{\alpha}_{x_j,-} \right), \tag{50} $$

$$ {}^{GL}_{B}\mathbb{I}^{\pm}_{L}\begin{bmatrix} \alpha \\ j \end{bmatrix} = \frac{1}{2}\left({}^{GL}_{x_j^1}I^{\alpha}_{x_j,+} \pm {}^{GL}_{x_j^2}I^{\alpha}_{x_j,-} \right), \tag{51} $$

which contain the Grünwald-Letnikov fractional operators defined on the finite interval $[x_j^1, x_j^2]$, *where* $x_j^1 = m_j^1 a_j$ *and* $x_j^1 = m_j^2 a_j$, *in the form*

$$ {}^{GL}_{B}D^{\alpha}_{x_j,\pm} u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{|a_j|^{\alpha}} \sum_{m_j=0}^{M_j^{\pm}} \frac{(-1)^{m_j}\Gamma(\alpha+1)}{\Gamma(m_j+1)\Gamma(\alpha-m_j+1)} u(\mathbf{r} \mp m_j \mathbf{a}_j), \tag{52} $$

$$ {}^{GL}_{B}I^{\alpha}_{x_j,\pm} u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{|a_j|^{\alpha}} \sum_{n_j=0}^{M_j^{\pm}} \frac{(-1)^{m_j}\Gamma(1-\alpha)}{\Gamma(m_j+1)\Gamma(1-\alpha-m_j)} u(\mathbf{r} \mp m_j \mathbf{a}_j), \tag{53} $$

where

$$ M_j^{+} = \left[\frac{x_j - x_j^1}{a_j} \right], \quad M_j^{-} = \left[\frac{x_j^2 - x_j}{a_j} \right]. \tag{54} $$

Proof. The proof follows directly from the definitions of the Grünwald-Letnikov fractional operators defined on the finite interval (see Section 20.4 in Ref. 1). □

In Secs. V-VII, we consider some lattice models with fractional-order difference equations and corresponding continuum models with power-law nonlocality that are described by derivatives and integrals of non-integer orders.

V. FRACTIONAL NAVIER-STOKES EQUATIONS FOR PHYSICAL LATTICE AND NONLOCAL CONTINUUM

In this section, we give an example of the fractional-order difference equation for physical lattice. A lattice model with long-range interactions and correspondent fractional nonlocal continuum models are suggested for the fractional Navier-Stokes equations of non-local fluids. The fractional generalization of Navier-Stokes equations for fractional hydrodynamic of fractal media are suggested in Ref. 35. The fractional Navier-Stokes equations with fractional Laplacian in the Riesz form have been suggested in Refs. 37 and 38. The Navier-Stokes equations, which contain the local fractional derivatives, have been suggested in Ref. 36 to describe fluid flows in fractal media. In this section, we consider the Navier-Stokes equations with the fractional-order difference operators of the Grünwald-Letnikov type to describe incompressible flows of nonlocal fluids in the framework of physical lattice with long-range interactions. In the continuum limits, these lattice Navier-Stokes equations give the fractional Navier-Stokes equations with continuum fractional derivatives of the Grünwald-Letnikov type that describe fluids with power-law nonlocality. Note that we should take into account a violation of the usual Leibniz rule and the chain rule for the fractional-order derivatives.[40,41]

A motion of incompressible viscous fluids as local continua is described by the Navier-Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}, \mathrm{Grad}\mathbf{v}\right) = -\frac{1}{\rho}\mathrm{Grad}p + \nu\,\Delta\mathbf{v} + \mathbf{f}, \tag{55}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$ is the flow velocity, $\mathbf{f} = \mathbf{f}(\mathbf{r},t)$ is the vector field of mass forces, ρ is the density of the fluid, ν is the kinematic viscosity which is the ratio of the dynamic viscosity μ to density ρ.

Let us define the velocity and force fields on the three-dimensional lattice

$$\mathbf{v}(\mathbf{m},t) = \sum_{j=1}^{3} \mathbf{e}_j\, v_j(\mathbf{m},t), \quad \mathbf{f}(\mathbf{m},t) = \sum_{j=1}^{3} \mathbf{e}_j\, f_j(\mathbf{m},t), \tag{56}$$

where $v_i(\mathbf{m},t)$ and $f_i(\mathbf{m},t)$ can be considered as components of the velocity and force fields for a lattice site that is defined by the spatial lattice points with the vector $\mathbf{m} = (m_1,m_2,m_3)$.

We can define the fractional-order difference analogs of the integer-order vector operators such as the nabla operator, the gradient, and the vector Laplacian. For simplification, we consider the case $\mathbf{a}_i = a_i\,\mathbf{e}_i$, where $a_i = |\mathbf{a}_i|$ and $\mathbf{e}_i$ are the vectors of the basis of the Cartesian coordinate system. This simplification means that our lattice model is based on the primitive orthorhombic Bravais lattice with long-range interactions. The fractional-order nabla operator, gradient, and vector Laplacian of the Grünwald-Letnikov type for physical lattice are defined by the following equations.

The lattice nabla operator of the Grünwald-Letnikov type is

$$^{GL}\nabla^{\alpha,\pm}_{L} = \sum_{j=1}^{3} \frac{\mathbf{a}_j}{|\mathbf{a}_j|}\, {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix}. \tag{57}$$

Note that this definition can be used for other types of Bravais lattices which are not orthorhombic.

The lattice gradient for the scalar field $u(\mathbf{m})$ is

$$^{GL}\mathrm{Grad}^{\alpha,\pm}_{L}\, u(\mathbf{m}) = \sum_{j=1}^{3} \mathbf{e}_j\, {}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \sum_{j=1}^{3} \frac{1}{a_j^{\alpha}}\mathbf{e}_j \sum_{m_j=-\infty}^{+\infty} {}^{GL}K^{\pm}_{\alpha}(n_j - m_j)\, u(\mathbf{m}). \tag{58}$$

The vector Laplacian[39] in 3-dimensional space $\mathbb{R}^3$ for the vector field $\mathbf{u} = \sum_{j=1}^{3} \mathbf{e}_j\, u_j(\mathbf{m})$ can be defined by two different equations with the repeated lattice derivative of orders α,

$$^{GL}\Delta^{\alpha,\alpha,\pm}_{L}\mathbf{u} = \mathrm{Div}^{\alpha,\pm}_{L}\,\mathrm{Grad}^{\alpha,\pm}_{L}\,\mathbf{u}(\mathbf{m}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}_j \left({}^{GL}\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ i\end{bmatrix}\right)^2 u_j(\mathbf{m}), \tag{59}$$

and by the derivative of the doubled order 2α,

$$ {}^{GL}\Delta_L^{2\alpha,\pm}\,\mathbf{u}(\mathbf{m}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{e}_j\,{}^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}2\alpha\\ i\end{bmatrix} u_j(\mathbf{m}). \tag{60} $$

The semigroup property for the Grünwald-Letnikov fractional differences leads to the fact that operators (59) and (60) coincide,

$$ {}^{GL}\Delta_L^{\alpha,\alpha,\pm} = {}^{GL}\Delta_L^{2\alpha,\pm}. \tag{61} $$

Using lattice operators (57), (58), and (60), we can write the equations

$$ \frac{\partial \mathbf{u}(\mathbf{m},t)}{\partial t} + \left(\mathbf{u}(\mathbf{m},t),\,{}^{GL}\nabla_L^{\alpha,\pm}\right)\mathbf{v}(\mathbf{m},t) = $$

$$ = -\frac{1}{\rho}\,{}^{GL}\mathrm{Grad}_L^{\alpha,\pm}\,p(\mathbf{m},t) + \nu\,{}^{GL}\Delta_L^{2\alpha,\pm}\,\mathbf{v}(\mathbf{m},t) + \mathbf{f}(\mathbf{m},t). \tag{62} $$

These equations can be considered as the Navier-Stokes equations for fluids on the physical lattice with long-range interaction of the Grünwald-Letnikov type.

The continuum limit ($a_j \to 0$) of lattice equation (62) gives the fractional Navier-Stokes equations for non-local continuous fluids in the form

$$ \frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial t} + \left(\mathbf{u}(\mathbf{r},t),\,{}^{GL}\nabla_C^{\alpha,\pm}\right)\mathbf{v}(\mathbf{r},t) = $$

$$ = -\frac{1}{\rho}\,{}^{GL}\mathrm{Grad}_C^{\alpha,\pm}\,p(\mathbf{r},t) + \nu\,{}^{GL}\Delta_C^{2\alpha,\pm}\,\mathbf{v}(\mathbf{r},t) + \mathbf{f}(\mathbf{r},t), \tag{63} $$

where ${}^{GL}\nabla_C^{\alpha,\pm}$, ${}^{GL}\mathrm{Grad}_C^{\alpha,\pm}\,{}^{GL}\Delta_C^{2\alpha,\pm}$ are the continuum fractional differential operators of the Grünwald-Letnikov type that are defined by the equations

$$ {}^{GL}\nabla_C^{\alpha,\pm} = \sum_{j=1}^{3}\mathbf{e}_j\,{}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}, \tag{64} $$

$$ {}^{GL}\mathrm{Grad}_C^{\alpha,\pm}\,p(\mathbf{r},t) = \sum_{j=1}^{3}\mathbf{e}_j\,{}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix} p(\mathbf{r},t), \tag{65} $$

$$ {}^{GL}\Delta_C^{2\alpha,\pm}\,\mathbf{v}(\mathbf{r},t) = \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{e}_j\,{}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}2\alpha\\ i\end{bmatrix} v_j(\mathbf{r},t). \tag{66} $$

We would like to have a fractional generalization of partial differential equations such that to obtain the original equations in the limit case, when the orders of fractional derivatives become equal to initial integer values. This correspondence principle and the fact that only the continuum fractional derivatives ${}^{GL}\mathbb{D}_C^{-}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ for $\alpha = 1$, and ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ for $\alpha = 2$ are the usual local derivatives of these integer orders allow us to consider Equation (62) with ${}^{GL}\nabla_L^{\alpha,-}$, ${}^{GL}\mathrm{Grad}_L^{\alpha,-}$, and ${}^{GL}\Delta_L^{2\alpha,+}$ as basic lattice equations,

$$ \frac{\partial \mathbf{u}(\mathbf{m},t)}{\partial t} + \left(\mathbf{u}(\mathbf{m},t),\,{}^{GL}\nabla_L^{\alpha,-}\right)\mathbf{v}(\mathbf{m},t) = $$

$$ = -\frac{1}{\rho}\,{}^{GL}\mathrm{Grad}_L^{\alpha,-}\,p(\mathbf{m},t) + \nu\,{}^{GL}\Delta_L^{2\alpha,+}\,\mathbf{v}(\mathbf{m},t) + \mathbf{f}(\mathbf{m},t). \tag{67} $$

The continuum limit ($a_j \to 0$) of these equations gives

$$ \frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial t} + \left(\mathbf{u}(\mathbf{r},t),\,{}^{GL}\nabla_C^{\alpha,-}\right)\mathbf{v}(\mathbf{r},t) = $$

$$ = -\frac{1}{\rho}\,{}^{GL}\mathrm{Grad}_C^{\alpha,-}\,p(\mathbf{r},t) + \nu\,{}^{GL}\Delta_C^{2\alpha,+}\,\mathbf{v}(\mathbf{r},t) + \mathbf{f}(\mathbf{r},t). \tag{68} $$

For $\alpha = 1$, only these equations that contain ${}^{GL}\nabla_L^{\alpha,-}$, ${}^{GL}\mathrm{Grad}_L^{\alpha,-}$, and ${}^{GL}\Delta_L^{2\alpha,+}$ give the usual Navier-Stokes equation (55).

The suggested fractional Navier-Stokes equation (68) with fractional derivatives of the Grünwald-Letnikov type differs from the fractional Navier-Stokes equations proposed in Ref. 36, where the local fractional derivatives are used. The main advantage of fractional Navier-Stokes equation (68) is the direct connection of these continuum equations with fractional-order difference equation (67) for the physical lattice models.

VI. FRACTIONAL MAXWELL EQUATIONS FOR PHYSICAL LATTICE AND NONLOCAL CONTINUUM

In this section, we give a second example of the fractional-order difference equation for physical lattice. A lattice model with long-range interactions and correspondent fractional nonlocal continuum models is suggested for the fractional Maxwell equations of non-local continuous media.

The usual Maxwell equations with derivatives of integer order for electrodynamics of continuous media[42,43] have the form

$$\mathrm{div}\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t), \quad \mathrm{curl}\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{69}$$

$$\mathrm{div}\mathbf{B}(\mathbf{r},t) = 0, \quad \mathrm{curl}\mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}, \tag{70}$$

where $\mathbf{E}$ is the electric field strength, $\mathbf{D}$ is the electric displacement field, $\mathbf{B}$ is the magnetic induction (the magnetic flux density), $\mathbf{H}$ is the magnetic field strength, ρ is the electric charge density, $\mathbf{J}$ is the electric current density.

Let us define the electric and magnetic fields on the three-dimensional lattice by the following equations:

$$\mathbf{E}(\mathbf{m},t) = \sum_{j=1}^{3} \mathbf{e}_j \, E_j(\mathbf{m},t), \quad \mathbf{B}(\mathbf{m},t) = \sum_{j=1}^{3} \mathbf{e}_j \, B_j(\mathbf{m},t), \tag{71}$$

where $E_i(\mathbf{m},t)$ and $B_i(\mathbf{m},t)$ are the components of the electric and magnetic fields for a lattice site that is defined by the spatial lattice points with the vector $\mathbf{m} = (m_1, m_2, m_3)$. The other fields $\mathbf{D}$, $\mathbf{H}$, $\mathbf{j}$, ρ for the three-dimensional physical lattice are defined by analogy.

We can define the fractional-order difference analogs of the first-order operators such as the divergence, and the circulation. For simplification, we consider the case $\mathbf{a}_i = a_i \, \mathbf{e}_i$, where $a_i = |\mathbf{a}_i|$ and $\mathbf{e}_i$ are the vectors of the basis of the Cartesian coordinate system. The fractional-order divergence and circulation of the Grünwald-Letnikov type for physical lattice are defined by the following equations. The lattice divergence for the vector lattice field $\mathbf{E} = \sum_{j=1}^{3} \mathbf{e}_j \, E_j(\mathbf{m},t)$ is

$${}^{GL}\mathrm{Div}_L^{\alpha,\pm}\mathbf{E} = \sum_{j=1}^{3} {}^{GL}\mathbb{D}_L^{\pm} \begin{bmatrix} \alpha \\ j \end{bmatrix} E_j(\mathbf{m},t) = \sum_{i=1}^{3} \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{GL}K_{\alpha}^{\pm}(n_j - m_j) \, E_j(\mathbf{m},t). \tag{72}$$

The lattice curl operator for the vector lattice field $\mathbf{E} = \sum_{j=1}^{3} \mathbf{e}_j \, E_j(\mathbf{m},t)$ is

$${}^{GL}\mathrm{Curl}_L^{\alpha,\pm}\mathbf{E} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, \mathbf{e}_i \, {}^{GL}\mathbb{D}_L^{\pm} \begin{bmatrix} \alpha \\ j \end{bmatrix} E_k(\mathbf{m},t), \tag{73}$$

where ϵ_{ijk} denotes the Levi-Civita symbol.

Using lattice operators (72) and (73), we can write the equations

$${}^{GL}\mathrm{Div}_L^{\alpha,\pm}\mathbf{D}(\mathbf{m},t) = \rho(\mathbf{m},t), \quad {}^{GL}\mathrm{Curl}_L^{\alpha,\pm}\mathbf{E}(\mathbf{m},t) = -\frac{\partial \mathbf{B}(\mathbf{m},t)}{\partial t}, \tag{74}$$

$${}^{GL}\mathrm{Div}_L^{\alpha,\pm}\mathbf{B}(\mathbf{m},t) = 0, \quad {}^{GL}\mathrm{Curl}_L^{\alpha,\pm}\mathbf{H}(\mathbf{m},t) = \mathbf{J}(\mathbf{m},t) + \frac{\partial \mathbf{D}(\mathbf{m},t)}{\partial t}. \tag{75}$$

These equations can be considered as the Maxwell equations for the lattice with long-range interaction of the Grünwald-Letnikov type.

The continuum limit of lattice equations (74) and (75) gives the fractional Maxwell equations for electrodynamics of non-local continuous media,

$$ {}^{GL}\mathrm{Div}^{\alpha,\pm}_{C}\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t), \quad {}^{GL}\mathrm{Curl}^{\alpha,\pm}_{C}\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{76} $$

$$ {}^{GL}\mathrm{Div}^{\alpha,\pm}_{C}\mathbf{B}(\mathbf{r},t) = 0, \quad {}^{GL}\mathrm{Curl}^{\alpha,\pm}_{C}\mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}, \tag{77} $$

where $\mathrm{Div}^{\alpha,\pm}_{C}$ and $\mathrm{Curl}^{\alpha,\pm}_{C}$ are the continuum fractional differential operators of order $\alpha > 0$ that are defined by the equations

$$ {}^{GL}\mathrm{Div}^{\alpha,\pm}_{C}\mathbf{u} = \sum_{j=1}^{3} {}^{GL}\mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} E_j(\mathbf{r},t), \tag{78} $$

$$ {}^{GL}\mathrm{Curl}^{\alpha,\pm}_{C}\mathbf{E} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\,\mathbf{e}_i\, {}^{GL}\mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} E_k(\mathbf{r},t). \tag{79} $$

It is obvious that we would like to have a fractional generalization of partial differential equations such that to obtain the original equations in the limit case, when the orders of fractional derivatives become equal to initial integer values. This correspondence principle and the fact that only the continuum fractional derivatives ${}^{GL}\mathbb{D}^{-}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for $\alpha = 1$ are the usual local derivatives of first order allow us to consider Equations (74) and (75) with ${}^{GL}\mathrm{Div}^{\alpha,-}_{L}$ and ${}^{GL}\mathrm{Curl}^{\alpha,-}_{L}$ as basic lattice equations,

$$ {}^{GL}\mathrm{Div}^{\alpha,-}_{L}\mathbf{D}(\mathbf{m},t) = \rho(\mathbf{m},t), \quad {}^{GL}\mathrm{Curl}^{\alpha,-}_{L}\mathbf{E}(\mathbf{m},t) = -\frac{\partial \mathbf{B}(\mathbf{m},t)}{\partial t}, \tag{80} $$

$$ {}^{GL}\mathrm{Div}^{\alpha,-}_{L}\mathbf{B}(\mathbf{m},t) = 0, \quad {}^{GL}\mathrm{Curl}^{\alpha,-}_{L}\mathbf{H}(\mathbf{m},t) = \mathbf{J}(\mathbf{m},t) + \frac{\partial \mathbf{D}(\mathbf{m},t)}{\partial t}. \tag{81} $$

For $\alpha = 1$, only these equations with ${}^{GL}\mathrm{Div}^{\alpha,-}_{L}$ and ${}^{GL}\mathrm{Curl}^{\alpha,-}_{L}$ give Maxwell equations (69) and (70) in the continuous limit. For components, these fractional Maxwell equations for nonlocal continua have the form

$$ \sum_{i=1}^{3} {}^{GL}\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} D_j(\mathbf{r},t) = \rho(\mathbf{r},t), \quad \sum_{j,k=1}^{3} \epsilon_{ijk}\, {}^{GL}\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} E_k(\mathbf{r},t) = -\frac{\partial B_i(\mathbf{r},t)}{\partial t}, \tag{82} $$

$$ \sum_{i=1}^{3} {}^{GL}\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} B_j(\mathbf{r},t) = 0, \quad \sum_{j,k=1}^{3} \epsilon_{ijk}\, {}^{GL}\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} H_k(\mathbf{r},t) = J_i(\mathbf{r},t) + \frac{\partial D_i(\mathbf{r},t)}{\partial t}, \tag{83} $$

where ${}^{GL}\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ are the continuum fractional derivatives of the Grünwald-Letnikov type of order $\alpha > 0$. For $\alpha = 1$, Equations (82) and (83) are the usual Maxwell equations (69) and (70).

Fractional Maxwell equations (82) and (83) with the fractional derivatives of the Grünwald-Letnikov type of non-integer orders $\alpha > 0$ can be considered as main equations of fractional nonlocal electrodynamics, and these equations correspond to the lattice model described by Equations (74) and (75) with ${}^{GL}\mathrm{Div}^{\alpha,-}_{L}$ and ${}^{GL}\mathrm{Curl}^{\alpha,-}_{L}$.

The suggested fractional Maxwell equations (82) and (83) with the continuum fractional derivatives of the Grünwald-Letnikov type differ from the fractional Maxwell equations proposed in Refs. 44 and 9, where the Caputo fractional derivatives are used. The main advantage of fractional Maxwell equations (82) and (83) is direct connection of these equations with the physical lattice model that is described by fractional-order difference equations (80) and (81).

VII. LATTICE AND CONTINUUM FRACTIONAL INTEGRAL MAXWELL EQUATIONS

Using the lattice fractional integral operators of the Grünwald-Letnikov type for bounded lattice ${}^{GL}_{B}\mathbb{I}^{\pm}_{L}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$, we can consider lattice analogs of the fractional integral Maxwell equations and correspondent fractional integral Maxwell equations with continuum fractional integrals.

Let us define fractional-order difference generalizations of circulation, flux, and volume integral for physical lattice with long-range interaction. We will consider an anisotropic physical lattice, where the properties are different for different directions $\mathbf{a}_j$ such that the parameters α_x, α_y, α_z do not coincide.

We give the definitions by expressions for the Cartesian coordinate system. The lattice fractional-order circulation, flux, and volume integral use the difference analogs of the operators

$$ {}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[L] = \sum_{j=1}^{3} \mathbf{e}_j \, {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_j \\ j \end{bmatrix}, \tag{84} $$

$$ {}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[S] = \mathbf{e}_x \, {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_y \, \alpha_z \\ y \; z \end{bmatrix} + \mathbf{e}_y \, {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_z \, \alpha_x \\ z \; x \end{bmatrix} + \mathbf{e}_z \, {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_x \, \alpha_y \\ x \; y \end{bmatrix}, \tag{85} $$

$$ {}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[V] = {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_x \, \alpha_y \, \alpha_z \\ x \; y \; z \end{bmatrix} = {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_x \\ x \end{bmatrix} {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_y \\ y \end{bmatrix} {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_z \\ z \end{bmatrix}, \tag{86} $$

for the lattice vector and scalar fields

$$ \mathbf{E} = \sum_{j=1}^{3} \mathbf{e}_j \, E_j(\mathbf{m},t), \quad f = f(\mathbf{m},t), $$

where

$$ \mathbf{m} = \sum_{j=1}^{3} m_j \, \mathbf{e}_j = m_x \, \mathbf{e}_x + m_y \, \mathbf{e}_y + m_z \, \mathbf{e}_z . $$

A lattice fractional circulation is a fractional "line" difference operator of the lattice vector field $\mathbf{E}(\mathbf{m})$ along a line L that is defined by

$$ \mathcal{E}^{\alpha,\pm}_{L}[L] \, \mathbf{E}(\mathbf{m}.t) = \left({}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[L], \mathbf{E} \right) = \sum_{j=1}^{3} {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_j \\ j \end{bmatrix} E_j(\mathbf{m},t) = $$

$$ = {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_x \\ x \end{bmatrix} E_x(\mathbf{m},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_y \\ y \end{bmatrix} E_y(\mathbf{m},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_z \\ z \end{bmatrix} E_z(\mathbf{m},t). \tag{87} $$

In the continuum limit ($a_j \to 0$), Equation (87) gives

$$ \mathcal{E}^{\alpha,\pm}_{C}[L] \, \mathbf{E}(\mathbf{r},t) = \left({}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[L], \mathbf{E} \right) = \sum_{j=1}^{3} {}^{GL}_{B}\mathbb{I}^{\pm}_{C} \begin{bmatrix} \alpha_j \\ j \end{bmatrix} E_j(\mathbf{r},t) = $$

$$ = {}^{GL}_{B}\mathbb{I}^{\pm}_{C} \begin{bmatrix} \alpha_x \\ x \end{bmatrix} E_x(\mathbf{r},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{C} \begin{bmatrix} \alpha_y \\ y \end{bmatrix} E_y(\mathbf{r},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{C} \begin{bmatrix} \alpha_z \\ z \end{bmatrix} E_z(\mathbf{r},t), \tag{88} $$

where

$$ \mathbf{r} = \sum_{j=1}^{3} x_j \, \mathbf{e}_j = x \, \mathbf{e}_x + y \, \mathbf{e}_y + z \, \mathbf{e}_z . $$

For $\alpha = 1$, expression (88) with ${}^{GL}_{B}\mathbb{I}^{-}_{C}$ gives

$$ \mathcal{E}^{1,-}_{C}[L]\mathbf{E}(\mathbf{r},t) = \left({}^{GL}_{B}\mathbf{I}^{1,-}_{L}[L], \mathbf{E} \right) = \int_L \left(\mathbf{dL}, \mathbf{E}(\mathbf{r},t) \right) = \int_L (E_x dx + E_y dy + E_z dz), \tag{89} $$

where $\mathbf{dL} = \mathbf{e}_1 dx + \mathbf{e}_2 dy + \mathbf{e}_3 dz$.

A fractional flux of the lattice vector field $\mathbf{E}(\mathbf{m})$ across a surface S is a lattice analog of the fractional surface integral such that

$$ \Phi^{\alpha,\pm}_{L}[S] \, \mathbf{E}(\mathbf{m},t) = \left({}^{GL}_{B}\mathbf{I}^{\alpha,\pm}_{L}[S], \mathbf{E} \right) = $$

$$ = {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_y \, \alpha_z \\ y \; z \end{bmatrix} E_x(\mathbf{m},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_z \, \alpha_x \\ z \; x \end{bmatrix} E_y(\mathbf{m},t) + {}^{GL}_{B}\mathbb{I}^{\pm}_{L} \begin{bmatrix} \alpha_x \, \alpha_y \\ x \; y \end{bmatrix} E_z(\mathbf{m},t). \tag{90} $$

In the continuum limit ($a_j \to 0$), Equation (90) gives

$$\Phi_C^{\alpha,\pm}[S](\mathbf{E}) = \left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{E} \right) =$$

$$= {}_B^{GL}\mathbb{I}_C^{\pm} \begin{bmatrix} \alpha_y \, \alpha_z \\ y \; z \end{bmatrix} E_x(\mathbf{m},t) + {}_B^{GL}\mathbb{I}_C^{\pm} \begin{bmatrix} \alpha_z \, \alpha_x \\ z \; x \end{bmatrix} E_y(\mathbf{m},t) + {}_B^{GL}\mathbb{I}_C^{\pm} \begin{bmatrix} \alpha_x \, \alpha_y \\ x \; y \end{bmatrix} E_z(\mathbf{m},t). \quad (91)$$

For $\alpha = 1$, expression (91) with ${}_B^{GL}\mathbb{I}_C^{-}$ gives

$$\Phi_C^{1,-}[S]\mathbf{E}(\mathbf{r},t) = \left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{E} \right) = \int\int_S \left(d\mathbf{S}, \mathbf{E} \right) = \int\int_S (E_x dy dz + E_y dz dx + E_z dx dy), \quad (92)$$

where $\mathbf{dS} = \mathbf{e}_1 dy dz + \mathbf{e}_2 dz dx + \mathbf{e}_3 dx dy$.

A lattice fractional volume integral is a triple fractional-order difference integral operator within a region V in $\mathbb{R}^3$ of a scalar field $f = f(\mathbf{r},t)$,

$$V_L^{\alpha,\pm}[V]\, f(\mathbf{m},t) = {}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[V]\, f(x,y,z,) = {}_B^{GL}\mathbb{I}_C^{\alpha,\pm,} \begin{bmatrix} \alpha_x \, \alpha_y \, \alpha_z \\ x \; y \; x \end{bmatrix} f(\mathbf{m},t). \quad (93)$$

In the continuum limit, Equation (93) gives

$$V_C^{\alpha,\pm}[V] f(\mathbf{r}) = {}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[V]\, f(\mathbf{r},t) = {}_B^{GL}\mathbb{I}_C^{\pm} \begin{bmatrix} \alpha_x \, \alpha_y \, \alpha_z \\ x \; y \; z \end{bmatrix} f(\mathbf{r},t). \quad (94)$$

For $\alpha = 1$, Equation (94) with ${}_B^{GL}\mathbb{I}_C^{-}$ gives

$$V_C^{1,-}[V]\, f(\mathbf{r},t) = \int\int\int_W dx\, dy\, dz\, f(x,y,z,t). \quad (95)$$

This is the usual volume integral for the function $f(\mathbf{r})$.

A fractional difference analog of the integral Maxwell's equations can be presented in the form

$$\left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[S], \mathbf{D}(\mathbf{m},t) \right) = {}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[V] \rho(\mathbf{m},t), \quad (S = \partial V), \quad (96)$$

$$\left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[L], \mathbf{E}(\mathbf{m},t) \right) = -\frac{d}{dt} \left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[S], \mathbf{B}(\mathbf{m},t) \right), \quad (L = \partial S), \quad (97)$$

$$\left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[V], \mathbf{B}(\mathbf{m},t) \right) = 0, \quad (98)$$

$$\left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[L], \mathbf{H}(m,t) \right) = \left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[S], \mathbf{j}(\mathbf{m},t) \right) + \frac{d}{dt} \left({}_B^{GL}\mathbf{I}_L^{\alpha,\pm}[S], \mathbf{D}(\mathbf{m},t) \right), \quad (L = \partial S). \quad (99)$$

In the continuum limit ($a_j \to 0$), lattice equations (96)–(99) give the continuum fractional integral Maxwell's equations with integration of non-integer orders of the Grünwald-Letnikov type,

$$\left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{D}(\mathbf{r},t) \right) = {}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[V] \rho(\mathbf{r},t), \quad (S = \partial V), \quad (100)$$

$$\left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[L], \mathbf{E}(\mathbf{r},t) \right) = -\frac{d}{dt} \left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{B}(\mathbf{r},t) \right), \quad (L = \partial S), \quad (101)$$

$$\left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[V], \mathbf{B}(\mathbf{r},t) \right) = 0, \quad (102)$$

$$\left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[L], \mathbf{H}(\mathbf{r},t) \right) = \left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{j}(\mathbf{r},t) \right) + \frac{d}{dt} \left({}_B^{GL}\mathbf{I}_C^{\alpha,\pm}[S], \mathbf{D}(\mathbf{r},t) \right), \quad (L = \partial S). \quad (103)$$

To get the integral Maxwell's equations for the case of first order operators ($\alpha_x = 1$, $\alpha_y = 1$, $\alpha_y = 1$), we should use the continuum fractional integration ${}_B^{GL}\mathbf{I}_C^{\alpha,-}$.

VIII. CONCLUSION

In this paper, we consider the fractional-order difference operators and equations for N-dimensional physical lattice with long-range interactions of the Grünwald-Letnikov type. The main advantage of the suggested approach is a possibility to consider fractional-order difference equations as tools for formulation of a microstructural basic model of fractional nonlocal continua. The fractional-order difference analogs of fractional derivatives and integrals are represented by kernels of long-range interactions of lattice particles. The suggested long-range interactions can be used for integer and fractional orders of suggested operators. The continuous limits for these fractional-order difference analogs of the derivatives and integrals give the continuum fractional derivatives and integrals of the Grünwald-Letnikov type with respect to space coordinates. The obtained fractional dynamics of nonlocal continua can be considered as a continuous limit of dynamics of the suggested physical lattice models, where the sizes of continuum elements are much larger than the distances between particles of the lattice.

The proposed fractional-order difference operators and equations allow us to construct different lattice models for wide class of media with power-law nonlocality. These models can serve as new microstructural basis for the fractional nonlocal continuum mechanics and physics. Fractional-order difference equations can be used to formulate adequate lattice models for nanomechanics.[52,53] The suggested fractional-order difference operators and equations are formulated for lattice systems with the long-range interparticle interactions. Therefore, these operators can be important to describe the non-local properties of materials at nano-scale, where the intermolecular interactions are crucial for properties of these materials.

In addition, we assume that the proposed approach to the fractional-order difference operators can be generalized for lattices with fractal properties,[45,46] and correspondent fractional continuum models of fractal materials and media that can be described by different mathematical methods (for example, see Refs. 9 and 47–51).

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