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WHAT DISCRETE MODEL CORRESPONDS EXACTLY TO A GRADIENT ELASTICITY EQUATION?

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In this paper, we obtain exact discrete analogs of the gradient elasticity equations. The suggested discrete equations have differences represented by infinite series. Physically, these equations describe models of lattices with long-range interactions. Mathematically, unique difference equations correspond exactly to continuum gradient elasticity equations.

1. Introduction

There are two basic approaches to describe elasticity of solid states: a microscopic approach based on the classical and quantum theory of crystal lattices and solids [Born and Huang 1998; Böttger 1983; Kittel 1987] and a macroscopic phenomenological approach based on the classical mechanics of continua [Sedov 1971]. On the one hand, continuum equations can be considered as a limit case of discrete (lattice) equations when the primitive lattice vectors tend to zero. On the other hand, different discretizations of the continuum equations can be used to get discrete (difference) equations, which allow us to apply computer simulations. Usually discretization of differential equations is realized by using the standard difference operators. In some cases, the corresponding difference equations are similar to the lattice equations. For example, the standard finite difference of second order corresponds to the nearest-neighbor interaction of lattice particles. The standard finite difference of fourth order describes the next-nearest-neighbor interaction [Tarasov 2014a; 2015b].

In this paper, we are not trying to find out which type of equation is primary and which is secondary. We do not try to argue that discrete or continuous equations are primary. The main goal of our paper is to define an exact correspondence between continuum and discrete (lattice) equations. We would like to describe the correspondence without using approximations and limit passages that discard some terms. The mathematical basis of our consideration is the following correspondence principle: the correspondence between the discrete (lattice) theory and the continuum theory lies not so much in the limiting condition when the steps (or primitive lattice vectors) tend to zero as in the fact that mathematical operations on these theories should obey the same laws in many cases. We will use a new type of difference operator, which can be considered an exact discretization of partial derivatives and a lattice operator on physical lattices with long-range interactions [Tarasov 2015a]. The proposed T-differences satisfy the same algebraic relations as the corresponding derivatives. The suggested difference operators allow us to have difference (lattice) equations, whose solutions are equal to the solutions of corresponding continuum differential equations.

Keywords: elasticity, gradient elasticity, long-range interactions, exact discretization, difference equation.

This article focuses on gradient elasticity models first suggested by Mindlin [1964; 1965; 1968; Eringen 1983]. These models can be considered a special type of theory of nonlocal elastic continua and continuous media with internal degrees of freedom [Carcaterra et al. 2015; Auffray et al. 2015; dell'Isola et al. 2016b; Sedov 1968; Eringen 1972; 2002; Rogula 1982]. The theory of nonlocal continuum mechanics was initiated by Piola [dell'Isola et al. 2015; 2016a; Rahali et al. 2015]. Nonlocal elasticity theory is based on the assumption that the forces between particles are long-range types that correspond to the long-range character of interatomic forces.

There are three main approaches to derive equations of gradient elasticity: a phenomenological continuum approach based on the postulation of equations, an approach based on homogenization of discrete models, and an approach based on continuum limit of lattice models.

The first approach to derive equations of gradient elasticity in the framework of phenomenological consideration was suggested by [Mindlin 1964; 1965; 1968], which suggest an elasticity theory of materials with microstructure, where two different types of quantities are used for the micro- and macroscales. In the phenomenological approach, the gradient elasticity models differ in the assumed relation between the microscopic deformation and the macroscopic displacement. It is important to note that, despite the theoretical differences between these models, the equations for displacements of these models are identical [Mindlin 1964]. Using the phenomenology approach, the simplest equation of the one-dimensional gradient elasticity has the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + l^2 \frac{\partial^4 u(x,t)}{\partial x^4}, \tag{1-1}$$

where l^2 is the scale parameter.

The second approach to obtain equations of gradient elasticity is the continualization (homogenization) of a lattice with nearest-neighbor interactions [Mindlin 1968]. Usually a basis of this approach is models of systems of particles and springs. In the simplest case of one-dimensional system of particles and springs, where all particles have mass M and all springs have spring stiffness K, the equations of motion have the form

$$M \frac{d^2 u_n(t)}{dt^2} = K(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)). \tag{1-2}$$

In the homogenization procedure, it is assumed that the continuum displacement $u(x,t)$ is equal to the lattice displacement $u_n(t)$ of particle n by $u_n(t) = u(nh,t)$, where h is the particle spacing. In this case, the displacements $u_{n\pm1}(t)$ are expressed in terms of the continuum displacement $u(x \pm h, t)$. Then the Taylor series is used in the form

$$u_{n\pm1}(t) = u(x \pm h, t) = \sum_{m=0}^{\infty} \frac{(\pm h)^m}{m!} \frac{\partial^m u(x,t)}{\partial x^m} \tag{1-3}$$

to substitute (1-3) into (1-2). Note that all odd-order derivatives of $u(x,t)$ have canceled. As a result, the division by the cross-section area A of the medium and the interparticle distance h gives the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{c^2}{h^2} \sum_{m=0}^{\infty} \frac{2h^{2m}}{(2m)!} \frac{\partial^{2m} u(x,t)}{\partial x^{2m}}, \tag{1-4}$$

where $\rho = M/(Ah)$ is the mass density and $E = (Kh)/A$ is the Young's modulus, where $c = \sqrt{E/\rho}$ is the elastic bar velocity. Equation (1-1) is obtained by deleting all terms $O(h^6)$, and we get the equation of the gradient elasticity

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{h^2 c^2}{12} \frac{\partial^4 u(x,t)}{\partial x^4}. \tag{1-5}$$

It is easy to see that continualization (homogenization) by Taylor series cannot give the equation of the gradient elasticity exactly. Therefore, the discrete equation (1-1) cannot be considered as an exact analog to (1-2) of the gradient elasticity.

The third approach for obtaining the gradient elasticity equations has been suggested in [Tarasov 2014a; 2015b]. This approach is based on the models of lattices with the nearest-neighbor and next-nearest-neighbor interactions, instead of the case (1-2) that corresponds to the lattice with the nearest-neighbor interactions only. It was proved that two classes of the gradient models (with positive and negative signs in front of the gradient term) can have a general lattice model as a microstructural basis. To obtain the gradient elasticity equations, we consider a lattice model with the nearest-neighbor and next-nearest-neighbor interactions with two different coupling constants. A generalization of this approach to lattice models with long-range interactions has been suggested in recent papers [Tarasov 2013; 2014b; 2014c; 2016b] to describe the fractional nonlocal elastic materials. Note that the models of lattices with the nearest-neighbor and next-nearest-neighbor interactions also cannot be considered as exact discrete models of gradient elasticity.

In this paper, we do not plan to discuss these approaches of obtaining the gradient elasticity equations in detail. Some aspects of this question have been discussed in the cited articles and in [Tarasov 2006a; 2006b; 2014d; 2015a]. We will solve an inverse problem. Considering these continuum equations as already specified, we would like to get an exact discrete analog to the equations. In this paper, we get discrete (lattice) equations that correspond exactly to the continuum gradient elasticity equation (1-1). Physically, these discrete equations describe lattice models with long-range interactions [Tarasov 2006a; 2006b]. Mathematically, unique difference equations correspond exactly to the continuum gradient elasticity equations. Note that exact correspondence means also that the difference equations have the same general solutions as the associated differential equations. In the beginning, we will consider one-dimensional gradient elasticity equations for simplification. Then we suggest a generalization for the three-dimensional case by using an approach proposed in [Tarasov 2014d; 2015a].

2. Transformation of a discrete equation into a continuum gradient elasticity equation

Let us describe in details a transform of a discrete equation into a continuum gradient elasticity equation to fix notation for further consideration.

Usually a discrete analog to the gradient elasticity equation (1-1) is considered in the form of the equation with finite difference of second order

$$\frac{d^2 u_n(t)}{dt^2} = \frac{c^2}{h^2}(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)). \tag{2-1}$$

Another discrete analog to the gradient elasticity equation (1-1) is considered in the form of the equation with finite difference of second and fourth orders [Tarasov 2014a; 2015b]. As we shortly describe in

Section 1, the discrete equation (2-1) cannot be an exact analog to the gradient elasticity equation (1-1). Let us give some details to explain a connection between (1-1) and (2-1). The Fourier series transform $F_{h,\Delta}$, which is defined by

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-iknh} = F_{h,\Delta}\{u_n(t)\}, \tag{2-2}$$

maps the difference equation (2-1) to

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = -\frac{2c^2}{h^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} (kh)^{2m} \hat{u}(k,t). \tag{2-3}$$

The inverse Fourier integral transform F^{-1}, which is defined by

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, \hat{u}(k,t) e^{ikx} = F^{-1}\{\hat{u}(k,t)\}, \tag{2-4}$$

gives

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{2c^2}{h^2} \sum_{m=1}^{\infty} \frac{h^{2m}}{(2m)!} \frac{\partial^{2m} u(x,t)}{\partial x^{2m}}. \tag{2-5}$$

Equation (2-5) also can be obtained (for details, see Section 8 of [Maslov 1976]) by using the well-known relation

$$\exp\left(h \frac{\partial}{\partial x}\right) f(x) = f(x+h).$$

It is easy to see that (1-1) can be obtained only in approximation of (2-5) by deleting all $O(h^6)$ terms. It is important to note that the limit $h \to 0$ of (2-5) gives only the wave equation since

$$\lim_{h\to 0} \frac{2}{h^2} \sum_{m=1}^{\infty} \frac{h^{2m}}{(2m)!} \frac{\partial^{2m} u(x,t)}{\partial x^{2m}} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{2-6}$$

It is important to emphasize that the gradient elasticity equation (1-1) cannot be obtained by the limit $h \to 0$ [Tarasov 2014a]. Equation (2-5) gives (1-1) only by deleting all $O(h^6)$ terms. Therefore, (2-1) cannot be considered as an exact discretization of (1-1) or its microstructural basis to derive equations of gradient elasticity.

It should be noted that approaches based on models of lattices with the nearest-neighbor and next-nearest-neighbor interactions [Tarasov 2014a; 2015b] can give (1-1) in the limit $h \to 0$ in contrast to approaches based on lattice equation (2-1) with the nearest-neighbor interactions. At the same time, the lattice equations with nearest-neighbor and next-nearest-neighbor interactions have infinite series of even-order derivatives similar to (2-5) before taking the limit.

3. Exact difference analogs of derivatives

To have an exact discrete analog to the gradient elasticity equations, we should consider a problem of discretization of these equations. Let us consider a problem of derivation of an exact discrete analog to the gradient elasticity equation (1-1). To solve this problem, we should find new types of differences, which will be denoted by ${}^T\Delta^{2n}$, that correspond exactly to the derivatives $\partial^{2n}/\partial x^{2n}$ with $n=1$ and $n=2$. In order for the difference ${}^T\Delta^{2n}$ of even orders $2n$ ($n \in \mathbb{N}$) to not correspond to the derivatives $\partial^{2n}/\partial x^{2n}$ approximately, these differences should satisfy the condition

$$\frac{1}{h^{2n}} F^{-1}(F_{h,\Delta}({}^T\Delta^{2n} u_n(t))) = \frac{\partial^{2n} u(x,t)}{\partial x^{2n}} \tag{3-1}$$

in contrast to the usual finite differences that are represented by infinite series of derivatives (see (2-5)).

Condition (3-1) can be realized if the difference ${}^T\Delta^{2n}$ has the Fourier series transform in the form

$$F_{h,\Delta}\{{}^T\mathbf{\Delta}^{2n} u_m(t)\} := \sum_{m=-\infty}^{+\infty} e^{-ikm}\, {}^T\mathbf{\Delta}^{2n} u_m(t) = (-1)^n (kh)^{2n} \hat{u}(k,t). \tag{3-2}$$

In order to get (3-2), the differences ${}^T\mathbf{\Delta}^{2n}$ should be represented by the convolution

$${}^T\mathbf{\Delta}^{2n} u_m(t) := \sum_{j=-\infty}^{+\infty} K_{2n}(j) u_{m-j}(t), \tag{3-3}$$

where

$$F_{1,\Delta}\{K_{2n}(j)\} = (-1)^n k^{2n} \tag{3-4}$$

and $K_{2n}(-m) = K_{2n}(m)$ hold for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

In order to apply $F_{1,\Delta}$ to the differences (3-3), we assume that u_m and $K_{2n}(m)$ are the real-valued functions of discrete variable $m \in \mathbb{Z}$ such that $u_m \in l^2$ and $K_{2n}(m) \in l^1$.

Using $K_{2n}(-m) = K_{2n}(m)$, the kernels $K_{2n}(m)$ can be defined by

$$K_{2n}(m) = F^{-1}_{1,\Delta}\{(-1)^n k^{2n}\} = (-1)^n \frac{1}{\pi} \int_0^{\pi} k^{2n} \cos(km)\, dk. \tag{3-5}$$

For $m=2$ and $m=4$, we get

$$K_2(n) = -\frac{2(-1)^n}{n^2} \quad (n \neq 0), \qquad K_2(0) = -\frac{\pi^2}{3}, \tag{3-6}$$

$$K_4(n) = +\frac{4\pi^2(-1)^n}{n^2} - \frac{24(-1)^n}{n^4} \quad (n \neq 0,\ n \in \mathbb{Z}), \qquad K_4(0) = +\frac{\pi^4}{5}. \tag{3-7}$$

As a result, the differences (3-3) of second and fourth orders are defined by

$${}^T\mathbf{\Delta}^2 u_n := -\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{2(-1)^m}{m^2} u_{n-m} - \frac{\pi^2}{3} u_n, \tag{3-8}$$

$${}^T\mathbf{\Delta}^4 u_n := \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \left(\frac{4\pi^2(-1)^m}{m^2} - \frac{24(-1)^m}{m^4} \right) u_{n-m} + \frac{\pi^4}{5} u_n. \tag{3-9}$$

In the general case, we can use Equation 2.5.3.5 of [Prudnikov et al. 1986], which gives

$$K_{2n}(m) = \sum_{k=0}^{n-1} \frac{(-1)^{m+k+n}(2n)!\,\pi^{2n-2k-2}}{(2n-2k-1)!} \frac{1}{m^{2k+2}} \quad (m \in \mathbb{Z},\ m \neq 0). \tag{3-10}$$

For $m = 0$, we have

$$K_{2n}(0) = \frac{(-1)^n \pi^{2n}}{2n+1}. \tag{3-11}$$

These kernels define the differences ${}^T\mathbf{\Delta}^{2n}$ of even orders $2n$ for all $n \in \mathbb{N}$ by (3-3).

4. Transformation of a gradient elasticity equation into a discrete equation

In Section 2, we demonstrate that discrete equation (2-1) cannot be considered as an exact discrete analog to (1-1).

Using the differences (3-8) and (3-9), we can consider an inverse problem. We will start with the equation of gradient elasticity and then try to get an exact discrete analog to this continuum equation without approximation by deleting terms. We would like to answer the questions: what do the gradient elasticity equations describe exactly at discrete (lattice) level and what is an exact analog to the gradient elasticity equations?

Let us consider the Fourier integral transform F, which is defined by

$$\hat{u}(k,t) = \int_{-\infty}^{+\infty} dx\, u(x,t) e^{-ikx} = F\{u(x,t)\}. \tag{4-1}$$

Applying this Fourier transform F to (1-1), we get

$$\frac{d^2 \hat{u}(k,t)}{dt^2} = -c^2 k^2 \hat{u}(k,t) + l^2 k^4 \hat{u}(k,t). \tag{4-2}$$

Using the inverse Fourier series transform $F^{-1}_{h,\Delta}$ such that

$$u_n(t) = \frac{h}{2\pi} \int_{-\pi/h}^{+\pi/h} dk\, \hat{u}(k,t) e^{ikhn} = F^{-1}_{h,\Delta}\{\hat{u}(k,t)\}, \tag{4-3}$$

(4-2) gives

$$\frac{d^2 u_n(t)}{dt^2} = \frac{c^2}{h^2} {}^T\mathbf{\Delta}^2 u_n(t) + \frac{l^2}{h^4} {}^T\mathbf{\Delta}^4 u_n(t), \tag{4-4}$$

where ${}^T\mathbf{\Delta}^2$ and ${}^T\mathbf{\Delta}^4$ are the differences that are defined by (3-8) and (3-9). Substitution of (3-8) and (3-9) into (4-4) gives

$$\frac{d^2 u_n(t)}{dt^2} = \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \left(\frac{4\pi^2 l^2 - 2c^2 h^2}{h^4} \frac{(-1)^m}{m^2} - \frac{24 l^2}{h^4} \frac{(-1)^m}{m^4} \right) u_{n-m}(t) + \left(\frac{\pi^4 l^2}{5h^4} - \frac{\pi^2 c^2}{3h^2} \right) u_n(t) \quad (n \in \mathbb{Z}). \tag{4-5}$$

These equations are an exact discrete analog to the equation of gradient elasticity (1-1).

Let us give some mathematical remarks about suggested difference equations. To use the Fourier series transform, we assume that the function $u_n(t)$ belongs to the Hilbert space l^2 of square-summable sequences, where the norm on the l^p-space is defined by

$$\|u\|_p := \left(\sum_{n=-\infty}^{+\infty} |u_n|^p \right)^{1/p}.$$

It is easy to see that the differences (3-8) and (3-9) are defined by convolutions of $u_m \in l^2$ and the functions

$$a_{2n}(m) = \frac{(-1)^m}{m^{2n}} \quad (m \neq 0,\ m \in \mathbb{Z})$$

that belong to the space l^1. Using the Young's inequality for convolutions [Young 1912a; 1912b; Hardy et al. 1952, Theorem 276] in the form

$$\| {}^T\mathbf{\Delta}^{2n} u \|_r = \| a_{2n} * u \|_r \leq \| a_{2n} \|_p \| u \|_q, \tag{4-6}$$

where

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}, \tag{4-7}$$

we get that the result of the action of operators ${}^T\mathbf{\Delta}^{2n}$ also belongs to the Hilbert space l^2 of square-summable sequences, i.e.,

$$g_m := {}^T\mathbf{\Delta}^{2n} u_m \in l^2 \tag{4-8}$$

since condition (4-7) holds.

As a result, the T-differences are the operators ${}^T\mathbf{\Delta}^{2n} : l^2 \to l^2$.

Note that, using Equation 5.1.2.3 of [Prudnikov et al. 1986], we can get

$$\sum_{m=1}^{\infty} K_{2n}(m) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^{2n}} = (2^{1-2n} - 1)\zeta(2n) = -\frac{1}{\Gamma(2n)} \int_0^{\infty} \frac{x^{2n-1}}{e^x + 1}\, dx = T_{2n}, \tag{4-9}$$

where $\zeta(z)$ is the Riemann zeta function, $\Gamma(z)$ is the gamma function, and

$$T_2 = -\frac{\pi^2}{12}, \qquad T_4 = -\frac{7\pi^4}{720}.$$

As a result, the T-differences acting on $u_m = 1$ converge.

The main property of the suggested differences (3-8) and (3-9) are that the Fourier series transform $F_{h,\Delta}$ of these differences is represented by

$$F_{h,\Delta}({}^T\mathbf{\Delta}^{2n} u_m(t)) = (ikh)^{2n} \hat{u}(k, t). \tag{4-10}$$

This equation leads us to the corresponding equality

$$\frac{1}{h^{2n}} F^{-1}(F_{h,\Delta}({}^T\mathbf{\Delta}^{2n} u_m(t))) = \frac{1}{h^{2n}} F^{-1}((ikh)^{2n} \hat{u}(k, t)) = \frac{\partial^{2n} u(x, t)}{\partial x^{2n}}, \tag{4-11}$$

which means that this difference of order $2n$ gives the derivative $\partial^{2n}/\partial x^{2n}$ exactly. The T-differences of orders $2n$ are connected with the derivatives $\partial^{2n}/\partial x^{2n}$ not only asymptotically by the limit $h \to 0$. It's obvious that the limit $h \to 0$ also gives this derivatives

$$\lim_{h \to 0} \frac{F^{-1}(F_{h,\Delta}({}^T\mathbf{\Delta}^{2n}))}{h^{2n}} = \frac{\partial^{2n}}{\partial x^{2n}}. \tag{4-12}$$

As a result, the suggested equations (4-5) with T-difference can be considered not only as approximations of the gradient elasticity equations. The suggested discrete equations (4-5) are exact discrete analogs to the continuum gradient elasticity equation (1-1).

5. Exact difference equations for three-dimensional gradient elasticity

In this section, we propose discrete equations of three-dimensional gradient elasticity by using the approach suggested in [Tarasov 2014d; 2015a].

The Mindlin equations [1964; 1965; 1968] of three-dimensional gradient elasticity have the form

$$\rho \frac{\partial^2 u_i(\boldsymbol{r},t)}{\partial t^2} - \rho l_1^2 \sum_{j=1}^{3} \frac{\partial^4 u_i(\boldsymbol{r},t)}{\partial x_j^2\, \partial t^2} = (\lambda+\mu) \sum_{j=1}^{3} \frac{\partial^2 u_j(\boldsymbol{r},t)}{\partial x_i\, \partial x_j} + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i(\boldsymbol{r},t)}{\partial x_j^2} - (\lambda+\mu) l_2^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^4 u_j(\boldsymbol{r},t)}{\partial x_k^2\, \partial x_i\, \partial x_j} - \mu l_3^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^4 u_i(\boldsymbol{r},t)}{\partial x_k^2\, \partial x_j^2} + f_i(\boldsymbol{r},t), \tag{5-1}$$

where the $u_i(\boldsymbol{r},t)$ are components of the displacement field for the continuum, $f_i(\boldsymbol{r},t)$ are the components of the body force, λ and μ are the Lame constants, l_i ($i = 1, 2, 3$) are the Mindlin scale parameters, ρ is the mass density, $\boldsymbol{r} = \sum_{j=1}^{3} x_j \boldsymbol{e}_j$, and $\boldsymbol{e}_j$ ($j = 1, 2, 3$) are the basis vectors of the Cartesian coordinate system of $\mathbb{R}^3$.

Using the Fourier transforms $F_{h,\Delta}^{-1} \circ F$, the equations with T-differences for (5-1) have the form

$$\rho \frac{\partial^2 u_i[\boldsymbol{n},t]}{\partial t^2} - \frac{\rho l_1^2}{h^2} \sum_{j=1}^{3} {}^{T}\boldsymbol{\Delta}_j^2 \frac{\partial^2 u_i[\boldsymbol{n},t]}{\partial t^2} = \frac{\lambda+\mu}{h^2} \sum_{j=1}^{3} {}^{T}\boldsymbol{\Delta}_i^1\, {}^{T}\boldsymbol{\Delta}_j^1 u_j[\boldsymbol{n},t] + \frac{\mu}{h^2} \sum_{j=1}^{3} {}^{T}\boldsymbol{\Delta}_j^2 u_i[\boldsymbol{n},t] - \frac{(\lambda+\mu) l_2^2}{h^4} \sum_{k=1}^{3}\sum_{j=1}^{3} {}^{T}\boldsymbol{\Delta}_k^2\, {}^{T}\boldsymbol{\Delta}_i^1\, {}^{T}\boldsymbol{\Delta}_j^1 u_j[\boldsymbol{n},t] - \frac{\mu l_3^2}{h^4} \sum_{k=1}^{3}\sum_{j=1}^{3} {}^{T}\boldsymbol{\Delta}_k^2\, {}^{T}\boldsymbol{\Delta}_j^2 u_i[\boldsymbol{n},t] + f_i[\boldsymbol{n},t], \tag{5-2}$$

where we assume $h_1 = h_2 = h_3 = h$ and $u_j[\boldsymbol{n},t] := F_{h,\Delta}^{-1} \circ F u_j(\boldsymbol{r},t)$ are discrete fields such that $u_j[\boldsymbol{n},t] = h u_j(h\boldsymbol{n},t)$. In (5-2), we use ${}^{T}\boldsymbol{\Delta}_j^1$ and ${}^{T}\boldsymbol{\Delta}_j^2$, which are the partial T-differences of first and second orders. The partial T-difference of first order is defined by

$${}^{T}\boldsymbol{\Delta}_j^1 u_i[\boldsymbol{n},t] := \sum_{\substack{m_j=-\infty \\ m_j \neq 0}}^{+\infty} \frac{(-1)^{m_j}}{m_j} u_i[\boldsymbol{n} - m_j \boldsymbol{e}_j, t]. \tag{5-3}$$

The partial T-difference of second order has the form

$${}^{T}\boldsymbol{\Delta}_j^2 u_i[\boldsymbol{n},t] := \sum_{\substack{m_j=-\infty \\ m_j \neq 0}}^{+\infty} \frac{2(-1)^{m_j+1}}{m_j^2} u_i[\boldsymbol{n} - m_j \boldsymbol{e}_j, t] - \frac{\pi^2}{3} u_i[\boldsymbol{n},t]. \tag{5-4}$$

Here $\boldsymbol{e}_j$ ($j = 1, 2, 3$) are the basis vectors of the Cartesian coordinate system of $\mathbb{R}^3$, and $\boldsymbol{n} = \sum_{j=1}^{3} n_j \boldsymbol{e}_j$, where $n_j \in \mathbb{Z}$.

Note that it is easy to generalize these difference equation to the case of various h_j ($j = 1, 2, 3$). For example, in this case, we should use $u_j[\boldsymbol{n},t] = h_j u_j(\boldsymbol{x}(\boldsymbol{n}),t)$, where $\boldsymbol{x}(\boldsymbol{n}) = \sum_{i=1}^{3} h_j n_j \boldsymbol{e}_j$.

For the three-dimensional case, the simplified continuum equations of gradient elasticity have the form

$$\rho \frac{\partial^2 u_i(\boldsymbol{r},t)}{\partial t^2} = \sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2}{\partial x_j\, \partial x_l} \left(1 + l^2 \sum_{m=1}^{3} \frac{\partial^2}{\partial x_m^2}\right) u_k(\boldsymbol{r},t) + f_i(\boldsymbol{r},t), \tag{5-5}$$

where $u_i(\boldsymbol{r}, t)$ are components of the displacement field for the continuum, $f_i(\boldsymbol{r}, t)$ are the components of the body force, and C_{ijkl} is the fourth-order elastic stiffness tensor. For isotropic materials, C_{ijkl} are expressed in terms of the Lame constants λ and μ by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5-6}$$

where λ and μ are the usual Lame constants.

The equations with T-differences for (5-5) of the three-dimensional gradient elasticity have the form

$$\rho\frac{\partial^2 u_i[\boldsymbol{n}, t]}{\partial t^2} = \frac{1}{h^2}\sum_{j,k,l=1}^{3} C_{ijkl}\,{}^T\boldsymbol{\Delta}_j^1\,{}^T\boldsymbol{\Delta}_l^1\left(1 + \frac{l^2}{h^2}\sum_{m=1}^{3}{}^T\boldsymbol{\Delta}_m^2\right)u_k[\boldsymbol{n}, t] + f_i[\boldsymbol{n}, t]. \tag{5-7}$$

If we consider the case with $u_x(\boldsymbol{r}, t) = u(x, t)$ and $f_x(\boldsymbol{r}, t) = f(x, t)$, where the other components, u_y, u_z, f_y, and f_z, are equal to zero, then we get the considered one-dimensional gradient elasticity equations.

Equations (5-2) and (5-7) are equations of exact discretization of the three-dimensional gradient elasticity equations. These equations with T-differences are connected with the partial differential equation of gradient elasticity without approximations.

To solve linear partial differential and difference equations of the gradient elasticity, we can use the method of separation of variables. For simplification, we will consider (1-1) and (4-4). For these equations, the fields $u(x, t)$ and $u_n(t)$ are represented in the forms

$$u(x, t) = u(x)T(t), \qquad u_n(t) = u[n]T(t). \tag{5-8}$$

Substitution of (5-8) into (1-1) and (4-4) gives equations of $u(x)$ and $u[n]$ that can be represented as

$$l^2\frac{\partial^4 u(x)}{\partial x^4} + c^2\frac{\partial^2 u(x)}{\partial x^2} + \omega^2 u(x) = 0, \tag{5-9}$$

$$\frac{l^2}{h^4}\,{}^T\boldsymbol{\Delta}^4 u[n] + \frac{c^2}{h^2}\,{}^T\boldsymbol{\Delta}^2 u[n] + \omega^2 u[n] = 0, \tag{5-10}$$

where ${}^T\boldsymbol{\Delta}^m$ is the T-difference of order m with respect to n. The equations for $T(t)$ are the same for (1-1) and (4-4).

To solve (5-10), we assume that the solution of (5-10) is proportional to $\exp(\lambda n)$ for some constant λ. Substitute $u[n] = \exp(\lambda n)$ into difference equation (5-10), and use the relation

$${}^T\boldsymbol{\Delta}^1 \exp(\lambda n) = \lambda \exp(\lambda n), \tag{5-11}$$

which is proved by the Poisson–Abel technique in [Tarasov 2016a]. Then we get a general solution of difference equation (5-10) in the form

$$u[n] = C_1 e^{\lambda_+ \cdot hn} + C_2 e^{-\lambda_+ \cdot hn} + C_3 e^{\lambda_- \cdot hn} + C_4 e^{-\lambda_- \cdot hn}, \tag{5-12}$$

where

$$\lambda_\pm := \sqrt{\pm\frac{1}{2l^2}\sqrt{c^4 - 4l^2\omega^2} - \frac{c^2}{2l^2}}. \tag{5-13}$$

Differential equation (5-9) has the general solution

$$u(x) = c_1 e^{\lambda_+ \cdot x} + c_2 e^{-\lambda_+ \cdot x} + c_3 e^{\lambda_- v x} + c_4 e^{-\lambda_- \cdot x}. \tag{5-14}$$

It is easy to see that solutions (5-12) and (5-14) are connected by the relation $u[n] = hu(hn)$ for all $n \in \mathbb{Z}$ and $h > 0$, where $C_k = hc_k$ ($k = 1, 2, 3, 4$).

Equation (5-10) can be considered as an exact discretization of differential equation (5-9). The exact discretization means that the difference equation has the same general solution as the associated differential equation. The criterion of exact discretization of differential equations can be formulated in the following form [Potts 1982; Mickens 2000; Tarasov 2016a].

An exact discretization is a map from a differential equation to a discrete (difference) equation, for which the solution $u[n]$ of the discrete equation and the solution $u(x)$ of the associated differential equation are the same, i.e., if and only if the discrete function $u[n]$ is exactly equal to the function $u(x)$ for $x = hn$, i.e., $u[n] = hu(hn)$ ($n \in \mathbb{Z}$) for arbitrary values of $h > 0$.

It should be noted that discretization of an equation by standard finite differences (5-9) cannot be considered as an exact discretization since ${}^f\Delta^1 \exp(\lambda n) \neq \lambda \exp(\lambda n)$ and (5-12) is not the solution of the corresponding finite difference equation.

In elasticity theory, the boundary conditions play an important role. The boundary conditions for T-difference equations have a form that is similar to the boundary conditions of the corresponding differential equations. In these boundary conditions, the function $u_i[n]$ should be used instead of the function $u_i(x)$ and the T-differences ${}^T\mathbf{\Delta}^m_j$ instead of the partial derivatives $\partial^m/\partial x^m_j$ of order $m \in \mathbb{N}$. For a simple example, the exact discrete analog to the boundary conditions

$$\left(\frac{\partial^m u(x)}{\partial x^m}\right)_{x=0} = 0, \qquad \left(\frac{\partial^m u(x)}{\partial x^m}\right)_{x=L} = 0, \tag{5-15}$$

for some values $m \in \{0, 1, 2, 3\}$, where $\partial^0 u(x)/\partial x^0 := u(x)$, have the form

$${}^T\mathbf{\Delta}^m u[0] = 0, \qquad {}^T\mathbf{\Delta}^m u[N] = 0, \tag{5-16}$$

where $hN = L$, $N \in \mathbb{N}$, $u[n] = hu(hn)$, and ${}^T\mathbf{\Delta}^0 u[n] = u[n]$. For example, the discrete analog to the periodic boundary condition $u(x + L) = u(x)$ takes the form $u[n + N] = u[n]$. The boundary conditions for the difference equations define the constants of the corresponding general solution.

As a result, we can see that discrete (lattice) equations with T-differences can be solved analytically. Thus, obtained solutions of these discrete equations are the same as those of the associated differential equations of continuum models.

6. Physical interpretation of the difference equations

In this section, we describe a direct connection between the proposed T-differences and lattice models with long-range interactions. We prove that the discrete (lattice) equations with T-differences, which are suggested for the gradient elasticity models, correspond to lattice models with long-range interactions of power-law type.

From a mathematical point of view, the previous discrete (lattice) models of the gradient elasticity are based on the standard (forward, backward, and central) finite differences. These models assume that we consider nearest-neighbor and next-nearest-neighbor interactions only, which do not correspond exactly to real physical properties of interactions of particles. The characteristic properties of the underlying physical interactions, which are electromagnetic interactions, are of a long-range nature. The models will more adequately describe elastic materials and media, if these models take into account the long-range character of interatomic forces. One of the most widely used long-range interactions is the interaction of the type $1/|n|^{\alpha}$, or equivalently $1/|n-m|^{\alpha}$. The integer values of α correspond to the well-known physical cases that correspond to the Coulomb potential for $\alpha = 1$ and the dipole-dipole interaction for $\alpha = 3$. Moreover, in various cases, these interactions are crucial. For example, the excitation transfer in molecular crystals and the vibron energy transport in polymers are due to the transition dipole-dipole interaction of the type $1/|n|^3$. Polyatomic molecules contain charged groups with a long-range Coulomb interaction $1/|n|^1$ between them. For excitons and phonons in semiconductors and molecular crystals, the dispersion curves of two elementary excitations intersect or are close, which leads to an effective long-range transfer.

It should be noted that classical and quantum descriptions of media with long-range interactions are the subject of continued interest in physics. The long-range interactions have been studied in discrete systems as well as in their continuous analogs. For example, discrete and lattice models with long-range interactions have been studied in the references below. An infinite one-dimensional model with long-range interactions is described in [Dyson 1969a; 1969b; 1971]. Two-dimensional and three-dimensional classical models with long-range interactions are considered in [Joyce 1969], and their quantum generalization has been suggested in [Nakano and Takahashi 1994a; 1994b; 1995; Sousa 2005]. Kinks, solitons, breathers, dynamical chaos, and synchronization in lattice models with long-range particle interactions are studied in different papers (for example, see [Gaididei et al. 1995; Mingaleev et al. 1998; Rasmussen et al. 1998; Gorbach and Flach 2005; Korabel and Zaslavsky 2007; Korabel et al. 2007; Zaslavsky et al. 2007] and references therein).

It should be noted that the kernels (3-10), (3-8), (5-3), and (5-4) of the suggested T-differences can be considered linear combinations of kernels of the type $1/|n|^{\alpha}$, with integer $\alpha \in \mathbb{N}$. We can assume that the suggested T-differences (3-8) of second and fourth orders in gradient elasticity equations correspond to the well-known underlying interatomic and intermolecular forces such as the Coulomb force of the type $1/|n|^2$ and the dipole-dipole force of the type $1/|n|^4$. From a mathematical point of view, these linear combinations are selected from the set of other combinations by the fact that they exactly correspond to the continuum models, which are described by differential equations of integer orders. The suggested type of long-range interactions, which are described by the kernels of the suggested T-differences, is distinguished from other interactions by exact correspondence to continuum differential equations and by preservation of the main characteristic properties of differential equations and corresponding solutions.

7. Conclusion

In this paper, we focused our consideration on gradient elasticity models that were suggested by Mindlin [1964; 1965; 1968; Eringen 1983]. It should be noted that the proposed approach can also be applied to the gradient elasticity models suggested by Aifantis [1994; 1992; 2011; Metrikine and Askes 2002; Askes and Aifantis 2011]. The standard approach has certain disadvantages compared to the proposed approach of obtaining exact discrete (lattice) analogs of continuum equations. Let us explain this point in more detail. From a mathematical point of view, the standard discrete (lattice) models of the gradient elasticity are based on a mathematical approach that uses the forward, backward, and central finite differences. From the physical point of view, these models assume nearest-neighbor and next-nearest-neighbor interactions in the media [Tarasov 2014a; 2015b; Askes and Aifantis 2011]. These models are not quite adequate for the following reasons. From a mathematical point of view, it is well-known that the finite differences of cannot be considered an exact discretization of the derivatives: solutions of equations with standard finite differences do not coincide with solutions of the corresponding differential equations, and the standard finite differences do not satisfy the same algebraic relations as the operators of differentiation. The correspondence between the discrete (lattice) theory and the continuum theory lies not so much in the limiting condition of the steps (or primitive lattice vectors) as in the fact that mathematical operations on these theories should obey the same laws in many cases. From a physical point of view, the standard discrete (lattice) models, which are based on an assumption of nearest-neighbor and next-nearest-neighbor interactions only, do not fully reflect the physical reality. The characteristic properties of the underlying physical interactions, which are electromagnetic interactions, are of a long-range nature. The models will more adequately describe elastic materials and media, when these models will take into account the long-range character of interatomic forces that can be characterized as $1/|n|^{\alpha}$. The integer values of α correspond to the well-known physical cases that correspond to the Coulomb force for $\alpha = 1$ and the dipole-dipole force for $\alpha = 4$. Moreover, in various cases, these interactions are crucial.

In this paper, we propose discrete (lattice) equations that correspond exactly to the gradient elasticity equations. From a mathematical point of view, these discrete equations are uniquely equations with differences that correspond exactly to the continuum equations. From a physical point of view, these equations describe microstructural models of lattices with long-range interactions of the type $1/|n|^{\alpha}$ with integer α.

The main advantage of the suggested discrete (lattice) equations is the connection with continuum equations without any approximation. Moreover, these discrete (difference) equations have the same general solutions as the associated differential equations. The exact discretization means that the difference equation has the same general solution as the associated differential equation. It should also be emphasized that these discrete equations allow us to obtain analytical solutions. This is based on the fact that the proposed T-differences satisfy the same algebraic relations as the operators of differentiation.

The computer simulations of discrete systems with long-range interactions of the form $1/|n|^{\alpha}$ are actively used for integer and noninteger values of α (for example, see [Gaididei et al. 1995; Mingaleev et al. 1998; Rasmussen et al. 1998; Gorbach and Flach 2005; Korabel and Zaslavsky 2007; Korabel et al. 2007; Zaslavsky et al. 2007]). The suggested T-differences can be considered linear combinations of interactions of the type $1/|n|^{\alpha}$ with integer α. Therefore, we assume that computer simulations of the suggested T-difference (lattice) equations, which are exact discretizations of corresponding differential equations of continua, can be successfully realized.

We assume that the suggested equations with T-differences can be important in application since they allow us to reflect characteristic properties of complex elastic materials and media at the microscale and nanoscale, where long-range interactions play a crucial role in determining the properties of these materials and media (see [Ostoja-Starzewski 2002; Tarasov 2010] and references therein).

References

[Aifantis 1992] E. C. Aifantis, "On the role of gradients in the localization of deformation and fracture", *Int. J. Eng. Sci.* **30**:10 (1992), 1279–1299.

[Aifantis 1994] E. C. Aifantis, "Gradient effects at macro, micro, and nano scales", *J. Mech. Behav. Mater.* **5**:3 (1994), 355–375.

[Aifantis 2011] E. C. Aifantis, "Gradient nanomechanics: applications to deformation, fracture, and diffusion in nanopolycrystals", *Metall. Mater. Trans. A* **42**:10 (2011), 2985–2998.

[Askes and Aifantis 2011] H. Askes and E. C. Aifantis, "Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results", *Int. J. Solids Struct.* **48**:13 (2011), 1962–1990.

[Auffray et al. 2015] N. Auffray, F. dell'Isola, V. A. Eremeyev, A. Madeo, and G. Rosi, "Analytical continuum mechanics *à la* Hamilton–Piola least action principle for second gradient continua and capillary fluids", *Math. Mech. Solids* **20**:4 (2015), 375–417.

[Born and Huang 1998] M. Born and K. Huang, *Dynamical theory of crystal lattices*, Clarendon, Oxford, 1998.

[Böttger 1983] H. Böttger, *Principles of the theory of lattice dynamics*, Physik-Verlag, Weinheim, Germany, 1983.

[Carcaterra et al. 2015] A. Carcaterra, F. dell'Isola, R. Esposito, and M. Pulvirenti, "Macroscopic description of microscopically strongly inhomogenous systems: a mathematical basis for the synthesis of higher gradients metamaterials", *Arch. Ration. Mech. Anal.* **218**:3 (2015), 1239–1262.

[dell'Isola et al. 2015] F. dell'Isola, U. Andreaus, and L. Placidi, "At the origins and in the vanguard of peridynamics, nonlocal and higher-gradient continuum mechanics: an underestimated and still topical contribution of Gabrio Piola", *Math. Mech. Solids* **20**:8 (2015), 887–928.

[dell'Isola et al. 2016a] F. dell'Isola, A. Della Corte, and I. Giorgio, "Higher-gradient continua: the legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives", *Math. Mech. Solids* (online publication January 2016).

[dell'Isola et al. 2016b] F. dell'Isola, I. Giorgio, M. Pawlikowski, and N. L. Rizzi, "Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium", *Proc. R. Soc. Lond. A* **472**:2185 (2016), article 20150790.

[Dyson 1969a] F. J. Dyson, "Existence of a phase-transition in a one-dimensional Ising ferromagnet", *Commun. Math. Phys.* **12**:2 (1969), 91–107.

[Dyson 1969b] F. J. Dyson, "Non-existence of spontaneous magnetization in a one-dimensional Ising ferromagnet", *Commun. Math. Phys.* **12**:3 (1969), 212–215.

[Dyson 1971] F. J. Dyson, "An Ising ferromagnet with discontinuous long-range order", *Commun. Math. Phys.* **21** (1971), 269–283.

[Eringen 1972] A. C. Eringen, "Nonlocal polar elastic continua", *Int. J. Eng. Sci.* **10**:1 (1972), 1–16.

[Eringen 1983] A. C. Eringen, "On differential-equations of nonlocal elasticity and solutions of screw dislocation and surface-waves", *J. Appl. Phys.* **54**:9 (1983), 4703–4710.

[Eringen 2002] A. C. Eringen, *Nonlocal continuum field theories*, Springer, New York, 2002.

[Gaididei et al. 1995] Yu. B. Gaididei, N. Flytzanis, A. Neuper, and F. G. Mertens, "Effect of nonlocal interactions on soliton dynamics in anharmonic lattices", *Phys. Rev. Lett.* **75**:11 (1995), 2240–2243.

[Gorbach and Flach 2005] A. V. Gorbach and S. Flach, "Compactlike discrete breathers in systems with nonlinear and nonlocal dispersive terms", *Phys. Rev. E* **72**:5 (2005), 056607.

[Hardy et al. 1952] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1952.

[Joyce 1969] G. S. Joyce, "Absence of ferromagnetism or antiferromagnetism in the isotropic Heisenberg model with long-range interactions", *J. Phys. C Solid State Phys.* **2**:8 (1969), 1531–1533.

[Kittel 1987] C. Kittel, *Quantum theory of solids*, 2nd ed., Wiley, New York, 1987.

[Korabel and Zaslavsky 2007] N. Korabel and G. M. Zaslavsky, "Transition to chaos in discrete nonlinear Schrödinger equation with long-range interaction", *Physica A* **378**:2 (2007), 223–237.

[Korabel et al. 2007] N. Korabel, G. M. Zaslavsky, and V. E. Tarasov, "Coupled oscillators with power-law interaction and their fractional dynamics analogues", *Commun. Nonlinear Sci. Numer. Simul.* **12**:8 (2007), 1405–1417.

[Maslov 1976] V. P. Maslov, *Operational methods*, Mir, Moscow, 1976.

[Metrikine and Askes 2002] A. V. Metrikine and H. Askes, "One-dimensional dynamically consistent gradient elasticity models derived from a discrete microstructure, 1: Generic formulation", *Eur. J. Mech. A Solids* **21**:4 (2002), 555–572.

[Mickens 2000] R. E. Mickens, "Nonstandard finite difference schemes", Chapter 1, pp. 1–54 in *Applications of nonstandard finite difference schemes* (Atlanta, 1999), edited by R. E. Mickens, World Scientific, River Edge, NJ, 2000.

[Mindlin 1964] R. D. Mindlin, "Micro-structure in linear elasticity", *Arch. Ration. Mech. Anal.* **16**:1 (1964), 51–78.

[Mindlin 1965] R. D. Mindlin, "Second gradient of strain and surface-tension in linear elasticity", *Int. J. Solids Struct.* **1**:4 (1965), 417–438.

[Mindlin 1968] R. D. Mindlin, "Theories of elastic continua and crystal lattice theories", Chapter 3, pp. 312–320 in *Mechanics of generalized continua* (Stuttgart, 1967), edited by E. Kröner, Springer, Berlin, 1968.

[Mingaleev et al. 1998] S. F. Mingaleev, Yu. B. Gaididei, and F. G. Mertens, "Solitons in anharmonic chains with power-law long range interactions", *Phys. Rev. E* **58**:3 (1998), 3833–3842.

[Nakano and Takahashi 1994a] H. Nakano and M. Takahashi, "Quantum Heisenberg chain with long-range ferromagnetic interactions at low temperature", *J. Phys. Soc. Japan* **63**:3 (1994), 926–933.

[Nakano and Takahashi 1994b] H. Nakano and M. Takahashi, "Quantum Heisenberg model with long-range ferromagnetic interactions", *Phys. Rev. B* **50**:14 (1994), 10331–10334.

[Nakano and Takahashi 1995] H. Nakano and M. Takahashi, "Magnetic-properties of quantum Heisenberg ferromagnets with long-range interactions", *Phys. Rev. B* **52**:9 (1995), 6606–6610.

[Ostoja-Starzewski 2002] M. Ostoja-Starzewski, "Lattice models in micromechanics", *Appl. Mech. Rev.* (*ASME*) **55**:1 (2002), 35–60.

[Potts 1982] R. B. Potts, "Differential and difference equations", *Amer. Math. Monthly* **89**:6 (1982), 402–407.

[Prudnikov et al. 1986] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*, vol. 1: Elementary functions, Gordon and Breach Science, New York, 1986.

[Rahali et al. 2015] Y. Rahali, I. Giorgio, J. F. Ganghoffer, and F. dell'Isola, "Homogenization à la Piola produces second gradient continuum models for linear pantographic lattices", *Int. J. Eng. Sci.* **97** (2015), 148–172.

[Rasmussen et al. 1998] K. Ø. Rasmussen, P. L. Christiansen, M. Johansson, Yu. B. Gaididei, and S. F. Mingaleev, "Localized excitations in discrete nonlinear Schrödinger systems: effects of nonlocal dispersive interactions and noise", *Physica D* **113**:2–4 (1998), 134–151.

[Rogula 1982] D. Rogula, *Nonlocal theory of material media*, International Centre for Mechanical Sciences **268**, Springer, Vienna, 1982.

[Sedov 1968] L. I. Sedov, "Модели сплошных сред с внутренними степенями свободы", *Prikl. Mat. Mekh.* **32**:5 (1968), 771–785. Translated as Models of continuous media with internal degrees of freedom in *Journal of Applied Mathematics and Mechanics* **32**:5 (1968), 803–819.

[Sedov 1971] L. I. Sedov, *A course in continuum mechanics*, vol. 1–4, Wolters-Noordhoff, Groningen, 1971.

[Sousa 2005] J. R. Sousa, "Phase diagram in the quantum XY model with long-range interactions", *Eur. Phys. J. B* **43**:1 (2005), 93–96.

[Tarasov 2006a] V. E. Tarasov, "Continuous limit of discrete systems with long-range interaction", *J. Phys. A Math. Gen.* **39**:48 (2006), 14895–14910.

[Tarasov 2006b] V. E. Tarasov, "Map of discrete system into continuous", *J. Math. Phys.* **47**:9 (2006), article 092901.

[Tarasov 2010] V. E. Tarasov, *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*, Springer, Heidelberg, 2010.

[Tarasov 2013] V. E. Tarasov, "Lattice model with power-law spatial dispersion for fractional elasticity", *Cent. Eur. J. Phys.* **11**:11 (2013), 1580–1588.

[Tarasov 2014a] V. E. Tarasov, "General lattice model of gradient elasticity", *Mod. Phys. Lett. B* **28**:7 (2014), article 1450054.

[Tarasov 2014b] V. E. Tarasov, "Lattice model of fractional gradient and integral elasticity: long-range interaction of Grünwald–Letnikov–Riesz type", *Mech. Mater.* **70** (2014), 106–114.

[Tarasov 2014c] V. E. Tarasov, "Lattice with long-range interaction of power-law type for fractional non-local elasticity", *Int. J. Solids Struct.* **51**:15–16 (2014), 2900–2907.

[Tarasov 2014d] V. E. Tarasov, "Toward lattice fractional vector calculus", *J. Phys. A Math. Gen.* **47**:35 (2014), article 355204.

[Tarasov 2015a] V. E. Tarasov, "Lattice fractional calculus", *Appl. Math. Comput.* **257** (2015), 12–33.

[Tarasov 2015b] V. E. Tarasov, "Lattice model with nearest-neighbor and next-nearest-neighbor interactions for gradient elasticity", *Discontin. Nonlinearity Complex.* **4**:1 (2015), 11–23.

[Tarasov 2016a] V. E. Tarasov, "Exact discretization by Fourier transforms", *Commun. Nonlinear Sci. Numer. Simul.* **37** (2016), 31–61.

[Tarasov 2016b] V. E. Tarasov, "Three-dimensional lattice models with long-range interactions of Grünwald–Letnikov type for fractional generalization of gradient elasticity", *Meccanica* (*Milano*) **51**:1 (2016), 125–138.

[Young 1912a] W. H. Young, "On classes of summable functions and their Fourier series", *Proc. R. Soc. Lond. A* **87**:594 (1912), 225–229.

[Young 1912b] W. H. Young, "On the multiplication of successions of Fourier constants", *Proc. R. Soc. Lond. A* **87**:596 (1912), 331–339.

[Zaslavsky et al. 2007] G. M. Zaslavsky, M. Edelman, and V. E. Tarasov, "Dynamics of the chain of forced oscillators with long-range interaction: from synchronization to chaos", *Chaos* **17**:4 (2007), 043124.

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