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Path integral for quantum operations

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Abstract
In this paper we consider a phase space path integral for general time-dependent quantum operations, not necessarily unitary. We obtain the path integral for a completely positive quantum operation satisfied Lindblad equation (quantum Markovian master equation). We consider the path integral for quantum operation with a simple infinitesimal generator.

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1. Introduction

Unitary evolution is not the most general type of state change possible for quantum systems. The most general state change of a quantum system is a quantum operation [1–5]. One can describe a quantum operation for a quantum system starting from a unitary evolution of some closed system if the quantum system is a part of the closed system [6–14]. However, situations can arise where it is difficult or impossible to find a closed system comprising the given quantum system [15–19]. This would render the theory of quantum operations a fundamental generalization of the unitary evolution of the closed quantum system.

The usual models of a quantum computer deal only with unitary quantum operations on pure states. In these models it is difficult or impossible to deal formally with measurements, dissipation, decoherence and noise. It turns out that the restriction to pure states and unitary gates is unnecessary [20]. In [20], a model of quantum computations by quantum operations with mixed states was constructed. The computations are realized by quantum operations, not necessarily unitary. Mixed states subjected to general quantum operations could increase efficiency. This increase is connected with the increasing number of computational basis elements for the Hilbert space. A pure state of n two level quantum systems is an element of the 2^n-dimensional functional Hilbert space. A mixed state of the system is an element of the 4^n-dimensional operator Hilbert space. Therefore, the increased efficiency can be formalized in terms of a four-valued logic replacing the conventional two-valued logic. Unitary gates and quantum operations for a quantum computer with pure states can be considered as quantum

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gates of a mixed state quantum computer. Quantum algorithms on a quantum computer with mixed states are expected to run on a smaller network than with pure state implementation.

The path integral for quantum operations can be useful for the continuous-variable generalization of quantum computations by quantum operations with mixed states. The usual models of a quantum computer deal only with discrete variables. Many quantum variables such as position and momentum are continuous. The use of continuous-variable quantum computing [21–23] allows information to be encoded and processed much more compactly and efficiently than with discrete variable computing. Quantum computation using continuous variables is an alternative approach to quantum computations with discrete variables.

All processes occur in time. It is natural to consider time dependence for quantum operations. In this paper we consider the path-integral approach to general time-dependent quantum operations. We use the operator space [24–38] and superoperators on this space. The path integral for unitary evolution from the operator (Liouville) space was derived in [33]. The quantum operation is considered as a real completely positive trace-preserving superoperator on the operator space. We derive a path integral for a completely positive quantum operation satisfied Lindblad equation (quantum Markovian master equation) [38–42, 18]. For example, we consider a path integral for a quantum operation with a simple infinitesimal generator.

In section 2, the requirements for a superoperator to be a generalized quantum operation are discussed. In section 3, the general Liouville–von Neumann equation and quantum Markovian (Lindblad) master equation are considered. In section 4, we derive a path integral for quantum operation satisfied Liouville–von Neumann equation. In section 5, we obtain a path integral for time-dependent quantum operation with an infinitesimal generator such that the adjoint generator is completely dissipative. In section 6, the continuous-variable quantum computation by quantum operations with mixed states is discussed. In the appendix, the mathematical background (Liouville space, superoperators) is considered.

2. Quantum operations as superoperators

Unitary evolution is not the most general type of state change possible for quantum systems. The most general state change of a quantum system is a positive trace-preserving map which is called a quantum operation. For the concept of quantum operations, see [1–5].

A quantum operation is a superoperator $\hat{\mathcal{E}}$ which maps the density matrix operator $|\rho)$ to the density matrix operator $|\rho')$. For the concept of superoperators and operator space see the appendix and [24–38].

If $|\rho)$ is a density matrix operator, then $\hat{\mathcal{E}}|\rho)$ should also be a density matrix operator. Any density matrix operator ρ is a self-adjoint $\left(\rho_t^\dagger = \rho_t\right)$, positive $(\rho_t > 0)$ operator with unit trace $(\mathrm{Tr}\,\rho_t = 1)$. Therefore, the requirements for a superoperator $\hat{\mathcal{E}}$ to be the quantum operation are as follows:

1. The superoperator $\hat{\mathcal{E}}$ is a *real* superoperator, i.e. $(\hat{\mathcal{E}}(A))^\dagger = \hat{\mathcal{E}}(A^\dagger)$ for all A. The real superoperator $\hat{\mathcal{E}}$ maps the self-adjoint operator ρ to the self-adjoint operator $\hat{\mathcal{E}}(\rho)$: $(\hat{\mathcal{E}}(\rho))^\dagger = \hat{\mathcal{E}}(\rho)$.
2. The superoperator $\hat{\mathcal{E}}$ is a *positive* superoperator, i.e. $\hat{\mathcal{E}}$ maps positive operators to positive operators: $\hat{\mathcal{E}}(A^2) > 0$ for all $A \neq 0$ or $\hat{\mathcal{E}}(\rho) \geqslant 0$.
3. The superoperator $\hat{\mathcal{E}}$ is a *trace-preserving* map, i.e. $(I|\hat{\mathcal{E}}|\rho) = (\hat{\mathcal{E}}^\dagger(I)|\rho) = 1$ or $\hat{\mathcal{E}}^\dagger(I) = I$.

We have to assume the superoperator $\hat{\mathcal{E}}$ to be not merely positive but completely positive [43]. The superoperator $\hat{\mathcal{E}}$ is a *completely positive* map of the operator space, if

$$\sum_{k=1}^{n}\sum_{l=1}^{n} B_k^{\dagger}\hat{\mathcal{E}}\big(A_k^{\dagger}A_l\big)B_l \geqslant 0$$

for all operators A_k, B_k and all n.

Let the superoperator $\hat{\mathcal{E}}$ be a *convex linear* map on the set of density matrix operators, i.e.

$$\hat{\mathcal{E}}\left(\sum_s \lambda_s \rho_s\right) = \sum_s \lambda_s \hat{\mathcal{E}}(\rho_s)$$

where all λ_s are $0 < \lambda_s < 1$ and $\sum_s \lambda_s = 1$. Any convex linear map of density matrix operators can be uniquely extended to a *linear* map on Hermitian operators. Note that any linear completely positive superoperator can be represented by

$$\hat{\mathcal{E}} = \sum_{k=1}^{m} \hat{L}_{A_k}\hat{R}_{A_k^{\dagger}} : \hat{\mathcal{E}}(\rho) = \sum_{k=1}^{m} A_k \rho A_k^{\dagger}.$$

If this superoperator is a trace-preserving superoperator, then

$$\sum_{k=1}^{m} A_k^{\dagger} A_k = I.$$

The restriction to linear quantum operations is unnecessary. Let us consider a linear real completely positive superoperator $\hat{\mathcal{E}}$ which is not trace preserving. Let $(I|\hat{\mathcal{E}}|\rho) = \mathrm{Tr}(\hat{\mathcal{E}}(\rho))$ be the probability that the process represented by the superoperator $\hat{\mathcal{E}}$ occurs. Since the probability is non-negative and never exceeds 1, it follows that the superoperator $\hat{\mathcal{E}}$ is a trace-decreasing superoperator: $0 \leqslant (I|\hat{\mathcal{E}}|\rho) \leqslant 1$ or $\hat{\mathcal{E}}^{\dagger}(I) \leqslant I$. In general, any real linear completely positive trace-decreasing superoperator is not a quantum operation, since it cannot be trace preserving. The quantum operation can be defined as a *nonlinear trace-preserving* operation $\hat{\mathcal{N}}$ by

$$\hat{\mathcal{N}}|\rho) = \hat{\mathcal{E}}|\rho)(I|\hat{\mathcal{E}}|\rho)^{-1} \qquad \text{or} \qquad \hat{\mathcal{N}}(\rho) = \frac{\hat{\mathcal{E}}(\rho)}{\mathrm{Tr}(\hat{\mathcal{E}}(\rho))} \tag{1}$$

where $\hat{\mathcal{E}}$ is a real linear completely positive trace-decreasing superoperator.

All processes occur in time. It is natural to consider time dependence for quantum operations $\hat{\mathcal{E}}(t, t_0)$. Let the linear superoperators $\hat{\mathcal{E}}(t, t_0)$ form a completely positive quantum semigroup [42] such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}(t, t_0) = \hat{\Lambda}_t \hat{\mathcal{E}}(t, t_0) \tag{2}$$

where $\hat{\Lambda}^{\dagger}$ is a completely dissipative superoperator [39, 42, 19]. We would like to consider the path integral for quantum operations $\hat{\mathcal{E}}(t, t_0)$ with the infinitesimal generator $\hat{\Lambda}$, where the adjoint superoperator $\hat{\Lambda}^{\dagger}$ is *completely dissipative*, i.e.

$$\hat{\Lambda}^{\dagger}(A_k A_l) - \hat{\Lambda}^{\dagger}(A_k)A_l - A_k\hat{\Lambda}^{\dagger}(A_l) \geqslant 0.$$

3. Evolution equations

An important property of most open and dissipative quantum systems is the entropy variation. Nevertheless the unitary quantum evolution of a mixed state ϱ_t described by the von Neumann equation

$$\frac{\partial \varrho_t}{\partial t} = -\frac{\mathrm{i}}{\hbar}[H, \varrho_t] \tag{3}$$

leaves the entropy $\langle S\rangle = -\operatorname{Tr}(\varrho_t \ln \varrho_t)$ unchanged. Therefore, to describe general quantum systems, one normally uses [47, 38] a generalization of (3).

To describe dissipative quantum systems one usually considers [47] the following equation:

$$\frac{\partial \varrho_t}{\partial t} = -\frac{\mathrm{i}}{\hbar}[H, \varrho_t] + D(\varrho). \tag{4}$$

3.1. Liouville–von Neumann equation

Let us consider a generalization of equation (3). The Liouville–von Neumann equation [16, 38, 44] can be represented as the linear equation

$$\frac{\mathrm{d}\varrho_t}{\mathrm{d}t} = \Lambda_t(\varrho_t). \tag{5}$$

Using the superoperator formalism, this equation can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}|\varrho_t) = \hat{\Lambda}_t|\varrho_t). \tag{6}$$

The superoperator language allows one to use the analogy with Dirac notation. This leads quite simply to the derivation of the appropriate equations.

Here $\hat{\Lambda}_t$ is a linear Liouville superoperator on the operator space $\overline{\mathcal{H}}$. For the Hamiltonian (closed) quantum systems (3) this superoperator is defined by the Hamiltonian H:

$$\hat{\Lambda}_t = -\frac{\mathrm{i}}{\hbar}(\hat{L}_H - \hat{R}_H). \tag{7}$$

For equation (4) the Liouville superoperator has the form

$$\hat{\Lambda}_t = -\frac{\mathrm{i}}{\hbar}(\hat{L}_H - \hat{R}_H) + \hat{D}. \tag{8}$$

In general, the operator $|\varrho_t)$ is an unnormalized density matrix operator, i.e. $\operatorname{Tr}\varrho_t = (I|\varrho_t) \neq 1$.

Equation (6) has a formal solution

$$|\varrho_t) = \hat{\mathcal{E}}(t, t_0)|\varrho_{t_0}) \tag{9}$$

where $\hat{\mathcal{E}}(t, t_0)$ is a linear quantum operation defined by

$$\hat{\mathcal{E}}(t, t_0) = T \exp \int_{t_0}^{t} \mathrm{d}\tau\, \hat{\Lambda}_\tau. \tag{10}$$

The symbol T is a Dyson's time-ordering operator [45]. The quantum operation (10) satisfies the Liouville–von Neumann equation (2). We can define a normalized density matrix operator $|\rho_t)$ by

$$|\rho_t) = |\varrho_t)(I|\varrho_t)^{-1} \qquad \text{or} \qquad |\rho_t) = \frac{\hat{\mathcal{E}}(t, t_0)|\varrho)}{(I|\hat{\mathcal{E}}(t, t_0)|\varrho)}$$

i.e. $\rho_t = \varrho_t / \operatorname{Tr}\varrho_t$. The evolution equation for the normalized density matrix operator ρ_t can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho_t) = \hat{\Lambda}_t|\rho_t) - |\rho_t)(I|\hat{\Lambda}_t|\rho_t). \tag{11}$$

In general, this equation is a nonlinear equation [19]. A formal solution of equation (11) is connected with the nonlinear quantum operation (1) by

$$|\rho_t) = \hat{\mathcal{N}}(t, t_0)|\rho_{t_0}).$$

3.2. Quantum Markovian equation

Lindblad [39] has shown that there exists a one-to-one correspondence between the completely positive norm continuous semigroup of superoperators $\hat{\mathcal{E}}(t, t_0)$ and superoperator $\hat{\Lambda}$ such that the adjoint superoperator $\hat{\Lambda}^{\dagger}$ is completely dissipative. The structural theorem of Lindblad gives the most general form of the bounded adjoint completely dissipative Liouville superoperator $\hat{\Lambda}$. The Liouville–von Neumann equation (11) for a completely positive evolution is a quantum Markovian master equation (Lindblad equation) [39–41]:

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H, \rho_t] + \frac{1}{2\hbar}\sum_{k=1}^{m}\left(\left[V_k\rho_t, V_k^{\dagger}\right] + \left[V_k, \rho_t V_k^{\dagger}\right]\right). \tag{12}$$

This equation in the Liouville space can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho_t) = \hat{\Lambda}|\rho_t)$$

where the Liouville superoperator $\hat{\Lambda}$ is given by

$$\hat{\Lambda} = -\frac{\mathrm{i}}{\hbar}(\hat{L}_H - \hat{R}_H) + \frac{1}{2\hbar}\sum_{k=1}^{m}\left(2\hat{L}_{V_k}\hat{R}_{V_k^{\dagger}} - \hat{L}_{V_k}\hat{L}_{V_k^{\dagger}} - \hat{R}_{V_k^{\dagger}}\hat{R}_{V_k}\right). \tag{13}$$

The basic assumption is that the general form of a bounded superoperator $\hat{\Lambda}$, given by the Lindblad theorem, is also valid for an unbounded superoperator [42, 46]. Another condition imposed on the operators H, V_k, $V_k^{\dagger}$ is that they are functions of the observables P and Q (with $[Q, P] = \mathrm{i}\hbar I$) of the one-dimensional quantum system. Let us consider $V_k = a_k P + b_k Q$, where $k = 1, 2$, and a_k, b_k are complex numbers, and the Hamiltonian operator H is

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2 + \frac{\mu}{2}(PQ + QP)$$

Then with the notation [46]:

$$\mathrm{d}_{qq} = \frac{\hbar}{2}\sum_{k=1,2}|a_k|^2 \qquad \mathrm{d}_{pp} = \frac{\hbar}{2}\sum_{k=1,2}|b_k|^2$$

$$\mathrm{d}_{pq} = -\frac{\hbar}{2}\,\mathrm{Re}\left(\sum_{k=1,2}a_k^* b_k\right) \qquad \lambda = -\mathrm{Im}\left(\sum_{k=1,2}a_k^* b_k\right)$$

equation (12) in the Liouville space can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho_t) = \frac{1}{2m}\hat{L}_P^{+}\hat{L}_P^{-} + \frac{m\omega^2}{2}\hat{L}_Q^{+}\hat{L}_Q^{-} - (\lambda - \mu)\hat{L}_P^{-}\hat{L}_Q^{+}|\rho_t) + (\lambda + \mu)\hat{L}_Q^{-}\hat{L}_P^{+}|\rho_t)\\ + \mathrm{d}_{pp}\hat{L}_Q^{-}\hat{L}_Q^{-}|\rho_t) + \mathrm{d}_{qq}\hat{L}_P^{-}\hat{L}_P^{-}|\rho_t) - 2\mathrm{d}_{pq}\hat{L}_P^{-}\hat{L}_Q^{-}|\rho_t)\end{aligned} \tag{14}$$

where $L^{\pm}$ are the multiplication superoperators defined by

$$\hat{L}_A^{-} = \frac{1}{\mathrm{i}\hbar}(\hat{L}_A - \hat{R}_A) \qquad \hat{L}_A^{+} = \frac{1}{2}(\hat{L}_A + \hat{R}_A).$$

The properties of these superoperators are considered in the appendix. Equation (14) is a superoperator form of the well-known phenomenological dissipative model [47, 46].

4. Path integral in the general form

In the coordinate representation the kernel

$$\varrho(q, q', t) = (q, q'|\varrho_t)$$

of the density operator $|\varrho_t)$ evolves according to the equation

$$\varrho(q, q', t) = \int \mathrm{d}q_0\, \mathrm{d}q'_0\, \mathcal{E}(q, q', q_0, q'_0, t, t_0)\varrho(q_0, q'_0, t_0).$$

The function

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = (q, q'|\hat{\mathcal{E}}(t, t_0)|q_0, q'_0) \tag{15}$$

is a kernel of the linear quantum operation $\hat{\mathcal{E}}(t, t_0)$. Let the Liouville superoperator $\hat{\Lambda}_t$ be time independent, i.e. the quantum operation $\hat{\mathcal{E}}(t, t_0)$ is given by

$$\hat{\mathcal{E}}(t, t_0) = \exp(t - t_0)\hat{\Lambda}. \tag{16}$$

Proposition 1. *Let $\{\hat{\mathcal{E}}(t, t_0), t \geqslant t_0\}$ be a superoperator semigroup on the operator space $\overline{\mathcal{H}}$*

$$\hat{\mathcal{E}}(t_0, t_0) = \hat{I} \qquad \hat{\mathcal{E}}(t, t_0) = \hat{\mathcal{E}}(t, t_1)\hat{\mathcal{E}}(t_1, t_0)$$

where $t \geqslant t_1 \geqslant t_0$ such that the infinitesimal generator $\hat{\Lambda}$ of this semigroup is defined by (16). Then the path integral for kernel (15) of the quantum operation $\hat{\mathcal{E}}(t, t_0)$ has the following form:

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \mathcal{D}q\mathcal{D}q'\mathcal{D}p\mathcal{D}p' \exp \int_{t_0}^{t} \mathrm{d}t \left(\frac{\mathrm{i}}{\hbar}[\dot{q}p - \dot{q}'p'] + \Lambda_S(q, q', p, p')\right). \tag{17}$$

This form is the integral over all trajectories in the double phase space with the constraints that $q(t_0) = q_0$, $q(t) = q$, $q'(t_0) = q'_0$, $q'(t) = q'$ and the measure

$$\mathcal{D}q = \prod_t \mathrm{d}q(t) \qquad \mathcal{D}p = \prod_t \frac{\mathrm{d}p(t)}{2\pi\hbar}.$$

The symbol $\Lambda_S(q, q', p, p')$ of the Liouville superoperator $\hat{\Lambda}$ is connected with the kernel $\Lambda(q, q', y, y')$ by

$$\Lambda_S(q, q', p, p') = \int \mathrm{d}y\, \mathrm{d}y' \Lambda(q, q', y, y') \exp -\frac{\mathrm{i}}{\hbar}[(q - y)p - (q' - y')p']$$

where $\Lambda(q, q', y, y') = (q, q'|\hat{\Lambda}|y, y')$ and

$$\Lambda(q, q', y, y') = \frac{1}{(2\pi\hbar)^{2n}} \int \mathrm{d}p\, \mathrm{d}p' \Lambda_S(q, q', p, p') \exp \frac{\mathrm{i}}{\hbar}[(q - y)p - (q' - y')p'].$$

Proof. Let us derive the path integral form (17) for quantum operation (16).

1. Let time interval $[t_0, t]$ have $n + 1$ equal parts

$$\tau = \frac{t - t_0}{n + 1}.$$

Using the superoperator semigroup composition rule

$$\hat{\mathcal{E}}(t, t_0) = \hat{\mathcal{E}}(t, t_n)\hat{\mathcal{E}}(t_n, t_{n-1}) \cdots \hat{\mathcal{E}}(t_1, t_0)$$

where $t \geqslant t_n \geqslant t_{n-1} \geqslant \cdots \geqslant t_1 \geqslant t_0$, we obtain the following integral representation:

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \mathrm{d}q_n\, \mathrm{d}q'_n \cdots \mathrm{d}q_1\, \mathrm{d}q'_1 \mathcal{E}(q, q', q_n, q'_n, t, t_n) \cdots \mathcal{E}(q_1, q'_1, q_0, q'_0, t_1, t_0).$$

This representation can be written in the form

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \prod_{k=1}^{n} \mathrm{d}q_k\, \mathrm{d}q'_k \prod_{k=1}^{n+1} \mathcal{E}(q_k, q'_k, q_{k-1}, q'_{k-1}, t_k, t_{k-1}).$$

Here $q_{n+1} = q$ and $q'_{n+1} = q'$.

2. Let us consider the kernel

$$\mathcal{E}(q_k, q'_k, q_{k-1}, q'_{k-1}, t_k, t_{k-1})$$

of the quantum operation $\hat{\mathcal{E}}(t_k, t_{k-1})$. If the time interval $[t_{k-1}, t_k]$ is small, then in the coordinate representation we have

$$\varrho(q_k, q'_k, t_k) = \int \mathrm{d}q_{k-1}\, \mathrm{d}q'_{k-1} \mathcal{E}(q_k, q'_k, q_{k-1}, q'_{k-1}, t_k, t_{k-1}) \varrho(q_{k-1}, q'_{k-1}, t_{k-1})$$

and

$$\begin{aligned}
\varrho(q_k, q'_k, t_k) &= \big(q_k, q'_k \big| \varrho_{t_k}\big) = (q_k, q'_k | \hat{\mathcal{E}}(t_k, t_{k-1}) \big| \varrho_{t_{k-1}}\big) \\
&= \sum_{n=0}^{\infty} \frac{\tau^n}{n!} (q_k, q'_k | \hat{\Lambda}^n \big| \varrho_{t_{k-1}}\big) = \big(q_k, q'_k \big| \varrho_{t_{k-1}}\big) + (q_k, q'_k | \hat{\Lambda} \big| \varrho_{t_{k-1}}\big)\tau + O(\tau^2) \\
&= \varrho(q_k, q'_k, t_{k-1}) + \tau \int \mathrm{d}q_{k-1}\, \mathrm{d}q'_{k-1} \Lambda(q_k, q'_k, q_{k-1}, q'_{k-1}) \\
&\quad \times \varrho(q_{k-1}, q'_{k-1}, t_{k-1}) + O(\tau^2) \\
&= \int \mathrm{d}q_{k-1}\, \mathrm{d}q'_{k-1} (\delta(q_k - q_{k-1}) \delta(q'_{k-1} - q'_k) \\
&\quad + \tau \Lambda(q_k, q'_k, q_{k-1}, q'_{k-1}) + O(\tau^2)) \varrho(q_{k-1}, q'_{k-1}, t_{k-1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathcal{E}(q_k, q'_k, q_{k-1}, q'_{k-1}, t_k, t_{k-1}) \\
&\quad = \delta(q_k - q_{k-1}) \delta(q'_{k-1} - q'_k) + \tau \Lambda(q_k, q'_k, q_{k-1}, q'_{k-1}) + O(\tau^2).
\end{aligned}$$

3. Delta-functions can be written in the form

$$\delta(q_k - q_{k-1}) \delta(q'_{k-1} - q'_k) = \int \frac{\mathrm{d}p_k\, \mathrm{d}p'_k}{(2\pi\hbar)^{2n}} \exp \frac{\mathrm{i}}{\hbar} [(q_k - q_{k-1}) p_k - (q'_k - q'_{k-1}) p'_k].$$

Using the relations

$$(q, q' | p, p') = \langle q | p \rangle \langle p' | q' \rangle = \frac{1}{(2\pi\hbar)^n} \exp \frac{\mathrm{i}}{\hbar} (qp - q'p')$$

$$(p, p' | q, q') = \langle p | q \rangle \langle q' | p' \rangle = \frac{1}{(2\pi\hbar)^n} \exp - \frac{\mathrm{i}}{\hbar} (qp - q'p')$$

we obtain the symbol $\Lambda_S(q_k, q'_k, p_k, p'_k)$ of the Liouville superoperator by

$$\begin{aligned}
\Lambda(q_k, q'_k, q_{k-1}, q'_{k-1}) &= (q_k, q'_k | \hat{\Lambda} | q_{k-1}, q'_{k-1}) \\
&= \int \mathrm{d}p_k\, \mathrm{d}p'_k (q_k, q'_k | \hat{\Lambda} | p_k, p'_k)(p_k, p'_k | q_{k-1}, q'_{k-1}) \\
&= \int \mathrm{d}p_k\, \mathrm{d}p'_k \Lambda_S(q_k, q'_k, p_k, p'_k)(q_k, q'_k | p_k, p'_k)(p_k, p'_k | q_{k-1}, q'_{k-1}) \\
&= \frac{1}{(2\pi\hbar)^{2n}} \int \mathrm{d}p_k\, \mathrm{d}p'_k \Lambda_S(q_k, q'_k, p_k, p'_k) \\
&\quad \times \exp \frac{\mathrm{i}}{\hbar} [(q_k - q_{k-1}) p_k - (q'_k - q'_{k-1}) p'_k].
\end{aligned}$$

4. The kernel of the quantum operation $\hat{\mathcal{E}}(t_k, t_{k-1})$ is

$$\mathcal{E}(q_k, q'_k, q_{k-1}, q'_{k-1}, t_k, t_{k-1}) = \frac{1}{(2\pi\hbar)^{2n}} \int \mathrm{d}p_k\, \mathrm{d}p'_k (1 + \tau\Lambda_S(q_k, q'_k, p_k, p'_k) + O(\tau^2)) \times \exp \frac{\mathrm{i}}{\hbar}((q_k - q_{k-1})p_k - (q'_k - q'_{k-1})p'_k).$$

5. The kernel of the quantum operation $\hat{\mathcal{E}}(t, t_0)$ is

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \prod_{k=1}^{n} \mathrm{d}q_k\, \mathrm{d}q'_k \prod_{k=1}^{n+1} \frac{\mathrm{d}p_k\, \mathrm{d}p'_k}{(2\pi\hbar)^{2n}} \times \exp \frac{\mathrm{i}}{\hbar} \sum_{k=1}^{n+1} [(q_k - q_{k-1})p_k - (q'_k - q'_{k-1})p'_k] \times \prod_{k=1}^{n+1} (1 - \tau\Lambda_S(q_k, q'_k, p_k, p'_k) + O(\tau^2)).$$

6. Using

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{\Lambda_k}{n}\right) = \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(\frac{\Lambda_k}{n}\right)$$

we obtain

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \prod_{k=1}^{n} \mathrm{d}q_k\, \mathrm{d}q'_k \prod_{k=1}^{n+1} \frac{\mathrm{d}p_k\, \mathrm{d}p'_k}{(2\pi\hbar)^{2n}} \times \exp \sum_{k=1}^{n+1} \tau \left(\frac{\mathrm{i}}{\hbar} \left[\frac{q_k - q_{k-1}}{\tau} p_k - \frac{q'_k - q'_{k-1}}{\tau} p'{}_k \right] + \Lambda_S(q_k, q'_k, p_k, p'_k) \right).$$

7. Let q_k, q'_k, p_k, p'_k be the values of the functions $q(t), q'(t), p(t), p'(t)$ and $t_k = t_0 + k\tau$, i.e.

$$q_k = q(t_k) \qquad q'_k = q'(t_k) \qquad p_k = p(t_k) \qquad p'_k = p'(t_k)$$

where $k = 0, 1, 2, \ldots, n, n+1$. Using

$$\lim_{\tau\to 0} \frac{q_k - q_{k-1}}{\tau} = \dot{q}(t_{k-1}) \qquad \lim_{n\to\infty} \sum_{k=1}^{n+1} A(t_k)\tau = \int_{t_0}^{t} \mathrm{d}t\, A(t)$$

we obtain the kernel of the quantum operation $\hat{\mathcal{E}}(t, t_0)$ in the path integral form (17). □

Corollary. *If the dissipative quantum evolution is defined by equation (4), then the path integral for the quantum operation kernel has the form*

$$\mathcal{E}(q, q', q_0, q'_0, t, t_0) = \int \mathcal{D}q\mathcal{D}p\mathcal{D}q'\mathcal{D}p' \exp\left(\frac{\mathrm{i}}{\hbar}(\mathcal{A}(q, p) - \mathcal{A}(p', q')) + D(q, q', p, p')\right).$$

Here $\mathcal{A}(q, p)$ and $\mathcal{A}(p', q')$ are action functionals defined by

$$\mathcal{A}(q, p) = \int_{t_0}^{t} \mathrm{d}t\, (\dot{q}p - H(q, p)) \qquad \mathcal{A}(p', q') = \int_{t_0}^{t} \mathrm{d}t\, (\dot{q}'p' - H(p', q')). \tag{18}$$

The functional $D(q, q', p, p')$ is a time integral of the symbol D_S of the superoperator $\hat{D}$.

The functional $D(q, q', p, p')$ describes the dissipative part of evolution.

Corollary. *If the quantum system has no dissipation, i.e. the quantum system is a closed Hamiltonian system, then $D(q, q', p, p') = 0$ and the path integral for the quantum operation can be separated*

$$\mathcal{E}(q, q', q_0, q_0', t, t_0) = U^*(q, q_0, t, t_0)U(q', q_0', t, t_0)$$

where

$$U(q', q_0', t, t_0) = \int \mathcal{D}q'\mathcal{D}p' \exp -\frac{\mathrm{i}}{\hbar}\mathcal{A}(p', q')$$

$$U^*(q, q_0, t, t_0) = \int \mathcal{D}q\mathcal{D}p \exp \frac{\mathrm{i}}{\hbar}\mathcal{A}(p, q).$$

The path integral for the dissipative quantum systems and the corresponding quantum operations cannot be separated, i.e. this path integral is defined in the double phase space.

5. Path integral for completely positive quantum operation

Let us consider the Liouville superoperator (13) for the Lindblad equation.

Proposition 2. *Let $\{\hat{\mathcal{E}}(t, t_0), t \geqslant t_0\}$ be a completely positive semigroup of linear real trace-preserving superoperators such that the infinitesimal generators $\hat{\Lambda}$ of this semigroup is defined by (13). Then the path integral for the kernel of the completely positive quantum operation $\hat{\mathcal{E}}(t, t_0)$ has the form*

$$\mathcal{E}(q, q', q_0, q_0', t, t_0) = \int \mathcal{D}q\mathcal{D}p\mathcal{D}q'\mathcal{D}p'\, \mathcal{F}(q, q', p, p') \exp \frac{\mathrm{i}}{\hbar}(\mathcal{A}(q, p) - \mathcal{A}(p', q')) \tag{19}$$

where $\mathcal{A}(q, p)$ and $\mathcal{A}(p', q')$ are action functionals (18) and the functional $\mathcal{F}(q, q', p, p')$ is defined as

$$\mathcal{F}(q, q', p, p') = \exp -\frac{1}{2\hbar}\int_{t_0}^{t} \mathrm{d}t \sum_{k=1}^{m} \big(\big(V_k^\dagger V_k\big)(q, p) + \big(V_k^\dagger V_k\big)(p', q') - 2V_k(q, p)V_k^\dagger(p', q')\big). \tag{20}$$

Proof. The kernel of superoperator (13) is

$$\Lambda(q, q', y, y') = (q, q_k'|\hat{\Lambda}|y, y') = -\frac{\mathrm{i}}{\hbar}(\langle y'|q'\rangle\langle q|H|y\rangle - \langle q|y\rangle\langle y'|H|q'\rangle) + \frac{1}{\hbar}\sum_{k=1}^{m}\langle q|V_k|y\rangle\langle y'|V_k^\dagger|q'\rangle - \frac{1}{2\hbar}\sum_{k=1}^{m}(\langle y'|q'\rangle\langle q|V_k^\dagger V_k|y\rangle - \langle q|y\rangle\langle y'|V_k^\dagger V_k|q'\rangle).$$

The symbol $\Lambda_S(q, q', p, p')$ of the Liouville superoperator $\hat{\Lambda}$ can be derived by

$$\Lambda(q, q', y, y') = \int \mathrm{d}p\,\mathrm{d}p' \left(-\frac{\mathrm{i}}{\hbar}(\langle y'|p'\rangle\langle p'|q'\rangle\langle q|H_1|p\rangle\langle p|y\rangle - \langle q|p\rangle\langle p|y\rangle\langle y'|p'\rangle\langle p'|H_2|q'\rangle) + \frac{1}{\hbar}\sum_{k=1}^{m}\langle q|V_k|p\rangle\langle p|y\rangle\langle y'|p'\rangle\langle p'|V_k^\dagger|q'\rangle\right)$$

where the operators H_1 and H_2 are defined by the relations

$$H_1 \equiv H - \frac{\mathrm{i}}{2}\sum_{k=1}^{m} V_k^\dagger V_k \qquad H_2 \equiv H + \frac{\mathrm{i}}{2}\sum_{k=1}^{m} V_k^\dagger V_k.$$

Then the symbol $\Lambda_S(q, q', p, p')$ of the Liouville superoperator (13) can be written in the form

$$\Lambda_S(q, q', p, p') = -\frac{\mathrm{i}}{\hbar}\left(H_1(q, p) - H_2(p', q') + \mathrm{i}\sum_{k=1}^{m} V_k(q, p) V_k^{\dagger}(p', q')\right)$$

or

$$\Lambda_S(q, q', p, p') = -\frac{\mathrm{i}}{\hbar}[H(q, p) - H(p', q')] - \frac{1}{2\hbar}\sum_{k=1}^{m}\left(\left(V_k^{\dagger}V_k\right)(q, p) + \left(V_k^{\dagger}V_k\right)(p', q') - 2V_k(q, p)V_k^{\dagger}(p', q')\right)$$

where $H(q, p)$ is a qp-symbol of the Hamiltonian operator H and $H(p, q)$ is a pq-symbol of the operator H.

In the Hamiltonian case ($V_k = 0$) the symbol is given by

$$\Lambda_S(q, q', p, p') = -\frac{\mathrm{i}}{\hbar}[H(q, p) - H(p', q')].$$

The path integral for a completely positive quantum operation kernel has the form

$$\mathcal{E}(q, q', q_0, q_0', t, t_0) = \int \mathcal{D}q\mathcal{D}p\mathcal{D}q'\mathcal{D}p'\, \mathcal{F}(q, q', p, p') \exp\frac{\mathrm{i}}{\hbar}(\mathcal{A}(q, p) - \mathcal{A}(p', q')).$$

Here $\mathcal{A}(q, p)$ and $\mathcal{A}(p', q')$ are action functionals (18), and the functional $\mathcal{F}(q, q', p, p')$ is defined by (20). □

The functional $\mathcal{F}(q, q', p, p')$ describes the dissipative part of the evolution and can be called a *(double) phase space influence functional*. The completely positive quantum operation is described by the functional (20).

Corollary. *For the phenomenological dissipative model (14) the double phase space path integral has the form (19) with the functional*

$$\mathcal{F}(q, q', p, p') = \exp\frac{1}{\hbar^2}\int_{t_0}^{t} \mathrm{d}t\,(2d_{qp}(q - q')(p - p') - d_{qq}(p - p')^2 - d_{pp}(q - q')^2 + \mathrm{i}\hbar\lambda(pq' - qp') + \mathrm{i}\hbar\mu(q'p' - qp)). \tag{21}$$

Using the well-known connection between the phase space (Hamiltonian) path integral and the configuration space (Lagrangian) path integral [51, 52], we can derive the following proposition.

Proposition 3. *If the symbol $\Lambda_s(q, q', p, p')$ of the Liouville superoperator can be represented in the form*

$$\Lambda_S(q, q', p, p') = -\frac{\iota}{\hbar}[H(q, p) - H(p', q')] + D_S(q, q', p, p') \tag{22}$$

where

$$H(q, p) = \tfrac{1}{2}a_{kl}^{-1}(q)p_k p_l - b_k(q)p_k + c(q) \tag{23}$$

$$D_S(q, q', p, p') = -d_k(q, q')p_k + d_k'(q, q')p_k' + e(q, q') \tag{24}$$

then the double phase space path integral (17) can be represented as a double configuration phase space path integral

$$\mathcal{E}(q, q', q_0, q_0', t, t_0) = \int \mathcal{D}q\mathcal{D}q'\, \mathcal{F}(q, q') \exp\frac{\mathrm{i}}{\hbar}(\mathcal{A}(q) - \mathcal{A}(q')) \tag{25}$$

where

$$\begin{aligned}\mathcal{A}(q) &= \int_{t_0}^{t} \mathrm{d}t\, \mathcal{L}(q,\dot{q}) \\ \mathcal{L}(q,\dot{q}) &= \frac{1}{2}a_{kl}(q)\dot{q}_k\dot{q}_l + a_{kl}(q)b_k(q)\dot{q}_l + \frac{1}{2}a_{kl}(q)b_k(q)b_l(q) - c(q).\end{aligned} \tag{26}$$

This Lagrangian $\mathcal{L}(q,\dot{q})$ is related to the Hamiltonian (23) by the usual relations

$$\mathcal{L}(q,\dot{q}) = \dot{q}_k p_k - H(q,p) \qquad p_k = \frac{\partial\mathcal{L}}{\partial\dot{q}}.$$

Proof. Substituting (22) into (17), we obtain the kernel of the corresponding quantum operation. Integrating (17) in p and p', we obtain relation (25) with the functional

$$\mathcal{F}(q,q') = \exp\frac{-1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}t\left(\mathrm{d}_k(q,q')a_{kl}(q)(b_l(q) - \frac{\mathrm{i}}{2\hbar}d_l(q,q')) + d'_k(q,q')a_{kl}(q')(b_l(q') - \frac{\mathrm{i}}{2\hbar}d'_l(q,q')) + e(q,q') + \delta(0)\Delta(q,q')\right)$$

where

$$\Delta(q,q') = -\frac{\hbar^2}{2}(\ln(\det(a_{kl}(q))) + \ln(\det(a_{kl}(q')))).$$

□

In equation (25) the functional $\mathcal{F}(q,q')$ can be considered as the Feynman–Vernon influence functional. It is known that this functional can be derived by eliminating the bath degrees of freedom, for example by taking a partial trace or by integrating them out. The Feynman–Vernon influence functional describes the dissipative dynamics of open systems when we assume the von Hove limit for a system–reservoir coupling. One can describe a quantum system starting from a unitary evolution of some closed system 'system–reservoir' if the quantum system is a part of this closed system. However, situations can arise where it is difficult or impossible to find a closed system comprising the given quantum system [6–14].

The Feynman path integral is defined for the configuration space. The most general form of quantum mechanical path integral is defined for the phase space. The Feynman path integral can be derived from the phase space path integral for the special form of the Hamiltonian [51–55]. It is known that the path integral for the configuration space is correct [51, 52] only for the Hamiltonian (23). The Feynman–Vernon path integral [6] is defined in the double configuration space. Therefore, this path integral is a special form of the double phase space path integral (17). The Feynman–Vernon path integral is correct only for the Liouville superoperator (22)–(24). Note that the symbol $\Lambda_s(q,q',p,p')$ for most of the dissipative and non-Hamiltonian systems (with completely positive quantum operations) cannot be represented in the form (22).

Corollary. *In the general case, the completely positive quantum operation cannot be represented as the double configuration space path integral (25).*

For example, the double phase space path integral (21) for the phenomenological dissipative model (14) has the term pp'. Therefore this model and the Liouville symbol $\Lambda_s(q,q',p,p')$ for this model cannot be represented in the form (22)–(24).

6. On the continuous-variable quantum computation by quantum operations with mixed states

The usual models of a quantum computer deal only with the discrete variables, unitary quantum operations (gates) and pure states. Many quantum variables such as position and momentum are continuous. The use of continuous-variable quantum computing [21–23] allows information to be encoded and processed much more compactly and efficiently than with discrete variable computing. Quantum computation using continuous variables is an alternative approach to quantum computations with discrete variables.

In the models with unitary quantum operations on pure states it is difficult or impossible to deal formally with measurements, dissipation, decoherence and noise. It turns out that the restriction to pure states and unitary gates is unnecessary [20]. In [20], a model of quantum computations by quantum operations with mixed states was constructed. It is known that the measurement is described by quantum operations. The measurement quantum operations are the special case of quantum operations on mixed states. The von Neumann measurement quantum operation as a nonlinear quantum gate is realized in [20]. The continuous quantum measurement is described by the path integrals [56–59]. Therefore, the path integral for quantum operations can be useful for continuous-variable quantum operations on mixed states. Quantum computation by quantum operations with mixed states is considered [20] for discrete variables only. Some points of the model of the continuous-variable quantum computations with mixed states are considered in this section. The double phase space path integral can be useful for the continuous-variable quantum gates on mixed states.

The main steps of the continuous-variable generalization of quantum computations by quantum operations with mixed states are the following.

(1) The state $|\rho(t))$ of the discrete-variable quantum computation with mixed states [20] is a superposition of basis elements

$$|\rho(t)) = \sum_{\mu=0}^{N-1} |\mu)\rho_\mu(t) \tag{27}$$

where $\rho_\mu(t) = (\mu|\rho(t))$ are real numbers (functions). The basis $|\mu)$ of the discrete-variable Liouville space $\overline{\mathcal{H}}^{(n)}$ is defined [20] by

$$|\mu) = |\mu_1 \ldots \mu_n) = \frac{1}{\sqrt{2^n}}|\sigma_\mu) = \frac{1}{\sqrt{2^n}}\big|\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}\big) \tag{28}$$

where σ_μ are Pauli matrices, $N = 4^n$, each $\mu_i \in \{0, 1, 2, 3\}$ and

$$(\mu|\mu') = \delta_{\mu\mu'} \qquad \sum_{\mu=0}^{N-1} |\mu)(\mu| = \hat{I} \tag{29}$$

is the discrete-variable computational basis.

The state $|\rho(t))$ of the continuous-variable quantum computation at any point of time can be considered as a superposition of basis elements

$$|\rho(t)) = \int \mathrm{d}x \int \mathrm{d}x' |x, x')\rho(x, x', t) \tag{30}$$

where $\rho(x, x', t) = (x, x'|\rho(t))$ are the density matrix elements. The basis $|x, x')$ of the continuous-variable operator space $\overline{\mathcal{H}}$ is defined by $|x, x') = \| x\rangle\langle x'|)$, where

$$(x, x'|y, y') = \delta(x - x')\delta(y - y') \qquad \int \mathrm{d}x \int \mathrm{d}x' |x, x')(x, x'| = \hat{I} \tag{31}$$

can be considered as a continuous-variable computational basis.

(2) In the discrete-variable computational basis $|\mu)$ any linear quantum operation $\hat{\mathcal{E}}$ acting on n-qubits mixed (or pure) states can be represented as a quantum four-valued logic gate [20]: $\hat{\mathcal{E}}$ on n-ququats can be given by

$$\hat{\mathcal{E}} = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu} |\mu)(\nu| \tag{32}$$

where $N = 4^n$

$$\mathcal{E}_{\mu\nu} = \frac{1}{2^n} \operatorname{Tr}(\sigma_\mu \hat{\mathcal{E}}(\sigma_\nu)) \tag{33}$$

and $\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$.

In the continuous-variable computational basis $|x, x')$ any linear quantum operation $\hat{\mathcal{E}}$ acting on mixed (or pure) states can be represented as a continuous-variable quantum gate:

$$\hat{\mathcal{E}}(t_2, t_1) = \int \mathrm{d}x\, \mathrm{d}x' \mathrm{d}y\, \mathrm{d}y' \mathcal{E}(x, x', y, y', t_2, t_1) |x, x')(y, y'| \tag{34}$$

where $\mathcal{E}(x, x', y, y', t_2, t_1) = (x, x'|\hat{\mathcal{E}}(t_2, t_1)|y, y')$ is a kernel of the real trace-preserving positive (or completely positive) superoperator $\hat{\mathcal{E}}(t_2, t_1)$. This quantum operation can be considered as a continuous-variable quantum gate.

The continuous quantum measurement which is described by the path integrals [56–59] is the special case of the continuous-variable quantum gate. The path integral for the quantum operations can be useful for all continuous-variable quantum operations on mixed states.

(3) Many quantum variables such as position and momentum are continuous. The use of continuous-variable quantum computing [21–23] allows information to be encoded and processed much more compactly and efficiently than with discrete variable computing.

Mixed states subjected to the general quantum operations could increase efficiency. This increase is connected with the increasing number of computational basis elements for the operator Hilbert space. A pure state of the quantum system is an element of the functional Hilbert space $\mathcal{H}$. A mixed state of the system is an element $|\rho)$ of the operator Hilbert space $\overline{\mathcal{H}}$. A mixed state of the system can be considered as an element $\rho(x, x', t)$ of the double functional Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

The use of continuous-variable quantum computation by quantum operations with mixed states can increase efficiency compared with discrete variable computing.

7. Conclusion

The usual quantum computer model deals only with discrete variables, unitary quantum operations and pure states. It is known that many quantum variables such as position and momentum are continuous. The use of continuous-variable quantum computing [21–23] allows information to be encoded and processed much more efficiently than in discrete-variable quantum computer. Quantum computation using continuous variables is an alternative approach to quantum computations with discrete variables.

The quantum computation by quantum operations with mixed states is considered in [20]. It is known that the measurement is described by quantum operations. The measurement quantum operations are the special case of quantum operations on mixed states. The von Neumann measurement quantum operation as a nonlinear quantum gate is realized in [20]. The continuous quantum measurement is described by the path integrals [56–59].

Therefore, the path integral for quantum operations can be useful for continuous-variable quantum operations on mixed states. Quantum computation by quantum operations with mixed states is considered [20] only for discrete variables. The model of continuous-variable quantum computations with mixed states will be suggested in the next publication. The double phase space path integral can be useful for continuous-variable quantum gates on mixed states.

Let us note the second application of double phase space path integral. The path integral formulation of the quantum statistical mechanics leads to the powerful simulation scheme [60] for the molecular dynamics. In the past few years the statistical mechanics of non-Hamiltonian systems was developed for the molecular dynamical simulation purpose [61–65]. The suggested path integral can be useful for the application in the non-Hamiltonian statistical mechanics of quantum [66, 67] and quantum–classical systems [68, 69].

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Appendix

For the concept of Liouville space and superoperators see [24–38].

A.1. Operator space

The space of linear operators acting on a Hilbert space $\mathcal{H}$ is a complex linear space $\overline{\mathcal{H}}$. We denote an element A of $\overline{\mathcal{H}}$ by a ket-vector $|A)$. The inner product of two elements $|A)$ and $|B)$ of $\overline{\mathcal{H}}$ is defined as $(A|B) = \mathrm{Tr}(A^\dagger B)$. The norm $\|A\| = \sqrt{(A|A)}$ is the Hilbert–Schmidt norm of operator A. A new Hilbert space $\overline{\mathcal{H}}$ with the inner product is called Liouville space attached to $\mathcal{H}$ or the associated Hilbert space, or Hilbert–Schmidt space [24–38].

The X-representation uses eigenfunctions $|x\rangle$ of the operator X. In general, the operator X can be an unbounded operator. This operator can have a continuous spectrum. This leads us to consider the rigged Hilbert space [48, 49, 19, 38] (Gelfand triplet) $\mathcal{B} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{B}^*$ and the associated operator space. The rigged operator Hilbert space can be considered as the usual rigged Hilbert space for the operator kernels.

Let the set $\{|x\rangle\}$ satisfy the following conditions:

$$\langle x|x'\rangle = \delta(x - x') \qquad \int \mathrm{d}x |x\rangle\langle x| = I.$$

Then $|x, x') = \||x\rangle\langle x'|)$ satisfies

$$(x, x'|y, y') = \delta(x - x')\delta(y - y') \qquad \int \mathrm{d}x \int \mathrm{d}x' |x, x')(x, x'| = \hat{I}.$$

For an arbitrary element $|A)$ of $\overline{\mathcal{H}}$ we have

$$|A) = \int \mathrm{d}x \int \mathrm{d}x' |x, x')(x, x'|A) \tag{35}$$

where $(x, x'|A)$ is a kernel of the operator A such that

$$(x, x'|A) = \mathrm{Tr}((|x\rangle\langle x'|)^\dagger A) = \mathrm{Tr}(|x'\rangle\langle x|A) = \langle x|A|x'\rangle = A(x, x').$$

An operator ρ of density matrix can be considered as an element $|\rho)$ of the Liouville (Hilbert–Schmidt) space $\overline{\mathcal{H}}$. Using equation (35), we obtain

$$|\rho) = \int \mathrm{d}x \int \mathrm{d}x' |x, x')(x, x'|\rho) \tag{36}$$

where the trace is represented by

$$\mathrm{Tr}\,\rho = (I|\rho) = \int \mathrm{d}x (x, x|\rho) = 1.$$

A.2. Superoperators

Operators, which act on $\overline{\mathcal{H}}$, are called superoperators and we denote them in general by the hat. A superoperator is a map which maps operator to operator.

For an arbitrary superoperator $\hat{\Lambda}$ on $\overline{\mathcal{H}}$, which is defined by $\hat{\Lambda}|A) = |\hat{\Lambda}(A))$, we have

$$(x, x'|\hat{\Lambda}|A) = \int \mathrm{d}y \int \mathrm{d}y' (x, x'|\hat{\Lambda}|y, y')(y, y'|A) = \int \mathrm{d}y \int \mathrm{d}y' \Lambda(x, x', y, y') A(y, y')$$

where $\Lambda(x, x', y, y') = (x, x'|\hat{\Lambda}|y, y')$ is a kernel of the superoperator $\hat{\Lambda}$.

Let A be a linear operator in the Hilbert space $\mathcal{H}$. We can define the multiplication superoperators $\hat{L}_A$ and $\hat{R}_A$ by the following equations:

$$\hat{L}_A|B) = |AB) \qquad \hat{R}_A|B) = |BA).$$

The superoperator kernels can be easily derived. For example, in the basis $|x, x')$ we have

$$(x, x'|\hat{L}_A|B) = \int \mathrm{d}y \int \mathrm{d}y' (x, x'|\hat{L}_A|y, y')(y, y'|B) = \int \mathrm{d}y \int \mathrm{d}y' L_A(x, x', y, y') B(y, y').$$

Using

$$(x, x'|AB) = \langle x|AB|x'\rangle = \int \mathrm{d}y \int \mathrm{d}y' \langle x|A|y\rangle\langle y|B|y'\rangle\langle y'|x'\rangle$$

we obtain the kernel of the left multiplication superoperator

$$L_A(x, x', y, y') = \langle x|A|y\rangle\langle x'|y'\rangle = A(x, y)\delta(x' - y').$$

A superoperator $\hat{\mathcal{E}}^\dagger$ is called the adjoint superoperator for $\hat{\mathcal{E}}$ if $(\hat{\mathcal{E}}^\dagger(A)|B) = (A|\hat{\mathcal{E}}(B))$ for all $|A)$ and $|B)$ from $\overline{\mathcal{H}}$. For example, if $\hat{\mathcal{E}} = \hat{L}_A \hat{R}_B$, then $\hat{\mathcal{E}}^\dagger = \hat{L}_{A^\dagger} \hat{R}_{B^\dagger}$. If $\hat{\mathcal{E}} = \hat{L}_A$, then $\hat{\mathcal{E}}^\dagger = \hat{L}_{A^\dagger}$.

Left superoperators $\hat{L}_A^\pm$ are defined as Lie and Jordan multiplication by the relations

$$\hat{L}_A^- B = \frac{1}{\mathrm{i}\hbar}(AB - BA) \qquad \hat{L}_A^+ B = \frac{1}{2}(AB + BA).$$

The left superoperator $\hat{L}_A^\pm$ and the right superoperator $\hat{R}_A^\pm$ are connected by $\hat{L}_A^- = -\hat{R}_A^-$, $\hat{L}_A^+ = \hat{R}_A^+$. An algebra of the superoperators $\hat{L}_A^\pm$ is defined [50] by

(1) the Lie relations

$$\hat{L}_{A\cdot B}^- = \hat{L}_A^- \hat{L}_B^- - \hat{L}_B^- \hat{L}_A^-$$

(2) the Jordan relations

$$\hat{L}_{(A\circ B)\circ C}^+ + \hat{L}_B^+ \hat{L}_C^+ \hat{L}_A^+ + \hat{L}_A^+ \hat{L}_C^+ \hat{L}_B^+ = \hat{L}_{A\circ B}^+ \hat{L}_C^+ + \hat{L}_{B\circ C}^+ \hat{L}_A^+ + \hat{L}_{A\circ C}^+ \hat{L}_B^+$$
$$\hat{L}_{(A\circ B)\circ C}^+ + \hat{L}_B^+ \hat{L}_C^+ \hat{L}_A^+ + \hat{L}_A^+ \hat{L}_C^+ \hat{L}_B^+ = \hat{L}_C^+ \hat{L}_{A\circ B}^+ + \hat{L}_B^+ \hat{L}_{A\circ C}^+ + \hat{L}_A^+ \hat{L}_{B\circ C}^+$$
$$\hat{L}_C^+ \hat{L}_{A\circ B}^+ + \hat{L}_B^+ \hat{L}_{A\circ C}^+ + \hat{L}_A^+ \hat{L}_{B\circ C}^+ = \hat{L}_{A\circ B}^+ \hat{L}_C^+ + \hat{L}_{B\circ C}^+ \hat{L}_A^+ + \hat{L}_{A\circ C}^+ \hat{L}_B^+$$

(3) the mixed relations

$$\hat{L}_{A\cdot B}^+ = \hat{L}_A^- \hat{L}_B^+ - \hat{L}_B^+ \hat{L}_A^- \qquad \hat{L}_{A\circ B}^- = \hat{L}_A^+ \hat{L}_B^- + \hat{L}_B^+ \hat{L}_A^-$$
$$\hat{L}_{A\circ B}^+ = \hat{L}_A^+ \hat{L}_B^+ - \frac{\hbar^2}{4} \hat{L}_B^- \hat{L}_A^- \qquad \hat{L}_B^+ \hat{L}_A^+ - \hat{L}_A^+ \hat{L}_B^+ = -\frac{\hbar^2}{4} \hat{L}_{A\cdot B}^-.$$

Here we use the notation

$$A \cdot B = \frac{1}{\mathrm{i}\hbar}(AB - BA) \qquad A \circ B = \frac{1}{2}(AB + BA).$$

References

[1] Hellwing K E and Kraus K 1969 *Commun. Math. Phys.* **11** 214
[2] Hellwing K E and Kraus K 1970 *Commun. Math. Phys.* **16** 142
[3] Kraus K 1971 *Ann. Phys.* **64** 311
[4] Kraus K 1983 *States, Effects and Operations. Fundamental Notions of Quantum Theory* (Berlin: Springer)
[5] Schumacher B 1996 *Phys. Rev.* A **54** 2614 (*Preprint* quant-ph/9604023)
[6] Feynman R P and Vernon F L 1963 *Ann. Phys.* **24** 118
[7] Caldeira A O and Leggett A J 1983 *Physica* A **121** 587
[8] Grabert H, Schramm P and Ingold G L 1988 *Phys. Rep.* **168** 115 and references therein
[9] Brink D M, Neto J and Weidenmuller H A 1979 *Phys. Lett.* B **80** 170 and references therein
[10] Pechukas P 1969 *Phys. Rev.* **181** 174
[11] Ankerhold J, Graber H and Ingold G L 1995 *Phys. Rev.* E **51** 4267
[12] Exner P 1985 *Open Quantum Systems and Feynman Integrals* (Dordrecht: Reidel)
[13] Exner P and Kolerov G I 1981 *Phys. Lett.* A **83** 203
[14] Strunz W T 1996 *Phys. Rev.* A **54** 2664
[15] Tarasov V E 1994 *Phys. Lett.* B **323** 296
[16] Tarasov V E 1994 *Theor. Math. Phys.* **100** 1100
[17] Tarasov V E 1994 *Proc. VIII Int. Workshop on High Energy Physics and Quantum Field Theory* (Moscow: MSU) pp 205–9
[18] Tarasov V E 2000 *Proc. XIV Int. Workshop on High Energy Physics and Quantum Field Theory* (Moscow: MSU) pp 637–40
[19] Tarasov V E 2000 *Quantum Mechanics. Lectures on Theory Foundation* (Moscow: Vuzovskaya kniga)
[20] Tarasov V E 2002 *J. Phys. A: Math. Gen.* **35** 5207
[21] Lloyd S and Braunstein S L 1999 *Phys. Rev. Lett.* **82** 1784 (*Preprint* quant-ph/9810082)
[22] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[23] Reid M D 1989 *Phys. Rev.* A **40** 913
[24] Crawford J A 1958 *Nuovo Cimento* **10** 698
[25] Schwinger J 1960 *Proc. Natl Acad. Sci.* **46** 257, 570
[26] Emch G 1964 *Helv. Phys. Acta* **37** 532
[27] Sauermann G 1966 *Physica* **32** 2017
[28] Gabriel H 1969 *Phys. Rev.* **181** 506
[29] Schwinger J 1970 *Quantum Kinematics and Dynamics* (New York: Benjamin) chapters 2.11, 2.12
[30] Schmutz M 1978 *Z. Phys.* B **30** 97
[31] Blum K 1981 *Density Matrix. Theory and Applications* (London: Plenum) chapter 7.6
[32] Abragam A and Goldman M 1982 *Nuclear Magnetism. Order and Disorder* (Oxford: Clarendon) chapter 1B
[33] Schmutz M 1983 *Lett. Nuovo Cimento* **37** 161
[34] Fick E and Sauermann G 1990 *The Quantum Statistics of Dynamic Processes* (Berlin: Springer) chapters 3.1., 8.1., 8.2.
[35] Suzuki M 1991 *Int. J. Mod. Phys.* B **5** 1821
[36] Tarasov V E 1997 *Proc. XI Int. Workshop on HEP and QFT* (Moscow: MSU) pp 368–71
[37] Caves C M 1999 *J. Supercond.* **12** 707 (*Preprint* quant-ph/9811082)
[38] Tarasov V E 2000 *Mathematical Introduction to Quantum Mechanics* (Moscow: MAI) chapters 2, 3, 5
[39] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
[40] Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821
[41] Gorini V, Frigerio A, Verri M, Kossakowski A and Sudarshan E C G 1978 *Rep. Math. Phys.* **13** 149
[42] Alicki R and Lendi K 1987 *Quantum Dynamical Semigroups and Applications* (Berlin: Springer)
[43] Arveson W 2000 *Preprint* math.OA/0005251 (chapter 3 and references therein)
[44] Tarasov V E 1997 *Theor. Math. Phys.* **110** 57
[45] Dyson F J 1949 *Phys. Rev.* **75** 486, 1736
[46] Sandulescu A and Scutaru H 1987 *Ann. Phys.* **173** 277
[47] Dekker H 1981 *Phys. Rep.* **80** 1
[48] Gel'fand I M and Vilenkin N Ya 1964 *Generalized Functions, Vol 4. Applications of Harmonic Analysis* (New York: Academic)
[49] Roberts J E 1966 *Commun. Math. Phys.* **3** 98
[50] Tarasov V E 2001 *Moscow Univ. Phys. Bull.* **56** 5
[51] Slavnov A and Fadeev L 1988 *Introduction to Quantum Theory of Gauge Fields* 2nd edn (Moscow: Nauka) chapter II.1

[52] Huang K 1992 *Quarks, Leptons and Gauge Fields* (New York: World Scientific) chapter 7.1
[53] Lee T D and Yang C N 1962 *Phys. Rev.* **128** 885
[54] Inomata A and Singh V A 1980 *Phys. Lett.* A **80** 105
[55] Ryder L H 1985 *Quantum Field Theory* (Cambridge: Cambridge University Press) chapter 5.1
[56] Caves C M 1986 *Phys. Rev.* D **33** 1643
[57] Mensky M B 2000 *Quantum Measurements and Decoherence: Models and Phenomenology* (Dordrecht: Kluwer) chapters 5.2, 5.3
[58] Mensky M B 1994 *Phys. Lett.* A **196** 159
[59] Mensky M B 1997 *Phys. Lett.* A **231** 1
[60] Marx M and Parrinello M 1996 *J. Chem. Phys.* **104** 4077
[61] Tuckerman M E, Mundy C J and Martyna G J 1999 *Europhys. Lett.* **45** 149
[62] Tuckerman M E, Liu Y, Ciccotti G and Martyna G J 2001 *J. Chem. Phys.* **115** 1678
[63] Ramshaw J D 2002 *Europhys. Lett.* **59** 319
[64] Tarasov V E 2003 *Mod. Phys. Lett.* B **17** 1219
[65] Sergi A 2003 *Phys. Rev.* E **67** 021101
[66] Tarasov V E 2002 *Phys. Lett.* A **299** 173
[67] Tarasov V E 2002 *Phys. Rev.* E **66** 056116
[68] Kapral R and Ciccotti G 1999 *J. Chem. Phys.* **110** 8919
[69] Nielsen S, Kapral R and Ciccotti G 2001 *J. Chem. Phys.* **115** 5805