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Psi-series solution of fractional Ginzburg–Landau equation

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Abstract
One-dimensional Ginzburg–Landau equations with derivatives of noninteger order are considered. Using psi-series with fractional powers, the solution of the fractional Ginzburg–Landau (FGL) equation is derived. The leading-order behaviours of solutions about an arbitrary singularity, as well as their resonance structures, have been obtained. It was proved that fractional equations of order α with polynomial nonlinearity of order s have the noninteger power-like behaviour of order $\alpha/(1-s)$ near the singularity.

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1. Introduction

Differential equations that contain derivatives of noninteger order [1, 2] are called fractional equations [3, 4]. The interest to fractional equations has been growing continually during the last few years because of numerous applications. In a fairly short period of time the areas of applications of fractional calculus have become broad. For example, the derivatives and integrals of fractional order are used in chaotic dynamics [5, 6], material sciences [7–9], mechanics of fractal and complex media [10, 11], quantum mechanics [12, 13], physical kinetics [5, 14–16], plasma physics [17, 18], astrophysics [19], long-range dissipation [20], non-Hamiltonian mechanics [21, 22], long-range interaction [23–25], anomalous diffusion and transport theory [5, 26, 27].

The fractional generalization of the Ginzburg–Landau equation was suggested in [28]. This equation can be used to describe the dynamical processes in continuums with fractal dispersion and the media with fractal mass dimension [29–31]. In this paper, we generalize the psi-series approach to the fractional differential equations. As an example, we consider a solution of the fractional Ginzburg–Landau (FGL) equation. We derive the psi-series for the one-dimensional FGL equation. The leading-order behaviours of solutions about an arbitrary singularity, as well as their resonance structures, have been obtained.

In section 2, we recall the appearance of the Ginzburg–Landau equation with fractional derivatives. In section 3, the singular behaviour of the FGL equation is considered. In section 4, we discuss the powers of series terms that have arbitrary coefficients that are called the resonances or Kovalevskaya exponents. In section 5, we derive the psi-series and recurrence relations for a one-dimensional FGL equation with rational order α ($1 < \alpha < 2$). In section 6, the example of an FGL equation with order $\alpha = 3/2$ is suggested. In section 7, the next to singular behaviour for arbitrary (rational or irrational) order is discussed. Finally, a short conclusion is given in section 8.

2. Fractional Ginzburg–Landau (FGL) equation

Let us recall the appearance of the nonlinear parabolic equation [32–35], and the FGL equation [28, 29, 31]. Consider wave propagation in some media and present the wave vector $\mathbf{k}$ in the form

$$\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa} = \mathbf{k}_0 + \boldsymbol{\kappa}_\parallel + \boldsymbol{\kappa}_\perp, \tag{1}$$

where $\mathbf{k}_0$ is the unperturbed wave vector and subscripts ($\parallel, \perp$) are taken respectively to the direction of $\mathbf{k}_0$. A symmetric dispersion law $\omega = \omega(k)$ for $\kappa \ll k_0$ can be written as

$$\omega(k) = \omega(|\mathbf{k}|) \approx \omega(k_0) + v_g(|\mathbf{k}| - k_0) + \tfrac{1}{2} v'_g(|\mathbf{k}| - k_0)^2, \tag{2}$$

where

$$v_g = \left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}, \qquad v'_g = \left(\frac{\partial^2 \omega}{\partial k^2}\right)_{k=k_0}, \tag{3}$$

and

$$|\mathbf{k}| = |\mathbf{k}_0 + \boldsymbol{\kappa}| = \sqrt{(\mathbf{k}_0 + \kappa_\parallel)^2 + \kappa_\perp^2} \approx k_0 + \kappa_\parallel + \frac{1}{2k_0}\kappa_\perp^2. \tag{4}$$

Substitution of (4) into (2) gives

$$\omega(k) \approx \omega_0 + v_g \kappa_\parallel + \frac{v_g}{2k_0}\kappa_\perp^2 + \frac{v'_g}{2}\kappa_\parallel^2, \tag{5}$$

where $\omega_0 = \omega(k_0)$. Expression (5) in the dual space ('momentum representation') corresponds to the following equation in the coordinate space:

$$\mathrm{i}\frac{\partial Z}{\partial t} = \omega_0 Z - \mathrm{i} v_g \frac{\partial Z}{\partial x} - \frac{v_g}{2k_0}\Delta_\perp Z - \frac{v'_g}{2}\Delta_\parallel Z \tag{6}$$

with respect to the field $Z = Z(t, x, y, z)$, where x is along $\mathbf{k}_0$, and we use the operator correspondence between the dual space and usual spacetime:

$$\begin{aligned} &\omega(k) \longleftrightarrow \mathrm{i}\frac{\partial}{\partial t}, && \kappa_\parallel \longleftrightarrow -\mathrm{i}\frac{\partial}{\partial x}, \\ &(\boldsymbol{\kappa}_\perp)^2 \longleftrightarrow -\Delta_\perp = -\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}, && (\boldsymbol{\kappa}_\parallel)^2 \longleftrightarrow -\Delta_\parallel = -\frac{\partial^2}{\partial x^2}. \end{aligned} \tag{7}$$

A generalization to the nonlinear case can be carried out similarly to (5) through a nonlinear dispersion law dependence on the wave amplitude:

$$\omega = \omega(k, |Z|^2) \approx \omega(k, 0) + b|Z|^2 = \omega(|\mathbf{k}|) + b|Z|^2 \tag{8}$$

with some constant $b = \partial\omega(k, |Z|^2)/\partial|Z|^2$ at $|Z| = 0$. In analogy to (6), we obtain from (5) and (7):

$$\mathrm{i}\frac{\partial Z}{\partial t} = \omega(k_0) Z - \mathrm{i} v_g \frac{\partial Z}{\partial x} - \frac{v_g}{2k_0}\Delta_\perp Z - \frac{v'_g}{2}\Delta_\parallel Z + b|Z|^2 Z. \tag{9}$$

This equation is known as the nonlinear parabolic equation [32–35]. The change of variables from (t, x, y, z) to $(t, x - v_g t, y, z)$ gives

$$-\mathrm{i}\frac{\partial Z}{\partial t} = \frac{v_g}{2k_0}\Delta_\perp Z + \frac{v_g'}{2}\Delta_\parallel Z - \omega(k_0)Z - b|Z|^2 Z \tag{10}$$

that is also known as the nonlinear Schrödinger (NLS) equation.

Wave propagation in media with fractal properties can be easily generalized by rewriting the dispersion law (5), (8) in the following way [28]:

$$\omega(k, |Z|^2) = \omega(k_0, 0) + v_g\kappa_\parallel + g_1\left(\boldsymbol{\kappa}_\perp^2\right)^{\alpha/2} + g_2\left(\boldsymbol{\kappa}_\parallel^2\right)^{\beta/2} + b|Z|^2, \quad (1 < \alpha, \beta < 2) \tag{11}$$

with new constants g_1, g_2.

Using the connection between Riesz fractional derivative and its Fourier transform [1]

$$(-\Delta_\perp)^{\alpha/2} \longleftrightarrow \left(\boldsymbol{\kappa}_\perp^2\right)^{\alpha/2}, \qquad (-\Delta_\parallel)^{\beta/2} \longleftrightarrow \left(\boldsymbol{\kappa}_\parallel^2\right)^{\beta/2}, \tag{12}$$

we obtain from (11)

$$\mathrm{i}\frac{\partial Z}{\partial t} = -\mathrm{i}v_g\frac{\partial Z}{\partial x} + g_1(-\Delta_\perp)^{\alpha/2}Z + g_2(-\Delta_\parallel)^{\beta/2}Z + \omega_0 Z + b|Z|^2 Z, \tag{13}$$

where $Z = Z(t, x, y, z)$. By changing the variables from (t, x, y, z) to (t, ξ, y, z), $\xi = x - v_g t$, and using

$$(-\Delta_\parallel)^{\beta/2} = \frac{\partial^\beta}{\partial|x|^\beta} = \frac{\partial^\beta}{\partial|\xi|^\beta}, \tag{14}$$

we obtain from (13) equation

$$\mathrm{i}\frac{\partial Z}{\partial t} = g_1(-\Delta_\perp)^{\alpha/2}Z + g_2(-\Delta_\parallel)^{\beta/2}Z + \omega_0 Z + b|Z|^2 Z, \tag{15}$$

that can be called the fractional nonlinear parabolic equation. For $g_2 = 0$ we get the nonstationary FGL equation (fractional NLS equation) suggested in [28]. Let us comment on the physical structure of (15). The first term on the right-hand side is related to wave propagation in media with fractal properties. The fractional derivative can also appear as a result of long-range interaction [23–25]. Other terms on the right-hand-side of equations (13) and (15) correspond to wave interaction due to the nonlinear properties of the media. Thus, equation (15) can describe fractal processes of self-focusing and related issues.

We may consider one-dimensional simplifications of (15), i.e.,

$$\mathrm{i}\frac{\partial Z}{\partial t} = g_2\frac{\partial^\beta Z}{\partial|\xi|^\beta} + \omega_0 Z + b|Z|^2 Z, \tag{16}$$

where $Z = Z(t, \xi)$, $\xi = x - v_g t$, or the equation

$$\mathrm{i}\frac{\partial Z}{\partial t} = g_1\frac{\partial^\alpha Z}{\partial|z|^\alpha} + \omega_0 Z + b|Z|^2 Z, \tag{17}$$

where $Z = Z(t, z)$. We can reduce (17) to the case of a propagating wave

$$Z = Z(z - v_g t) \equiv Z(\eta). \tag{18}$$

For the real field Z, equation (17) becomes

$$g_1\frac{\mathrm{d}^\alpha Z}{\mathrm{d}|\eta|^\alpha} + c\frac{\mathrm{d}Z}{\mathrm{d}\eta} + \omega_0 Z + bZ^3 = 0, \qquad \eta = z - v_g t, \tag{19}$$

where $c = \mathrm{i}v_g$. This equation takes the form of the fractional generalization of the Ginzburg–Landau equation, when $v_g = 0$.

It is well known that the nonlinear term in equations of type (10) leads to a steepening of the solution and its singularity. The steepening process may be stopped by a diffusional or dispersional term, i.e. by a higher derivative term. A similar phenomenon may appear for the fractional nonlinear equations (19).

3. Singular behaviour of FGL equation

There is an approach to the question of integrability which is not concerned with the display of explicit functions, but with the demonstration of a specific property. This is the existence of Laurent series for each of the dependent variables. The series may not be summable to an explicit form, but does represent an analytic function. The essential feature of this Laurent series is that it is an expansion about a particular type of movable singularity, i.e., a pole. Consequently the existence of these Laurent series is intimately concerned with the singularity analysis of differential equations initiated about a century ago by Painleve, Gambier and Garnier [36] and continued since by many workers including Bureau [37] and Cosgrove *et al* [38].

The connection of this type of singular behaviour and the solution of partial differential equations by the method of the inverse scattering transform was noticed by Ablowitz *et al* [39] who developed an algorithm, called the ARS algorithm, to test whether the solution of an ordinary differential equation was expressible in terms of a Laurent expansion. If this was the case, the ordinary differential equation was said to pass the Painleve test and was conjectured to be integrable. Under more precise conditions Conte [40] has shown that the equation is integrable. Psi-series solutions of differential equations are considered in [41–44].

In this paper, we consider the fractional equation

$$gD^{\alpha}_{x}Z(x) + cD^{1}_{x}Z(x) + aZ(x) + bZ^{3}(x) = 0, \qquad (20)$$

where $1 < \alpha < 2$, and D^{α}_{x} is the fractional Riemann–Liouville derivative:

$$D^{\alpha}_{x}Z(x) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_{x_0}^{x}\mathrm{d}y\frac{Z(y)}{(x-y)^{\alpha-m+1}}. \qquad (21)$$

Here, m is the first whole number greater than or equal to α. In our case $m = 2$. We detect possible singular behaviour in the solution of a differential equation by means of the leading-order analysis.

To determine the leading-order behaviour, we set

$$Z(x) = f(x-x_0)^p, \qquad (22)$$

where x_0 is an arbitrary constant (the location of the singularity). Then, we substitute (22) into the fractional differential equation (20) and look for two or more dominant terms. The detection of which terms are dominant is identical to the determination of which terms in an equation are self-similar.

Substituting (22) into equation (20), and using the relation

$$D^{\alpha}_{x}x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}x^{p-\alpha}, \quad (p > -1), \qquad (23)$$

we get

$$gf\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x-x_0)^{p-\alpha} + cpf(x-x_0)^{p-1} + af(x-x_0)^p + bf^3(x-x_0)^{3p} = 0. \qquad (24)$$

If $1 < \alpha < 2$, then $p-\alpha < p-1$. For the dominant terms,

$$gf\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x-x_0)^{p-\alpha} + bf^3(x-x_0)^{3p} = 0. \qquad (25)$$

As a result, we obtain

$$p - \alpha = 3p, \qquad (26)$$

$$g\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} + bf^2 = 0. \qquad (27)$$

Equation (26) gives

$$p = -\frac{\alpha}{2}. \tag{28}$$

If $1 < \alpha < 2$, then $-1 < p < -1/2$. Therefore the leading-order singular behaviour is found:

$$Z(x) = f(x - x_0)^{-\alpha/2}, \qquad f^2 + \frac{g\Gamma(1 - \alpha/2)}{b\Gamma(1 - 3\alpha/2)} = 0, \tag{29}$$

and the singularity is a pole of order $\alpha/2$. Evidently our psi-series starts at $(x - x_0)^{-\alpha/2}$. The resonance conditions and psi-series is considered in the next sections.

As a result, we get that fractional dinnerential equations of order α with polynomial nonlinearity of order s have the noninteger power behaviour of order $\alpha/(1 - s)$ near the singularity.

4. Resonance terms of FGL equation

The powers of $(x - x_0)$ that have arbitrary coefficients are called the resonances or Kovalevskaya's exponents. To find resonance, we consider the substitution

$$Z(x) = f(x - x_0)^p + l(x - x_0)^{p+r}, \tag{30}$$

and find the values of r. In equation (30) parameters p and f are defined by

$$p = -\frac{\alpha}{2}, \qquad f = \sqrt{-\frac{g\Gamma(1 - \alpha/2)}{b\Gamma(1 - 3\alpha/2)}}. \tag{31}$$

Substitution of equation (30) into (20) gives

$$\begin{aligned} gf&\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x - x_0)^{p-\alpha} + cpf(x - x_0)^{p-1} + af(x - x_0)^p + bf^3(x - x_0)^{3p} \\ &+ gl\frac{\Gamma(p+r+1)}{\Gamma(p+r+1-\alpha)}(x - x_0)^{p+r-\alpha} + cpl(x - x_0)^{p+r-1} + al(x - x_0)^{p+r} \\ &+ bl^3(x - x_0)^{3p+3r} + 3bl^2 f(x - x_0)^{2p+3r} + 3blf^2(x - x_0)^{p+3r} = 0. \end{aligned} \tag{32}$$

Using equation (31), and considering the linear with respect to l terms of (32), we have

$$\frac{\Gamma(1 + r - \alpha/2)}{\Gamma(1 + r - 3\alpha/2)} - 3\frac{\Gamma(1 - \alpha/2)}{\Gamma(1 - 3\alpha/2)} = 0. \tag{33}$$

This relation allows us to derive the values of r. Equation (33) can be directly derived by using the recurrence relations. In the general case, the values of r can be irrational or complex numbers. The solution of the FGL equation with $1 < \alpha \leqslant 2$ have two arbitrary parameters. Therefore, we must have two values of r that are the solutions of equation (33). It is interesting to note that (33) gives two real values of r only for

$$\alpha > \alpha_0, \tag{34}$$

where

$$\alpha_0 \approx 1.300\,588\,8986. \tag{35}$$

The order α_0 is an universal value that does not depend on values of parameters g, a, b, c of the FGL equation (20).

The plots of the function

$$y(r) = \frac{\Gamma(1 + r - \alpha/2)}{\Gamma(1 + r - 3\alpha/2)} - 3\frac{\Gamma(1 - \alpha/2)}{\Gamma(1 - 3\alpha/2)} \tag{36}$$

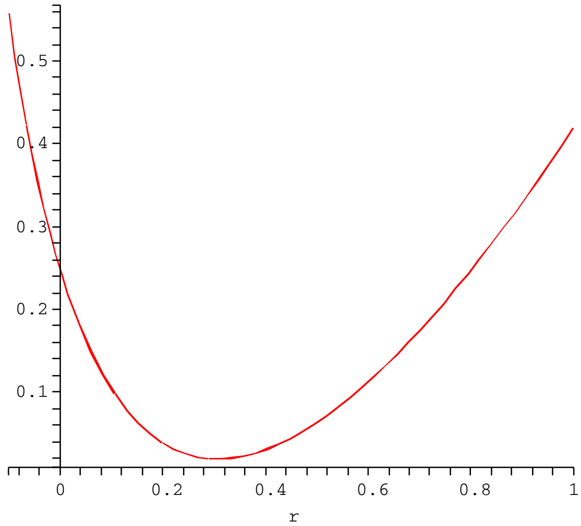

Figure 1. Plot for the order $\alpha = 1.30$.

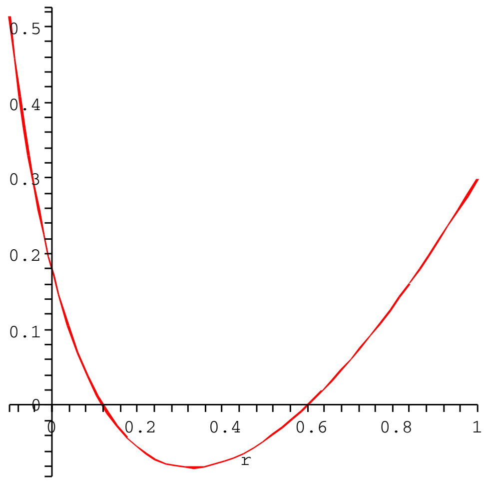

Figure 2. Plot for the order $\alpha = 1.31$.

are shown in figures 1 and 2. The solutions of equation (33) correspond to the points of intersection with the horizontal axis.

As a result, the nature of the resonances is summarized as follows:

(1) For α such that $1 < \alpha < \alpha_0$ the values of r are complex or $r < -\alpha/2$.

(2) For α such that $\alpha_0 < \alpha < 2$, we have two the real values of r. Note that for $\alpha_0 < \alpha < 1.999\,999\,9995$, the values r satisfy the inequality $r < 6.426$.

5. Psi-series for FGL equation of rational order

Let us consider the psi-series and recurrence relations for the one-dimensional FGL equation (20), where the order α is a rational number such that

$$\alpha = \frac{m}{n}, \quad (1 < \alpha < 2). \tag{37}$$

Following a standard procedure [41], we substitute

$$Z(x) = \frac{1}{(x - x_0)^{\alpha/2}} \sum_{k=0}^{\infty} e_k \phi^k (x - x_0), \tag{38}$$

where

$$e_0 = f = \sqrt{-\frac{g\Gamma(1 - \alpha/2)}{b\Gamma(1 - 3\alpha/2)}}, \tag{39}$$

into the fractional Ginzburg–Landau equation (20). Note that the coefficient e_0 is a real number for two cases: (1) $g/b \geqslant 0$ and $1 < \alpha < 4/3$; (2) $g/b \leqslant 0$ and $4/3 < \alpha < 2$.

For the rational order $\alpha = m/n$, we suggest to use the ϕ-function in the form

$$\phi(x - x_0) = (x - x_0)^{1/2n}. \tag{40}$$

Then

$$Z(x) = \frac{1}{(x - x_0)^{\alpha/2}} \sum_{k=0}^{\infty} e_k (x - x_0)^{\beta_k} = \sum_{k=0}^{\infty} e_k (x - x_0)^{\beta_k - \alpha/2}, \tag{41}$$

where

$$\beta_k = \frac{k}{2n}. \tag{42}$$

In this case, the action of the fractional derivative of order $\alpha = m/n$ can be represented as the change of the number of term $e_k \to e_{k-2m}$:

$$D_x^{\alpha} (x - x_0)^{\beta_k} = \frac{\Gamma(\beta_k + 1)}{\Gamma(\beta_{k-2m} + 1)} (x - x_0)^{\beta_{k-2m}}. \tag{43}$$

It allows us to derive the generalized psi-series solutions of the fractional Ginzburg–Landau equation.

Substitution of the series

$$Z(x) = \sum_{k=0}^{\infty} e_k (x - x_0)^{\beta_k - \alpha/2} = \sum_{k=0}^{\infty} e_k (x - x_0)^{\frac{k-m}{2n}} \tag{44}$$

into equation (20) gives

$$g \sum_{k=0}^{\infty} e_k \frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} (x - x_0)^{\frac{k-3m}{2n}} + c \sum_{k=0}^{\infty} e_k \frac{k - m}{2n} (x - x_0)^{\frac{k-m-2n}{2n}}$$
$$+ a \sum_{k=0}^{\infty} e_k (x - x_0)^{\frac{k-m}{2n}} + b \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} e_{k_1} e_{k_2} e_{k_3} (x - x_0)^{\frac{k_1+k_2+k_3-3m}{2n}} = 0. \tag{45}$$

Let us compute e_k $(k = 1, 2, \ldots)$ through the equation of coefficients of like power of $(x - x_0)$ to zero in (45):

$$g \sum_{k=0}^{\infty} e_k \frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} (x - x_0)^{\frac{k-3m}{2n}} + c \sum_{k=2m-2n}^{\infty} e_{k-2m+2n} \frac{k - 3m + 2n}{2n} (x - x_0)^{\frac{k-3m}{2n}}$$
$$+ a \sum_{k=2m}^{\infty} e_{k-2m} (x - x_0)^{\frac{k-3m}{2n}} + 3be_0^2 \sum_{k=0}^{\infty} e_k (x - x_0)^{\frac{k-3m}{2n}}$$
$$+ b \sum_{k=0}^{\infty} \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j (x - x_0)^{\frac{k-3m}{2n}} = 0. \tag{46}$$

Using $e_0^2 = f^2$, we get

$$g e_k \frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} + c e_{k-2m+2n} \frac{k-3m+2n}{2n} + a e_{k-2m} + 3bf^2 e_k + b \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j = 0. \tag{47}$$

Here $k = 0, 1, 2, \ldots$, and $e_k = 0$ for $k < 0$. We can rewrite the recurrence relation (47) as

$$e_k \left(g \frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} + 3bf^2 \right) = -c e_{k-2m+2n} \frac{k-3m+2n}{2n} - a e_{k-2m} - b \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j, \tag{48}$$

where

$$f^2 = -\frac{g\Gamma(1-\alpha/2)}{b\Gamma(1-3\alpha/2)} = -\frac{g\Gamma\left(\frac{2n-m}{2n}\right)}{b\Gamma\left(\frac{2n-3m}{2n}\right)}. \tag{49}$$

Substitution of equation (49) into equation (48) gives

$$e_k g \left(\frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} - \frac{3\Gamma\left(\frac{2n-m}{2n}\right)}{\Gamma\left(\frac{2n-3m}{2n}\right)} \right) = -c e_{k-2m+2n} \frac{k-3m+2n}{2n} - a e_{k-2m} - b \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j. \tag{50}$$

Note that we get resonances for the k that satisfies the condition

$$\frac{\Gamma\left(\frac{k-m+2n}{2n}\right)}{\Gamma\left(\frac{k-3m+2n}{2n}\right)} - \frac{3\Gamma\left(\frac{2n-m}{2n}\right)}{\Gamma\left(\frac{2n-3m}{2n}\right)} = 0. \tag{51}$$

In this case, the coefficient e_k can be arbitrary.

As a result, we obtain the nonresonance terms

$$e_k = -A(k, m, n) \left(c e_{k-2m+2n} \frac{k-3m+2n}{2n} + a e_{k-2m} + b \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j \right), \tag{52}$$

where

$$A(k, m, n) = -\frac{\Gamma\left(\frac{k-3m+2n}{2n}\right)\Gamma\left(\frac{2n-3m}{2n}\right)}{g\left[\Gamma\left(\frac{k-m+2n}{2n}\right)\Gamma\left(\frac{2n-3m}{2n}\right) - 3\Gamma\left(\frac{k-3m+2n}{2n}\right)\Gamma\left(\frac{2n-m}{2n}\right)\right]}. \tag{53}$$

6. Fractional Ginzburg–Landau equation with α = 1.5

Let us consider the FGL equation (20) with derivative of order $\alpha = 3/2$. In this case, $n = 2$, $m = 3$, and the coefficients e_k are defined by

$$e_k = -\frac{\left(c e_{k-2} \frac{k-5}{4} + a e_{k-6} + b \sum_{i=1}^{k-1} \sum_{j=1}^{i} e_{k-i-j} e_i e_j\right)\Gamma\left(\frac{k-5}{4}\right)\Gamma\left(\frac{-5}{4}\right)}{g\left[\Gamma\left(\frac{k+1}{4}\right)\Gamma\left(\frac{-5}{4}\right) - 3\Gamma\left(\frac{k-5}{4}\right)\Gamma\left(\frac{1}{4}\right)\right]}. \tag{54}$$

For $k = 0$, we have

$$e_0 = f = \sqrt{-\frac{g\Gamma\left(\frac{1}{4}\right)}{b\Gamma\left(\frac{-5}{4}\right)}}, \tag{55}$$

where $\Gamma\left(\frac{1}{4}\right)/\Gamma\left(\frac{-5}{4}\right) > 0$, and we suppose $g/b < 0$. For $k = 1$, equation (54) leads to $e_1 = 0$. For $k = 2$,

$$e_2 = \frac{3ce_0\Gamma\left(\frac{-3}{4}\right)\Gamma\left(\frac{-5}{4}\right)}{4g\left[\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{-5}{4}\right) - 3\Gamma\left(\frac{-3}{4}\right)\Gamma\left(\frac{1}{4}\right)\right]}. \tag{56}$$

Using relation (55), we get

$$e_2 = -\frac{c\sqrt{-5(g/b)}\pi^{3/2}2^{3/4}\Gamma(3/4)}{2g[2\Gamma^4(3/4) + 5\pi^2]}. \tag{57}$$

For $k = 3$, $e_3 = 0$. For $k = 4$,

$$e_4 = -\frac{\left(ce_2\frac{-1}{4} + be_0e_2^2\right)\Gamma\left(\frac{-1}{4}\right)\Gamma\left(\frac{-5}{4}\right)}{g\left[\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{-5}{4}\right) - 3\Gamma\left(\frac{-1}{4}\right)\Gamma\left(\frac{1}{4}\right)\right]}. \tag{58}$$

Substitution of (55), and (57) into (58) gives

$$e_4 = \frac{c^2\sqrt{5g\pi\sqrt{2}/b}\Gamma^7(3/4)}{4g^2[2\Gamma^4(3/4) + 5\pi^2]^2}. \tag{59}$$

For $k = 5$, we have $e_5 = 0$. For $k = 6$, equation (54) is

$$e_6 = -\frac{\left(ce_4\frac{1}{4} + ae_0 + be_2^3 + 2be_0e_2e_4\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{-5}{4}\right)}{g\left[\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{-5}{4}\right) - 3\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)\right]}. \tag{60}$$

Substituting e_0, e_2 and e_4 from equations (55), (57) and (59) into (60), we have

$$\begin{aligned} e_6 = -&\frac{\sqrt{-5g/b}\pi^{3/2}2^{3/4}\Gamma(3/4)}{3g^2[2\Gamma^4(3/4) + 5\pi^2]^3[2\Gamma^4(3/4) - 5\pi^2]}(c^3\Gamma^{12}(3/4) + 5c^3\Gamma^8(3/4)\pi^2 \\ &+ 20c^3\Gamma^4(3/4)\pi^4 + 16ag^2\Gamma^{12}(3/4) + 120ag^2\Gamma^8(3/4)\pi^2 \\ &+ 300ag^2\Gamma^4(3/4)\pi^4 + 250ag^2\pi^6). \end{aligned} \tag{61}$$

As a result, we obtain

$$\begin{aligned} Z(x) &= \sum_{k=0}^{\infty} e_k(x - x_0)^{\frac{k-3}{4}} \\ &= e_0(x - x_0)^{-3/4} + e_2(x - x_0)^{-1/4} + e_4(x - x_0)^{1/4} + e_6(x - x_0)^{3/4} + \cdots \end{aligned} \tag{62}$$

This is a power psi-series that presents the solution of the FGL equation of order $\alpha = 1.5$. The coefficients in equation (62) are defined by equations (55), (57), (59) and (61). For example, $a = -b = c = g = 1$ gives

$$e_0 \approx 0.961\,553\,9375, \qquad e_2 \approx -0.238\,224\,6293,$$
$$e_4 \approx 0.001\,685\,563\,496, \qquad e_6 \approx 0.387\,213\,4448.$$

7. Next to singular behaviour

In sections 5 and 6, we consider the rational α. In this section, we suppose that the order α is an arbitrary (rational or irrational). Instead of imposing a series commencing at the power indicated by the singularity found by the leading-order analysis, we can determine the next to singular behaviour by writing

$$Z(x) = f(x - x_0)^p + G(x), \tag{63}$$

where

$$p = -\frac{\alpha}{2}, \qquad f = \sqrt{-\frac{g\Gamma(1-\alpha/2)}{b\Gamma(1-3\alpha/2)}}. \tag{64}$$

We can always write $Z(x)$ in the form (63). To make the process useful, we require that the first term be the leading-order term, i.e.,

$$(x-x_0)^{-p}G(x) = (x-x_0)^{\alpha/2}G(x) \to 0 \qquad \text{if} \quad (x-x_0) \to 0. \tag{65}$$

Substituting (63) into (20), and using (23), we have

$$gf\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x-x_0)^{p-\alpha} + cpf(x-x_0)^{p-1} + af(x-x_0)^p + bf^3(x-x_0)^{3p}$$
$$+ gD_x^\alpha G(x) + cD_x^1 G(x) + aG(x) + bG^3(x)$$
$$+ 3bf^2(x-x_0)^{2p}G(x) + 3bf(x-x_0)^p G^2(x) = 0. \tag{66}$$

Equations (66) and (64) give

$$gD_x^\alpha G(x) + cD_x^1 G(x) + aG(x) + bG^3(x) + 3bf^2(x-x_0)^{2p}G(x)$$
$$+ 3bf(x-x_0)^p G^2(x) cpf(x-x_0)^{p-1} + af(x-x_0)^p = 0. \tag{67}$$

Multiplying this equation on $(x-x_0)^{-3p}$, and using condition (65), we get the equation without nonlinear terms for $(x-x_0) \to 0$.

As a result, we obtain

$$gD_x^\alpha G(x) + cD_x^1 G(x) + aG(x) + cpf(x-x_0)^{-\alpha/2-1} + af(x-x_0)^{-\alpha/2} = 0, \tag{68}$$

with condition (65) for the solutions. Equation (68) is a linear inhomogeneous fractional equation. The solution of this equation allows us to find the solution of the one-dimensional FGL equation.

Let us consider equation (68) with $c = 0$, and the boundary conditions

$$(D_x^{\alpha-k}G(x))_{x=x_0} = G_k, \qquad k = 1, 2. \tag{69}$$

Then the solution is

$$G(x) = \sum_{k=1}^{2} G_k (x-x_0)^{\alpha-k} E_{\alpha,\alpha+1-k}[-a(x-x_0)^\alpha]$$
$$+ af\int_0^x (x-x_0-y)^{\alpha-1} E_{\alpha,\alpha}[-a(x-x_0-y)^\alpha](y-x_0)^{-\alpha/2}\,\mathrm{d}y. \tag{70}$$

Here $E_{\alpha,\beta}[z]$ is a Mittag–Leffler function that is defined by

$$E_{\alpha,\beta}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{71}$$

Let us consider the integral representation for the Mittag–Leffler function

$$E_{\alpha,\beta}[z] = \frac{1}{2\pi \mathrm{i}} \int_{Ha} \frac{\xi^{\alpha-\beta} e^{\xi}}{\xi^\alpha - z}\,\mathrm{d}\xi, \tag{72}$$

where Ha denotes the Hankel path, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|\xi| \leqslant |z|^{1/\alpha}$ in the positive direction, and ends at $-\infty$ along the upper side of the negative real axis. By the replacement $\xi^\alpha \to \xi$ equation (72) transforms into [3, 45]:

$$E_{\alpha,\beta}[z] = \frac{1}{2\pi \mathrm{i}\alpha} \int_{\gamma(a,\delta)} \frac{\mathrm{e}^{\xi^{1/\alpha}} \xi^{(1-\beta)/\alpha}}{\xi - z}\,\mathrm{d}\xi, \qquad (1 < \alpha < 2), \tag{73}$$

where $\pi\alpha/2 < \delta < \min\{\pi, \pi\alpha\}$. The contour $\gamma(a, \delta)$ consists of two rays $S_{-\delta} = \{\arg(\xi) = -\delta, |\xi| \geqslant a\}$ and $S_{+\delta} = \{\arg(\xi) = +\delta, |\xi| \geqslant a\}$, and a circular arc $C_\delta = \{|\xi| = 1, -\delta \leqslant \mathrm{arc}(\xi) \leqslant \delta\}$. Let us denote the region on the left from $\gamma(a, \delta)$ as $G^-(a, \delta)$. Then [45]

$$E_{\alpha,\beta}[z] = -\sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(\beta - \alpha n)}, \qquad z \in G^-(a, \delta), \quad (|z| \to \infty), \tag{74}$$

and $\delta \leqslant |\arg(z)| \leqslant \pi$. In our case, $z = -a(x - x_0)^\alpha$, $\arg(z) = \pi$. As a result, we arrive at the asymptotic of the solution, which exhibits power-like tails for $x \to \infty$. These power-like tails are the most important effect of the noninteger derivative in the fractional equations.

8. Conclusion

In this paper, we generalize the psi-series approach to the fractional differential equations. As an example, we consider the fractional Ginzburg–Landau (FGL) equation [28–31]. The suggested psi-series approach can be used for a wide class of fractional nonlinear equations. The leading-order behaviours of solutions about an arbitrary singularity, as well as their resonance structures, can be derived for fractional equations by the suggested generalization of psi-series.

In the paper, we use the series

$$Z(x) = \frac{1}{(x - x_0)^{m/2n}} \sum_{k=0}^{\infty} e_k (x - x_0)^{k/2n} \tag{75}$$

where k, m, n are the integer numbers. For the order $\alpha = m/n$, the action of the fractional derivative

$$D_x^\alpha (x - x_0)^{k/2n} = \frac{\Gamma(k/2n + 1)}{\Gamma((k - 2m)/2n + 1)} (x - x_0)^{(k-2m)/2n} \tag{76}$$

can be represented as the change of the number of term $e_k \to e_{k-2m}$ in (75). It allows us to derive the psi-series for the fractional differential equation of order $\alpha = m/n$. For the FGL equation the leading-order singular behaviour is defined by power that is equal to the half of derivative order.

A remarkable property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails [25]. In this paper, we prove that fractional differential equations of order α with a polynomial nonlinear term of order s have the noninteger power-like behaviour of order $\alpha/(1 - s)$ near the singularity.

It is interesting to find barriers to integrability for fractional differential equations. In general, the integrability of fractional nonlinear equations is a very interesting object for future researches. It has many problems that are connected with specific properties of the fractional calculus. For example, we must derive a generalization of the Lie algebra for the vector fields that are defined by fractional derivatives. For this generalization, the Jacobi identity cannot be satisfied, and we have a non-Lie algebra. The definition of such 'fractional' Lie algebra is an open question and cannot be realized by a simple way. To formulate the fractional generalization of a Lie algebra for derivatives of noninteger order, we can use the representation of fractional derivatives as infinite series of derivatives of integer orders [1]. For example, the Riemann–Liouville fractional derivative can be represented as

$$D_x^\alpha = \sum_{n=0}^{\infty} A_n(x, x_0, \alpha) \frac{\mathrm{d}^n}{\mathrm{d}x^n}, \tag{77}$$

where

$$A_n(x, a, \alpha) = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} \frac{(x-x_0)^{n-\alpha}}{\Gamma(n+1-\alpha)}. \tag{78}$$

Then the possible realization of the generalization is connected with the special algebraic structures for infinite jets [46]. These structures and approaches can help to solve some problems that are connected with the integrability of fractional nonlinear equations.

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