J. Phys. A: Math. Gen. **39** (2006) 8409–8425 doi:10.1088/0305-4470/39/26/009

# Fractional variations for dynamical systems: Hamilton and Lagrange approaches

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Received 7 February 2006, in final form 16 May 2006
Published 14 June 2006
Online at stacks.iop.org/JPhysA/39/8409

**Abstract**

Fractional generalization of an exterior derivative for calculus of variations is defined. The Hamilton and Lagrange approaches are considered. Fractional Hamilton and Euler–Lagrange equations are derived. Fractional equations are obtained by fractional variation of Lagrangian and Hamiltonian that have only integer derivatives.

PACS numbers: 45.20.−d, 45.20.Jj

## 1. Introduction

The theory of derivatives of non-integer order [1, 2] goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov [2]. Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. In a fairly short period of time the list of such applications becomes long. For example, it includes chaotic dynamics [3, 4], mechanics of fractal media [5–8], quantum mechanics [9, 10], physical kinetics [3, 11–16], plasma physics [17, 18], astrophysics [19], long-range dissipation [20, 21], mechanics of non-Hamiltonian systems [22, 23], theory of long-range interaction [24–26], anomalous diffusion and transport theory [3, 27, 28] and many other physical topics.

In mathematics and theoretical physics, the variational (functional) derivative is a generalization of the usual derivative that arises in the calculus of variations. In a variation instead of differentiating a function with respect to a variable, one differentiates a functional with respect to a function. In this paper, we consider the fractional generalization of variational (functional) exterior derivatives.

The main results are derived in sections 4.2, 5.2, 6.2 and 6.3. In sections 2, 3, 4.1, 5.1 and 6.1, brief reviews of fractional derivatives, differential forms, Hamiltonian systems are considered to fix notation and provide convenient references. In section 2, a brief review of differential forms is considered. In section 3, we consider Hamiltonian and fractional Hamiltonian systems [23]. In section 4, we define the fractional variations in Hamilton's approach to describe the motion. The fractional generalization of stationary action principle is suggested. In section 5, we discuss the fractional variations in Lagrenge's approach to describe the motion, and the fractional generalization of stationary action principle is suggested. In section 6, we consider the generalization of action principle to non-Hamiltonain systems. The fractional equations of motion with friction are discussed. Finally, a short conclusion is given in section 7.

## 2. Fractional derivatives and differential forms

### *2.1. Differential forms*

In this subsection, a brief review of differential forms [29, 30] is considered to fix notation and provide a convenient reference.

**Definition 1.** *A differential 1-form*

$$\omega = F^i(x)\,\mathrm{d}x_i \tag{1}$$

*is called an exact 1-form in $R^n$ if the vector field $F^i(x)$ can be presented as*

$$F^i(x) = -\frac{\partial V}{\partial x_i}, \tag{2}$$

*where $V = V(x)$ is a continuously differentiable function.*

In this case, the differential 1-form (1) is an exact form $\omega = -\mathrm{d}V$, where $V = V(x)$ is a continuously differentiable function (0-form). Here $d$ is the exterior derivative [29]. The exterior derivative of the function $V$ is the 1-form $\mathrm{d}V = \mathrm{d}x_i\,\partial V/\partial x_i$ written in a coordinate chart $(x_1,\ldots,x_n)$. For the $k$-form $\omega_k$ and the $l$-form $\omega_l$, the exterior derivative obeys the relation

$$\mathrm{d}(\omega_k \wedge \omega_l) = (\mathrm{d}\omega_k)\wedge\omega_l + (-1)^k\omega_k \wedge \mathrm{d}\omega_l. \tag{3}$$

Here $k$ and $l$ are integers. Note that $\mathrm{d}\,\mathrm{d}\omega = 0$ for any form $\omega$. If $\mathrm{d}\omega = 0$, then $\omega$ is called a closed form.

In mathematics [29], the concepts of closed and exact forms are defined by the equation $\mathrm{d}\omega = 0$ for a given $\omega$ to be a closed form, and $\omega = \mathrm{d}h$ for an exact form. It is known that to be exact is a sufficient condition to be closed. In abstract terms the question of whether this is also a necessary condition is a way of detecting topological information by differential conditions.

**Proposition 1.** *If a smooth vector field $\mathbf{F} = \mathbf{e}_i F^i(x)$ satisfies the relations*

$$\frac{\partial F^i}{\partial x_j} - \frac{\partial F^j}{\partial x_i} = 0 \tag{4}$$

*on a contractible open subset $X$ of $R^n$, then (1) is the exact form such that*

$$\omega = -\mathrm{d}x_i \frac{\partial V(x)}{\partial x_i}. \tag{5}$$

**Proof.** Let us consider the form (1). The formula for the exterior derivative of (1) is

$$\mathrm{d}\omega = \frac{1}{2}\left(\frac{\partial F^i}{\partial x_j} - \frac{\partial F^j}{\partial x_i}\right)\mathrm{d}x_j \wedge \mathrm{d}x_i,$$

where $\wedge$ is the wedge product [29]. Therefore the condition for $\omega$ to be closed is (4). If $F^i = -\partial V/\partial x_i$, then the implication from èxact' to 'closed' is a consequence of the permutability of the second derivatives. For the smooth function $V = V(x)$, the second derivative commute, and equation (4) holds. □

### 2.2. *Fractional differential forms*

If the partial derivatives in the definition of the exterior derivative

$$\mathrm{d} = \mathrm{d}x_i \frac{\partial}{\partial x_i}$$

are allowed to assume fractional order, then a fractional exterior derivative is defined [31] by

$$d^\alpha = (\mathrm{d}x_i)^\alpha \mathbf{D}^\alpha_{x_i}. \tag{6}$$

Here we use

$$\mathbf{D}^\alpha_x f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{\mathrm{d}y}{(x-y)^{\alpha-m+1}} \frac{\partial^m f(y)}{\partial y^m}, \tag{7}$$

where $\alpha > 0$, and $m$ is the first whole number greater than or equal to $\alpha$. Equation (7) defines the Caputo fractional derivatives [5, 33–35] of order $\alpha > 0$.

**Definition 2.** *A differential 1-form*

$$\omega_\alpha = F^i(x)(\mathrm{d}x_i)^\alpha \tag{8}$$

*is called an exact fractional form if the vector field $F^i(x)$ can be represented as*

$$F^i(x) = -\mathbf{D}^\alpha_{x_i} V, \tag{9}$$

*where $V = V(x)$ is a continuously differentiable function, and $\mathbf{D}^\alpha_{x_i}$ is a derivative of order $\alpha$.*

Using (6) the exact fractional form can be represented as

$$\omega_\alpha = -d^\alpha V = -(\mathrm{d}x_i)^\alpha \mathbf{D}^\alpha_{x_i} V.$$

Therefore, we have (9).

Note that equation (8) is a fractional generalization of the differential form (1). Obviously that fractional 1-form $\omega_\alpha$ can be closed when the differential 1-form $\omega = \omega_1$ is not closed. The fractional analogue of proposition 1 has the form

**Proposition 2.** *If a smooth vector field $\mathbf{F} = \mathbf{e}_i F^i(x)$ on a contractible open subset $X$ of $R^n$ satisfies the relations*

$$\mathbf{D}^\alpha_{x_j} F^i - \mathbf{D}^\alpha_{x_i} F^j = 0, \tag{10}$$

*then the form (8) is an exact fractional 1-form such that*

$$\omega_\alpha = -\mathbf{D}^\alpha_{x_i} V(x), \tag{11}$$

*where $V(x)$ is a continuous differentiable function and $\mathbf{D}^\alpha_{x_i} V = -F^i$.*

**Proof.** This proposition is a corollary of the fractional generalization of the Poincare lemma [32]. The Poincare lemma is shown [31, 32] to be true for the exterior fractional derivative. □

Note that we can generalize the definition of fractional exterior derivative by the equation

$$d^\alpha = \sum_{i=1}^n (\mathrm{d}x_i)^{\alpha_i} D^{\alpha_i}_{x_i}, \tag{12}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and consider the fractional differential 1-forms:

$$\omega_\alpha = \sum_{i=1}^n \omega_i(x)(\mathrm{d}x_i)^{\alpha_i}. \tag{13}$$

In this case, we can derive equations with derivatives of different orders $\alpha_i$. For simplicity, we suppose that all $\alpha_i = \alpha$.

## 3. Hamiltonian systems

In this section, a brief review of Hamiltonian systems [29] and fractional Hamiltonian systems [23] is considered to fix notation and provide a convenient reference.

### *3.1. Definition and properties of Hamiltonian systems*

Let us consider the canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in the phase space $R^{2n}$. We consider a dynamical system that is defined by the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = G^i(q, p), \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = F^i(q, p). \tag{14}$$

The definition of Hamiltonian systems can be realized in the following form [23, 38, 39].

**Definition 3.** *The dynamical system (14) on the phase space $R^{2n}$ is called a Hamiltonian system if*

$$\beta = G^i \,\mathrm{d}p_i - F^i \,\mathrm{d}q_i \tag{15}$$

*is a closed form, $\mathrm{d}\beta = 0$. A dynamical system is called a non-Hamiltonian system if (15) is non-closed, $\mathrm{d}\beta \neq 0$.*

The exterior derivative for the phase space is defined as

$$\mathrm{d} = \mathrm{d}q_i \frac{\partial}{\partial q_i} + \mathrm{d}p_i \frac{\partial}{\partial p_i}. \tag{16}$$

Here and later we mean the sum on the repeated index $i$ from 1 to $n$.

**Proposition 3.** *If the right-hand sides of equations (14) satisfy the conditions*

$$\frac{\partial G^i}{\partial p_j} - \frac{\partial G^j}{\partial p_i} = 0, \qquad \frac{\partial G^j}{\partial q_i} + \frac{\partial F^i}{\partial p_j} = 0, \qquad \frac{\partial F^i}{\partial q_j} - \frac{\partial F^j}{\partial q_i} = 0, \tag{17}$$

*then the dynamical system (14) is a Hamiltonian system.*

**Proof.** In the canonical coordinates $(q, p)$, the vector fields that define the system have the components $(G^i, F^i)$, which are used in equation (14). Let us consider the 1-form (15). The exterior derivative of (15) is written by

$$\mathrm{d}\beta = \mathrm{d}(G^i \,\mathrm{d}p_i) - d(F^i \,\mathrm{d}q_i).$$

Then

$$\mathrm{d}\beta = \frac{\partial G^i}{\partial q_j} \mathrm{d}q_j \wedge \mathrm{d}p_i + \frac{\partial G^i}{\partial p_j} \mathrm{d}p_j \wedge \mathrm{d}p_i - \frac{\partial F^i}{\partial q_j} \mathrm{d}q_j \wedge \mathrm{d}q_i - \frac{\partial F^i}{\partial p_j} \mathrm{d}p_j \wedge \mathrm{d}q_i. \tag{18}$$

Here $\wedge$ is the wedge product. Equation (18) can be presented in an equivalent form

$$\mathrm{d}\beta = \left(\frac{\partial G^j}{\partial q_i} + \frac{\partial F^i}{\partial p_j}\right) \mathrm{d}q_i \wedge \mathrm{d}p_j + \frac{1}{2}\left(\frac{\partial G^j}{\partial p_i} - \frac{\partial G^i}{\partial p_j}\right) \mathrm{d}p_i \wedge \mathrm{d}p_j + \frac{1}{2}\left(\frac{\partial F^i}{\partial q_j} - \frac{\partial F^j}{\partial q_i}\right) \mathrm{d}q_i \wedge \mathrm{d}q_j.$$

Here we use the skew-symmetry of $\mathrm{d}q_i \wedge \mathrm{d}q_j$ and $\mathrm{d}p_i \wedge \mathrm{d}p_j$ with respect index $i$ and $j$. It is obvious that conditions (17) lead to the equation $\mathrm{d}\beta = 0$. Equations (17) are called the Helmholtz conditions [23, 36, 38, 39] for the phase space. □

**Proposition 4.** *The dynamical system (14) on the phase space $R^{2n}$ is a Hamiltonian system that is defined by the Hamiltonian $H = H(q, p)$ if the form (15) is an exact form $\beta = \mathrm{d}H$, where $H = H(q, p)$ is a continuous differentiable unique function on the phase space.*

**Proof.** Suppose that the form (15) is

$$\beta = \mathrm{d}H = \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\,\mathrm{d}q_i .$$

Then the vector fields $(G^i, F^i)$ are

$$G^i(q,p) = \frac{\partial H}{\partial p_i}, \qquad F^i(q,p) = -\frac{\partial H}{\partial q_i}. \tag{19}$$

If $H = H(q,p)$ is a continuous differentiable function, then conditions (17) are satisfied, and (14) is a Hamiltonian system. Substitution of (19) into (14) gives

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \tag{20}$$

As the result, the equations of motion are uniquely defined by the Hamiltonian $H$. □

### *3.2. Fractional Hamiltonian systems*

Fractional generalization of Hamiltonian systems has been suggested in [23]. The fractional analogue of the form (15) can be defined by

$$\beta_\alpha = G^i(\mathrm{d}p_i)^\alpha - F^i(\mathrm{d}q_i)^\alpha . \tag{21}$$

Let us consider the equations of motion

$$A_t q_i = G^i(q,p), \qquad B_t p_i = F^i(q,p), \tag{22}$$

where $A_t$ and $B_t$ are the linear (or nonlinear) operators that have derivatives (integer or fractional order) with respect to time. As the simple examples, we can consider the total time derivatives $A_t = B_t = \mathrm{d}/\mathrm{d}t$ and equation (14). For the fractional derivatives $A_t = B_t = D_t^\alpha$,

$$D_t^\alpha q_i = G^i(q,p), \qquad D_t^\alpha p_i = F^i(q,p). \tag{23}$$

**Definition 4.** *The dynamical system (22) on the phase space $R^{2n}$ is called a fractional Hamiltonian system if (21) is a closed fractional form*

$$d^\alpha \beta_\alpha = 0, \tag{24}$$

*where $d^\alpha$ is the fractional exterior derivative. The system is called a fractional non-Hamiltonian system if (21) is a non-closed fractional form, i.e., $d^\alpha\beta_\alpha \neq 0$.*

The fractional exterior derivative for the phase space $R^{2n}$ is defined as

$$d^\alpha = (\mathrm{d}q_i)^\alpha \mathbf{D}^\alpha_{q_i} + (\mathrm{d}p_i)^\alpha \mathbf{D}^\alpha_{p_i}, \qquad \alpha > 0. \tag{25}$$

Let us consider a fractional generalization of the Helmholtz conditions.

**Proposition 5.** *If the right-hand sides of equations (22) satisfy the conditions*

$$\mathbf{D}^\alpha_{p_j} G^i - \mathbf{D}^\alpha_{p_i} G^j = 0, \qquad \mathbf{D}^\alpha_{q_i} G^j + \mathbf{D}^\alpha_{p_j} F^i = 0, \qquad \mathbf{D}^\alpha_{q_j} F^i - \mathbf{D}^\alpha_{q_i} F^j = 0, \tag{26}$$

*then the dynamical system (22) is a fractional Hamiltonian system.*

**Proof.** This proposition has been proved in [23]. □

**Proposition 6.** *The dynamical system (22) on the phase space $R^{2n}$ is a fractional Hamiltonian system with the Hamiltonian $H = H(q,p)$ if (21) is an exact fractional form*

$$\beta_\alpha = \mathrm{d}^\alpha H, \tag{27}$$

*where $H = H(q,p)$ is a continuous differentiable function on the phase space.*

**Proof.** Suppose that the fractional form (21) is

$$\beta_\alpha = d^\alpha H = (dp_i)^\alpha \mathbf{D}^\alpha_{p_i} H + (dq_i)^\alpha \mathbf{D}^\alpha_{q_i} H.$$

Then

$$G^i(q, p) = \mathbf{D}^\alpha_{p_i} H, \qquad F^i(q, p) = -\mathbf{D}^\alpha_{q_i} H,$$

and equation (22) gives

$$A_t q_i = \mathbf{D}^\alpha_{p_i} H, \qquad B_t p_i = -\mathbf{D}^\alpha_{q_i} H. \tag{28}$$

These equations describe the motion of fractional Hamiltonian systems. □

## 4. Hamilton's approach

### *4.1. Hamilton's equations of integer order*

Let us consider Hamiltonian systems in the extended phase space $M^{2n+1} = R^1 \times R^n \times R^n$ of coordinates $(t, q, p)$. The motion of systems is defined by the stationary states of the action functional

$$S[q, p] = \int [p\dot{q} - H(t, q, p)]\, dt, \tag{29}$$

where $H$ is a Hamiltonian of the system, $\dot{q} = dq/dt$ and both $q$ and $p$ are assumed to be independent functions of time. In classical mechanics, the trajectory of an object is derived by finding the path for which the action integral (29) is stationary (a minimum or a saddle point).

In Hamilton's approach the action functional (29) can be written as

$$S[q, p] = \int \omega_h, \tag{30}$$

where

$$\omega_h = p\, dq - H\, dt. \tag{31}$$

The form (31) is called the *Poincare–Cartan 1-form* or the *action 1-form*.

The Poincare–Cartan form looks like the integrand of the action or the Lagrangian. However, it is a differential form on the extended phase space $M^{2n+1}$ of $(t, q, p)$, not a function. Once we integrate it over a curve $C$ in $M^{2n+1}$, we get the action

$$S[q, p] = \int_C \omega_h = \int_A^B [p\, dq - H(t, q, p)\, dt]. \tag{32}$$

The integration is taken from $A$ to $B$ in the extended phase space $M^{2n+1}$.

Now suppose we integrate from $A$ to $B$ along two slightly different paths and take the difference to get a close loop integral. To evaluate this integral we can use Stokes's theorem [29]. In the language of differential forms, Stokes's theorem is written as

$$\int_{\partial M} \omega = \int_M d\omega. \tag{33}$$

Here, $M$ is an $n$-dimensional compact orientable manifold with boundary $\partial M$ and $\omega$ is a $(n-1)$-form; 'd' is its exterior derivative. Note that $M$ can be a submanifold of a larger space, so that Stokes's theorem actually implies a whole set of relations including the familiar Gauss and Stokes laws of ordinary vector calculus.

Applying equation (33) to the difference of actions computed along two neighbouring paths with $(q, t)$ fixed at the endpoints, we get

$$\delta S[q, p] = \int_\sigma d\omega_h = \int_\sigma (dp \wedge dq - dH \wedge dt), \tag{34}$$

where $\sigma$ denotes the surface area in the extended phase space bounded by the two paths from $A$ to $B$.

The principal of stationary action states that $\delta S = 0$ for small variations about the true path, with $(q, t)$ fixed at the end points. This will be true for arbitrary small variations, if and only if $\mathrm{d}\omega_h = 0$ for the tangent vector along the extremal path.

We can consider the exterior derivative of Poincare–Cartan 1-form, and derive the equations of motion from the condition $\mathrm{d}\omega_h = 0$. Using this condition, we get Hamilton's equations of motion. This condition is equivalent to the stationary action principle $\delta S[q, p] = 0$.

**Proposition 7.** *The exterior derivative of the Poincare–Cartan 1-form (31) is defined by the equation*

$$\mathrm{d}\omega_h = [D_t p + D_q H]\mathrm{d}t \wedge \mathrm{d}q - [\mathrm{d}q - D_p H\,\mathrm{d}t] \wedge \mathrm{d}p, \tag{35}$$

*where*

$$D_t = \frac{\partial}{\partial t}, \qquad D_q = \frac{\partial}{\partial q}, \qquad D_p = \frac{\partial}{\partial p}. \tag{36}$$

**Proof.** The exterior derivative of the form (31) can be calculated from the equation

$$\mathrm{d}\omega_h = \mathrm{d}(p\,\mathrm{d}q) - \mathrm{d}(H\,\mathrm{d}t) = D_t p\,\mathrm{d}t \wedge \mathrm{d}q + D_q p\,\mathrm{d}q \wedge \mathrm{d}q + D_p p\,\mathrm{d}p \wedge \mathrm{d}q - D_t H\,\mathrm{d}t \wedge \mathrm{d}t - D_q H\,\mathrm{d}q \wedge \mathrm{d}t - D_p H\,\mathrm{d}p \wedge \mathrm{d}t. \tag{37}$$

Using $\mathrm{d}t \wedge \mathrm{d}t = 0$, $\mathrm{d}p \wedge \mathrm{d}t = -\mathrm{d}t \wedge \mathrm{d}p$, and $D_q p = 0$, we get

$$\mathrm{d}\omega_h = [D_t p + D_q H]\,\mathrm{d}t \wedge \mathrm{d}q - [D_p p\,\mathrm{d}q - D_p H\,\mathrm{d}t] \wedge \mathrm{d}p. \tag{38}$$

The relation $D_p p = 1$ gives

$$\mathrm{d}\omega_h = [D_t p + D_q H]\,\mathrm{d}t \wedge \mathrm{d}q - [\mathrm{d}q - D_p H\,\mathrm{d}t] \wedge \mathrm{d}p. \tag{39}$$

□

**Stationary action principle in Hamilton's approach.** *The trajectory of a Hamiltonian system can be derived by finding the path for which the Poincare–Cartan 1-form $\omega_h$ is closed, i.e.,*

$$\mathrm{d}\omega_h = 0. \tag{40}$$

Using the stationary action principle (40), we get the equations of motion

$$\mathrm{d}q - D_p H\,\mathrm{d}t = 0, \qquad D_t p = -D_q H. \tag{41}$$

As the result, we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} = D_p H, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -D_q H, \tag{42}$$

which are the well-known Hamilton's equations.

### *4.2. Fractional Hamilton's equations*

The fractional generalization of the form (31) can be defined by

$$\omega_{h,\alpha} = p(\mathrm{d}q)^\alpha - H(\mathrm{d}t)^\alpha. \tag{43}$$

Note that $\omega_{h,\alpha}$ is a fractional 1-form that can be called a *fractional Poincare–Cartan 1-form* or *fractional action 1-form*.

We can consider the fractional exterior derivative of the form (43), and use $d^\alpha \omega_{h,\alpha} = 0$ to obtain the fractional equations of motion.

**Proposition 8.** *The fractional exterior derivative of the fractional form (43) is*

$$d^\alpha \omega_{h,\alpha} = \left[D_t^\alpha p + D_q^\alpha H\right](\mathrm{d}t)^\alpha \wedge (\mathrm{d}q)^\alpha - \left[\frac{p^{1-\alpha}}{\Gamma(2-\alpha)}(\mathrm{d}q)^\alpha - D_p^\alpha H(\mathrm{d}t)^\alpha\right] \wedge (\mathrm{d}p)^\alpha. \qquad (44)$$

**Proof.** The fractional exterior derivative of the form (43) is calculated by using the rule

$$\mathbf{D}_x^\alpha(fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\mathbf{D}_x^{\alpha-k} f\right) \frac{\partial^k g}{\partial x^k},$$

and the relation

$$\frac{\partial^k}{\partial x^k}[(\mathrm{d}x)^\alpha] = 0 \quad (k \geqslant 1).$$

For example, we have

$$d^\alpha[A^i(\mathrm{d}x_i)^\alpha] = \sum_{k=0}^{\infty} (\mathrm{d}x_j)^\alpha \wedge \binom{\alpha}{k}\left(\mathbf{D}_{x_j}^{\alpha-k} A^i\right)\frac{\partial^k}{\partial x_j^k}(\mathrm{d}x_i)^\alpha$$
$$= (\mathrm{d}x_j)^\alpha \wedge (\mathrm{d}x_i)^\alpha \binom{\alpha}{0}\left(\mathbf{D}_{x_j}^\alpha A^i\right) = \left(\mathbf{D}_{x_j}^\alpha A^i\right)(\mathrm{d}x_j)^\alpha \wedge (\mathrm{d}x_i)^\alpha,$$

where

$$\binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}.$$

As the result,

$$d^\alpha\omega_{h,\alpha} = d^\alpha(p(\mathrm{d}q)^\alpha) - d^\alpha(H(\mathrm{d}t)^\alpha) = \left(D_t^\alpha p\right)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}q)^\alpha + \left(D_q^\alpha p\right)(\mathrm{d}q)^\alpha \wedge (\mathrm{d}q)^\alpha$$
$$+ \left(D_p^\alpha p\right)(\mathrm{d}p)^\alpha \wedge (\mathrm{d}q)^\alpha - \left(D_t^\alpha H\right)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}t)^\alpha$$
$$- \left(D_q^\alpha H\right)(\mathrm{d}q)^\alpha \wedge (\mathrm{d}t)^\alpha - \left(D_p^\alpha H\right)(\mathrm{d}p)^\alpha \wedge (\mathrm{d}t)^\alpha. \qquad (45)$$

Here $D^\alpha$ are the Riesz or Caputo fractional derivatives [5, 33–35]. Note that the Riemann–Liouville fractional derivative leads us to dependence of independent coordinates [22, 23]:

$$D_q^\alpha p = pD_q^\alpha 1 \neq 0. \qquad (46)$$

Therefore the fractional equations are more complicated for Riemann–Liouville derivatives. Using $(\mathrm{d}t)^\alpha \wedge (\mathrm{d}t)^\alpha = 0$, $(\mathrm{d}p)^\alpha \wedge (\mathrm{d}t)^\alpha = -(\mathrm{d}t)^\alpha \wedge (\mathrm{d}p)^\alpha$ and $D_q^\alpha p = 0$ for Riesz and Caputo derivativs, we can rewrite equation (45) in the form

$$d^\alpha\omega_{h,\alpha} = \left[D_t^\alpha p + D_q^\alpha H\right](\mathrm{d}t)^\alpha \wedge (\mathrm{d}q)^\alpha - \left[\left(D_p^\alpha p\right)(\mathrm{d}q)^\alpha - \left(D_p^\alpha H\right)(\mathrm{d}t)^\alpha\right] \wedge (\mathrm{d}p)^\alpha. \qquad (47)$$

Substitution of

$$D_p^\alpha p = \frac{p^{1-\alpha}}{\Gamma(2-\alpha)} \qquad (48)$$

into equation (47) gives (44). □

**Fractional action principle in Hamilton's approach.** *The trajectory of a dynamical system can be derived by finding the path for which the fractional Poincare–Cartan 1-form $\omega_{h,\alpha}$ is a fractional closed form, i.e.,*

$$d^\alpha\omega_{h,\alpha} = 0. \qquad (49)$$

Here, we consider only fractional Hamiltonian systems. The non-Hamiltonian systems are considered in section 6.

Using (44) and (49), we get

$$\frac{p^{1-\alpha}}{\Gamma(2-\alpha)}(\mathrm{d}q)^{\alpha} - D_p^{\alpha} H (\mathrm{d}t)^{\alpha} = 0, \qquad D_t^{\alpha} p = -D_q^{\alpha} H. \tag{50}$$

As the result, we obtain

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^{\alpha} = \Gamma(2-\alpha) p^{\alpha-1} D_p^{\alpha} H, \qquad D_t^{\alpha} p = -D_q^{\alpha} H. \tag{51}$$

These equations are the fractional generalization of Hamilton's equations.

For the fractional Poincare–Cartan 1-form

$$\omega_{h,\alpha} = p^{\beta}(\mathrm{d}q)^{\alpha} - H(\mathrm{d}t)^{\alpha}, \tag{52}$$

equations (51) are

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^{\alpha} = \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)} p^{\alpha-\beta} D_p^{\alpha} H, \qquad D_t^{\alpha} p^{\beta} = -D_q^{\alpha} H. \tag{53}$$

Note that we cannot use the rule of differentiating composite functions for fractional derivative $D_t^{\alpha} p^{\beta}$. Therefore equations (53) with $\beta \neq 1$ are more complicated than equation (51).

## 5. Lagrange's approach

### *5.1. Lagrange's equations of integer order*

Suppose $L(t, q, v)$ is a Lagrangian of dynamical system, where $q$ is coordinate and $v$ is the velocity. Let us consider the variational problem on the extremum of the action functional

$$S_0[q, v] = \int L(t, q, v)\,\mathrm{d}t, \tag{54}$$

under the additional condition

$$\dot{q} = v, \tag{55}$$

where both $q$ and $v$ are assumed to be independent functions of time. In this case, $p$ play the role of independent Lagrange multipliers. Obviously, the indicated variational problem is equivalent to the problem on the extremum of the action,

$$S[q, v, p] = \int [L(t, q, v) + p(\dot{q} - v)]\,\mathrm{d}t, \tag{56}$$

where already all the variables $q, v, p$ have to be varied. The corresponding Lagrange's equations are

$$\dot{q} = v, \qquad \dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}. \tag{57}$$

We can introduce the extended Hamiltonian in the space of variables $(t, q, p, v)$ as

$$H_*(t, q, p, v) = pv - L(t, q, v). \tag{58}$$

The corresponding extended Poincare–Cartan 1-form is defined by

$$\omega_{h*} = p\,\mathrm{d}q - H^*\,\mathrm{d}t = p\,\mathrm{d}q + L\,\mathrm{d}t - pv\,\mathrm{d}t. \tag{59}$$

**Proposition 9.** *The exterior derivative of the form (59) is defined by the equation*

$$\mathrm{d}\omega_{h*} = [D_t p - D_q L]\,\mathrm{d}t \wedge \mathrm{d}q - [\mathrm{d}q - v\,\mathrm{d}t] \wedge \mathrm{d}p - [p - D_v L]\,\mathrm{d}v \wedge \mathrm{d}t. \tag{60}$$

**Proof.** The exterior derivative of (59) is

$$\mathrm{d}\omega_{h*} = \mathrm{d}(pdq) - d(H^*\,\mathrm{d}t) = D_t p\,\mathrm{d}t \wedge dq + D_q p\,\mathrm{d}q \wedge dq + D_p p\,\mathrm{d}p \wedge \mathrm{d}q + D_v p\,\mathrm{d}v \wedge \mathrm{d}q \\ - D_t H^*\,\mathrm{d}t \wedge \mathrm{d}t - D_q H^*\,\mathrm{d}q \wedge \mathrm{d}t - D_p H^*\,\mathrm{d}p \wedge \mathrm{d}t - D_v H^*\,\mathrm{d}v \wedge \mathrm{d}t. \tag{61}$$

Using $\mathrm{d}t \wedge \mathrm{d}t = 0$, $dq \wedge \mathrm{d}t = -\mathrm{d}t \wedge dq$ and $D_q p = 0$, $D_v p = 0$, equation (61) gives

$$\mathrm{d}\omega_{h*} = [D_t p + D_q H^*]\,\mathrm{d}t \wedge \mathrm{d}q - [D_p p\,\mathrm{d}q - D_p H^*\,\mathrm{d}t] \wedge \mathrm{d}p - D_v H^*\,\mathrm{d}v \wedge \mathrm{d}t. \tag{62}$$

From (58), we get

$$D_q H^* = D_q[pv - L] = -D_q L,$$
$$D_p H^* = D_p[pv - L] = v D_p p - D_p L(t, q, v) = v D_p p = v,$$
$$D_v H^* = D_v[pv - L] = p D_v v - D_v L = p - D_v L.$$

As the result, we obtain (60). □

**Stationary action principle in Lagrange's approach.** *The trajectory of a dynamical system can be derived by finding the path for which the form (59) is closed , i.e.,*

$$\mathrm{d}\omega_{h*} = 0. \tag{63}$$

From equations (60) and (63), we get

$$D_t p - D_q L = 0, \qquad \mathrm{d}q - v\,\mathrm{d}t = 0, \qquad p - D_v L = 0. \tag{64}$$

It is easy to see that equation (64) coincides with Lagrange's equations (57) that can be presented as

$$D_q L - \left[\frac{\mathrm{d}}{\mathrm{d}t} D_v L\right]_{v=\dot{q}} = 0. \tag{65}$$

As the result, we obtain

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = 1, \ldots, n, \tag{66}$$

which are the Euler–Lagrange equations for the Lagrangian system.

### *5.2. Fractional Lagrange's equations*

Suppose $L(t, q, v)$ is a Lagrangian of the system, and the extended Hamiltonian is defined by

$$H_*(t, q, p, v) = pv^\beta - L(t, q, v), \tag{67}$$

where $\beta$ is a positive power. Let us define

$$\omega_{h*\alpha} = p(\mathrm{d}q)^\alpha - H^*(\mathrm{d}t)^\alpha = p(\mathrm{d}q)^\alpha + L(\mathrm{d}t)^\alpha - pv^\beta(\mathrm{d}t)^\alpha, \tag{68}$$

which is a fractional generalization of the extended Poincare–Cartan 1-form (59).

**Proposition 10.** *The fractional exterior derivative of the fractional 1-form (68) is defined by*

$$d^\alpha \omega_{h*\alpha} = \left[D_t^\alpha p - D_q^\alpha L\right](\mathrm{d}t)^\alpha \wedge (\mathrm{d}q)^\alpha - \left(D_p^\alpha p\right)[(\mathrm{d}q)^\alpha - v^\beta(\mathrm{d}t)^\alpha] \wedge (\mathrm{d}p)^\alpha \\ - \left[p D_v^\alpha v^\beta - D_v^\alpha L\right](dv)^\alpha \wedge (\mathrm{d}t)^\alpha. \tag{69}$$

**Proof.** The fractional exterior derivative of the form (68) gives

$$d^\alpha \omega_{h*\alpha} = d^\alpha[p d^\alpha q] - d[H^*(\mathrm{d}t)^\alpha] = \left(D_t^\alpha p\right)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}q)^\alpha + \left(D_q^\alpha p\right)(\mathrm{d}q)^\alpha \wedge (\mathrm{d}q)^\alpha \\ + \left(D_p^\alpha p\right)(\mathrm{d}p)^\alpha \wedge (\mathrm{d}q)^\alpha + \left(D_v^\alpha p\right)(\mathrm{d}v)^\alpha \wedge (\mathrm{d}q)^\alpha - \left(D_t^\alpha H^*\right)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}t)^\alpha \\ - \left(D_q^\alpha H^*\right)(\mathrm{d}q)^\alpha \wedge (\mathrm{d}t)^\alpha - \left(D_p^\alpha H^*\right)(\mathrm{d}p)^\alpha \wedge (\mathrm{d}t)^\alpha \\ - \left(D_v^\alpha H^*\right)(\mathrm{d}v)^\alpha \wedge (\mathrm{d}t)^\alpha. \tag{70}$$

Using $(dt)^\alpha \wedge (dt)^\alpha = 0$, $(dq)^\alpha \wedge (dt)^\alpha = -(dt)^\alpha \wedge (dq)^\alpha$ and $D_q p = 0$, $D_v p = 0$ for Riesz or Caputo fractional derivatives, we can rewrite (68) as

$$d^\alpha \omega_{h*\alpha} = \left[D_t^\alpha p + D_q^\alpha H^*\right](dt)^\alpha \wedge (dq)^\alpha - \left[\left(D_p^\alpha p\right)(dq)^\alpha - \left(D_p^\alpha H^*\right)(dt)^\alpha\right] \wedge (dp)^\alpha - \left(D_v^\alpha H^*\right)(dv)^\alpha \wedge (dt)^\alpha. \qquad (71)$$

Using the extended Hamiltonian (67) and properties of Riesz and Caputo derivatives, we have

$$D_q^\alpha H^* = D_q^\alpha[pv^\beta - L] = -D_q^\alpha L,$$
$$D_p^\alpha H^* = D_p^\alpha[pv^\beta - L] = v^\beta D_p^\alpha p - D_p^\alpha L(t, q, v) = v^\beta D_p^\alpha p,$$
$$D_v^\alpha H^* = D_v^\alpha[pv^\beta - L] = pD_v^\alpha v^\beta - D_v^\alpha L.$$

As the result, we obtain (69). □

**Fractional action principle in Lagrange's approach.** *The trajectory of fractional dynamical systems can be derived by finding the path for which the fractional extended Poincare–Cartan 1-form* (68) *is a closed form, i.e.,*

$$d^\alpha \omega_{h*\alpha} = 0. \qquad (72)$$

Using (69) and (72), we have

$$\left(D_t^\alpha p\right) - D_q^\alpha L = 0, \qquad (dq)^\alpha - v^\beta (dt)^\alpha = 0, \qquad pD_v^\alpha v^\beta - D_v^\alpha L = 0. \qquad (73)$$

From the relation

$$D_v^\alpha v^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} v^{\beta-\alpha}, \qquad (74)$$

where $\beta > -1$, we get

$$D_t^\alpha p = D_q^\alpha L, \qquad v^\beta = \frac{(dq)^\alpha}{(dt)^\alpha}, \qquad p = \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)} v^{\alpha-\beta} D_v^\alpha L. \qquad (75)$$

Substitution of the third equation from (75) into the first one gives

$$D_q^\alpha L - \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)} D_t^\alpha \left[v^{\alpha-\beta} D_v^\alpha L\right]_{v^\beta=(\dot q)^\alpha} = 0. \qquad (76)$$

Equation (76) has the dependence

$$v^\beta = (\dot q)^\alpha. \qquad (77)$$

It is easy to see that equation (76) looks unusually even for $\beta = 1$. Therefore we use $\beta = \alpha$ for the Hamiltonian (67) and the form (68).

Using $\beta = \alpha$ in equations (67), (68) and

$$D_v^\alpha v^\alpha = \frac{1}{\Gamma(2-\alpha)}, \qquad (78)$$

we have the fractional extended Lagrange's equations

$$(D_t^\alpha p) = D_q^\alpha L, \qquad v^\alpha = \frac{(dq)^\alpha}{(dt)^\alpha}, \qquad p = \Gamma(2-\alpha) D_v^\alpha L. \qquad (79)$$

Substituting the third equation from (80) into the first one, we obtain

$$D_q^\alpha L - \Gamma(2-\alpha) D_t^\alpha \left[D_v^\alpha L\right]_{v=\dot q} = 0, \qquad (80)$$

that is *fractional Euler–Lagrange equations*. As the result, the fractional equations of motion for Lagrangian systems are presented by

$$\hat{E}_i^\alpha L(t, q, \dot q) = 0, \qquad i = 1, \ldots, n, \qquad (81)$$

where

$$\hat{E}_i^\alpha = D_{q_i}^\alpha - \Gamma(2-\alpha) D_t^\alpha D_{\dot{q}_i}^\alpha. \quad (82)$$

For $\alpha = 1$, equations (81) are the Euler–Lagrange equations (66).

## 6. Fractional non-Hamiltonian systems

### *6.1. Helmholtz conditions*

In this subsection, we consider the brief review of Helmholtz conditions [36, 37] to fix notation and provide a convenient reference. The Helmholtz equations give the necessary and sufficient conditions for equations to be the Euler–Lagrange equations that can be derived from the stationary action principle.

**Proposition 11.** *The necessary and sufficient conditions for*

$$F_i(t, q, \dot{q}, \dots, q^{(N)}) = 0, \qquad i = 1, \dots, n \quad (83)$$

*to be equations that can be derived from the stationary action principle are*

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} - \sum_{k=1}^{N} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left( \frac{\partial F_j}{\partial q_i^{(k)}} \right) = 0, \quad (84)$$

$$\frac{\partial F_i}{\partial q_j^{(m)}} - \sum_{k=m}^{N} (-1)^k \binom{k}{m} \frac{\mathrm{d}^{k-m}}{\mathrm{d}t^{k-m}} \left( \frac{\partial F_j}{\partial q_i^{(k)}} \right) = 0, \qquad m = 1, \dots, N \quad (85)$$

*where* $q_i^{(k)} = \mathrm{d}^k q_i / \mathrm{d}t^k$, $i, j = 1, \dots, n$ *and*

$$\binom{k}{m} = \frac{k!}{m!(k-m)!}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{k=1}^{N} q_i^{(k)} \frac{\partial^k}{\partial q_i^{(k-1)}}. \quad (86)$$

**Proof.** This proposition is proved in [37].

For simple example, let us consider the equations

$$F_i(t, q, \dot{q}) = 0. \quad (87)$$

Conditions (84) and (85) have the form

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial F_j}{\partial \dot{q}_i} \right) = 0, \quad (88)$$

$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} = 0, \qquad i, j = 1, \dots, n, \quad (89)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i}. \quad (90)$$

Equations (89) gives

$$\frac{\partial^2 F_i}{\partial \dot{q}_j \partial \dot{q}_\kappa} = 0, \qquad i, j, \kappa = 1, \dots, n. \quad (91)$$

These conditions are satisfied for the linear dependence $F_i$ with respect to $\dot{q}$, i.e.,

$$F_i = C_{ij}(t, q)\dot{q}_j + D_i(t, q) = 0, \qquad i = 1, \dots, n. \quad (92)$$

□

**Corollary 1.** *The necessary and sufficient conditions to derive equations (92) from the stationary action principle have the form*

$$C_{ij} = -C_{ji}, \tag{93}$$

$$\frac{\partial C_{ij}}{\partial q_\kappa} + \frac{\partial C_{j\kappa}}{\partial q_i} + \frac{\partial C_{\kappa i}}{\partial q_j} = 0, \tag{94}$$

$$\frac{\partial C_{ij}}{\partial t} - \frac{\partial D_i}{\partial q_j} + \frac{\partial D_j}{\partial q_i} = 0. \tag{95}$$

**Proof.** Substitution of equation (92) into equations (88) and (89) yields (93), (94) and (95). □

**Corollary 2.** *The necessary and sufficient conditions for*

$$F_i(t, q, \dot{q}, \ddot{q}) = 0, \qquad i = 1, \ldots, n \tag{96}$$

*to be equations that can be derived from stationary action principle are*

$$\frac{\partial F_i}{\partial \ddot{q}_j} - \frac{\partial F_j}{\partial \ddot{q}_i} = 0, \tag{97}$$

$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} - 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_j}{\partial \ddot{q}_i}\right) = 0, \tag{98}$$

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_j}{\partial \dot{q}_i}\right) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial F_j}{\partial \ddot{q}_i}\right) = 0, \tag{99}$$

*where*

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i} + \ddot{q}_i\frac{\partial}{\partial \dot{q}_i}. \tag{100}$$

Note that using equation (98) condition (99) can be rewritten in the more symmetric form

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_i}{\partial \dot{q}_j} - \frac{\partial F_j}{\partial \dot{q}_i}\right) = 0. \tag{101}$$

### *6.2. Non-Lagrangian systems*

**Definition 5.** *A dynamical system is called non-Lagrangian system if the equations of motion (83) cannot be represented in the form*

$$\sum_{k=0}^{N}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\partial}{\partial q^{(k)}}L(t, q, \dot{q}, \ldots, q^{(N)}) = 0, \tag{102}$$

*with some function* $L = L(t, q, \dot{q}, \ldots, q^{(N)})$, *where* $q^{(k)} = \mathrm{d}^k q/\mathrm{d}t^k$.

It is well known that the equations of second order cannot be presented as

$$\hat{E}_i L(t, q, \dot{q}) = 0, \qquad i = 1, \ldots, n, \tag{103}$$

where $\hat{E}_i$ is the Euler–Lagrange operator

$$\hat{E}_i = D_{q_i} - D_t D_{\dot{q}_i}. \tag{104}$$

In the general case, Lagrange's equations have the additional term $Q_i(t, q, \dot{q})$ which is a generalized non-potential force. This force cannot be presented as $Q_i = \hat{E}_i U$ for some function $U = U(t, q, \dot{q})$. In general, the Euler–Lagrange equations [44] are

$$\hat{E}_i L(t, q, \dot{q}) + Q_i = 0. \tag{105}$$

If we consider non-potential forces and non-Lagrangian systems, then the non-holonomic variational equation suggested by L I Sedov [40–43] should be used instead of the stationary action principle.

### *6.3. Non-Hamiltonian systems and friction force*

In general, the phase-space equations of motion cannot be presented in the form

$$\dot{q}_i = D_{p_i} H, \qquad \dot{p}_i = -D_{q_i} H, \tag{106}$$

where $H = H(t, q, p)$ is a smooth function. Hamilton's equations are written as

$$\dot{q}_i = D_{p_i} H + G^i(t, q, p), \qquad \dot{p}_i = -D_{q_i} H + F^i(t, q, p), \tag{107}$$

where $H = H(t, q, p)$ is a Hamiltonian of the system. For example, $H(t, q, p) = T(p) + U(q)$, where $T(p)$ is a kinetic energy and $U(q)$ is a potential energy of the system. The functions $G^i(t, q, p)$ and $F^i(t, q, p)$ describe the non-potential forces which act on the system. For mechanical systems, we can consider $G^i(t, q, p) = 0$. If the functions $G^i(t, q, p)$ and $F^i(t, q, p)$ do not satisfy the Helmholtz conditions (17), then (107) is a non-Hamiltonian system.

In general, the exterior derivative of the Poincare–Cartan 1-form is not equal to zero ($\mathrm{d}\omega_h \neq 0$). This derivative is equal to differential 2-form $\theta$ that is defined by non-potential forces

$$\theta = F^i(t, q, p)\,\mathrm{d}t \wedge \mathrm{d}q_i - G^i(t, q, p)\,\mathrm{d}t \wedge \mathrm{d}p_i \tag{108}$$

for the non-Hamiltonian system (107). For example, the linear friction force $F^i = -\gamma p_i$ gives

$$\theta = -\gamma p_i \mathrm{d}t \wedge \mathrm{d}q_i. \tag{109}$$

**Proposition 12.** *The differential 2-form $\theta$ of non-potential forces is a non-closed form.*

**Proof.** If differential 2-form $\theta$ is a closed form ($\mathrm{d}\theta = 0$) on a contractible open subset X of $R^{2n}$, then the form is the exact form such that a function $h = h(t, q, p)$ exists, and $\theta = \mathrm{d}h$. In this case, we have a new Poincare–Cartan 1-form

$$\omega_h' = \omega_h + h,$$

such that $\mathrm{d}\omega' = 0$, and the system is Hamiltonian.

There is a generalization of the stationary action principle for the systems with non-potential forces. □

**Action principle for non-Hamiltonian systems.** *The trajectory of a non-Hamiltonian system can be derived by finding the path for which the exterior derivative of the action 1-form* (31) *is equal to the non-closed 2-form* (108)*, i.e.,*

$$\mathrm{d}\omega_h = \theta. \tag{110}$$

Equations (35), (108) and (110) give the equations of motion (107) for the non-Hamiltonian system.

### *6.4. Fractional generalization of non-Hamiltonian systems*

Let us define a fractional generalization of the form (108) by

$$\theta_\alpha = F^i(t, q, p)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}q_i)^\alpha - G^i(t, q, p)(\mathrm{d}t)^\alpha \wedge (\mathrm{d}p_i)^\alpha. \tag{111}$$

This form allows us to derive fractional equations of motion for non-Hamiltonian systems.

**Fractional action principle for non-Hamiltonian systems.** *The trajectory of a fractional system subjected by non-potential forces can be derived by finding the path for which the fractional exterior derivative of the fractional action 1-form* (43) *is equal to the non-closed fractional 2-form* (111)*, i.e.*

$$d^\alpha \omega_{h\alpha} = \theta_\alpha. \tag{112}$$

Using (44), (111) and (112), we get

$$\frac{p^{1-\alpha}}{\Gamma(2-\alpha)}(\mathrm{d}q)^\alpha - D^\alpha_p H(\mathrm{d}t)^\alpha = G(t, q, p)(\mathrm{d}t)^\alpha, \qquad D^\alpha_t p = -D^\alpha_q H + F(t, q, p). \tag{113}$$

As the result, we obtain

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^\alpha = \Gamma(2-\alpha)p^{\alpha-1}D^\alpha_p H + G(t, q, p), \qquad D^\alpha_t p = -D^\alpha_q H + F(t, q, p). \tag{114}$$

These equations are the fractional generalization of equations of motion for non-Hamiltonian systems.

## 7. Conclusion

In this paper, we define a fractional exterior derivative for calculus of variations. Hamiltonian and Lagrangian approaches are considered. Hamilton's and Lagrange's equations with fractional derivatives are derived from the stationary action principles. We prove that fractional equations can be derived from the action which has only integer derivatives. Derivatives of non-integer order appear by the fractional variation of Lagrangian and Hamiltonian.

Application of fractional variational calculus can be connected with a generalization of variational problems. The gradient systems (GS) form a restricted class of ordinary differential equations. Equations for GS can be defined by one function—potential. Therefore the study of GS can be reduced to research of potential. As a physical example, the ways of some chemical reactions are defined from the analysis of potential energy surfaces [45, 46]. The fractional gradient systems (FGS) have been suggested in [23]. It was proved that GS are a special case of such systems. FGS include a wide class of non-gradient systems. For example, the Lorenz and Rossler equations are fractional gradient systems [23]. Therefore the study of non-gradient systems which are FGS can be reduced to research of potential.

We can assume that the ways of some chemical reactions with dissipation and systems with deterministic chaos can be considered by the analysis of fractional potential surfaces. Let us note the interesting property of potential surfaces for systems with strange attractors. The surfaces of the stationary states of the Lorenz and Rossler equations separate the three-dimensional Euclidean space into some number of areas [23]. We have eight areas for the Lorenz equations and four areas for the Rossler equations. This separation has the interesting property: all regions are connected with each other [23]. Beginning movement from one of the areas, it is possible to appear in any other area, not crossing a surface. Any two points from different areas can be connected by a curve which does not cross a surface.

The fractional variations can be used to define the fractional generalization of gradient type equations that have wide application for the description dissipative structures [47, 48]. The fractional gradient type equations are generalization of FGS [23] from ordinary differential equations into partial differential equations. We plan to realize this generalization in the next paper by using the de Donder–Weyl Hamiltonian and the Poincare–Cartan $n$-form.

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