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Nonholonomic constraints with fractional derivatives

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Abstract
We consider the fractional generalization of nonholonomic constraints defined by equations with fractional derivatives and provide some examples. The corresponding equations of motion are derived using variational principle. We prove that fractional constraints can be used to describe the evolution of dynamical systems in which some coordinates and velocities are related to velocities through a power-law memory function.

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1. Introduction

The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov [1–3]. Fractional analysis proved to be useful in mechanics and physics. In a fairly short time the list of such applications continuously has grown. The applications include chaotic dynamics [4, 5], material sciences [6–9], mechanics of fractal and complex media [10–12], quantum mechanics [13, 14], physical kinetics [4, 15–17], plasma physics [18, 19], electromagnetic theory [19–21], long-range dissipation [22–24], non-Hamiltonian mechanics [25, 26], long-range interaction [27–29], anomalous diffusion and transport theory [4, 30–32].

Equations with fractional derivatives usually appear from some phenomenological models. In this paper, fractional equations are used to describe a motion that is restricted by constraints with power-law memory. The evolution of dynamical system in which some coordinates and velocities are related to other coordinates or velocities through a memory function can be considered. To describe constraints with power-like memory function, we use the fractional calculus. It allows us to take into account the memory effects and derive fractional equations of motions. We consider the fractional generalization of nonholonomic constraints such that the constraint equations consist of fractional derivatives, called fractional constraints. The corresponding equations of motion will be derived by the d'Alembert–Lagrange principle and some simple examples are considered.

It has been known for a long time that some objects related to chemistry or to the soft condensed matter cannot be characterized by an integer dimension. The most typical of them are colloidal aggregates [33] and so-called chemical surfaces [34]. It became more clear later that a useful tool describes the fractal features of the media and its dynamics by rewriting the corresponding equations using fractional derivatives. Particularly, the appearance of fractional derivative with respect to the coordinate follows immediately from a dispersion law $\omega = \omega(k)$ if it consists of fractional powers of $|k|$ (see for example applications to the modified Ginzburg–Landau equation [12, 35]). The appearance of fractional derivatives changes formally and physically different properties of the dynamics, kinetics, equilibrium states and others. Particularly with respect to the material of our paper, it will be shown in section 3.2 that fractional derivatives in the constraints are due to a distributed memory what is typical for the media or a set of particles (spins) with long-range interaction. A number of appearances of fractional derivatives in the reaction–diffusion-type systems can be found in [36].

In section 2, we provide a brief review of nonholonomic systems, fix notations and convenient references. In section 3, we consider fractional generalizations of nonholonomic constraints. In subsection 3.2, we prove that the fractional constraints can be used to describe the evolution of a dynamical system in which some coordinates and velocities are related to other coordinates or velocities through a power-law memory function. Some examples are considered. In section 4, we discuss the applicability of the stationary action principle for fractional constraints. In section 5, geometric methods for fractional nonholonomic mechanics are suggested. Finally, a short conclusion is given in section 6.

2. Nonholonomic constraints with integer derivatives

In this section, a brief review of nonholonomic systems is considered to fix notations and provide convenient references [39].

2.1. Lagrange equations for nonholonomic system

It is known that the d'Alembert–Lagrange principle allows us to derive equations of motion with holonomic and nonholonomic constraints. For N-particle system it has the form of the variation equation

$$\left(\frac{\mathrm{d}(m\mathbf{v}_i)}{\mathrm{d}t} - \mathbf{F}_i\right)\delta\mathbf{r}_i = 0, \tag{1}$$

where $\mathbf{r}_i$ $(i = 1, \ldots, N)$ is the radius vector of the ith particle, $\mathbf{v}_i = \dot{\mathbf{r}}_i$ is the velocity and $\mathbf{F}_i$ is the force that acts on the ith particle, and the sum over repeated index i is from 1 to N. To exclude holonomic constraints, the general coordinates q^k $(k = 1, \ldots, n)$ are used. Here, $n = 3N - m$ is the number of degrees of freedom, where m is the number of holonomic constraints. Then $\mathbf{r}_i$ is the function of generalized coordinates and time: $\mathbf{r}_i = \mathbf{r}_i(q, t)$. Using $\delta\mathbf{r}_i = (\partial\mathbf{r}_i/\partial q^k)\delta q^k$, equation (1) gives

$$\left(\frac{\mathrm{d}(m\mathbf{v}_i)}{\mathrm{d}t}\frac{\partial\mathbf{r}_i}{\partial q^k} - \mathbf{F}_i\frac{\partial\mathbf{r}_i}{\partial q^k}\right)\delta q^k = 0, \tag{2}$$

and the sum over repeated index k is from 1 to n. Then, we define [37] the generalized forces:

$$Q_k = \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q^k} \qquad k = 1, \ldots, n.$$

By usual transformations [37]

$$\begin{aligned}\frac{\mathrm{d}(m\mathbf{v}_i)}{\mathrm{d}t}\frac{\partial \mathbf{r}_i}{\partial q^k} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{v}_i\frac{\partial \mathbf{r}_i}{\partial q^k}\right) - (m\mathbf{v}_i)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{r}_i}{\partial q^k} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{v}_i\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k}\right) - m\mathbf{v}_i\frac{\partial \mathbf{v}_i}{\partial q^k} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^k}\left(\frac{m}{2}\mathbf{v}_i\mathbf{v}_i\right) - \frac{\partial}{\partial q^k}\left(\frac{m}{2}\mathbf{v}_i\mathbf{v}_i\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k},\end{aligned}$$

we transform equation (2) into

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k\right)\delta q^k = 0, \tag{3}$$

where $T = m\mathbf{v}^2/2$ is the kinetic energy. Using

$$\mathbf{v}_i = \frac{\mathrm{d}\mathbf{r}_i(q,t)}{\mathrm{d}t} = \frac{\partial \mathbf{r}_i}{\partial q^k}\frac{\mathrm{d}q^k}{\mathrm{d}t} + \frac{\partial \mathbf{r}_i}{\partial t}, \tag{4}$$

we get

$$T = \frac{m}{2}(g_{kl}(q,t)\dot{q}^k\dot{q}^l + 2g_k(q,t)\dot{q}^k + g(q,t)),$$

where

$$g_{kl}(q,t) = \frac{\partial \mathbf{r}_i}{\partial q^k}\frac{\partial \mathbf{r}_i}{\partial q^l}, \qquad g_k(q,t) = \frac{\partial \mathbf{r}_i}{\partial q^k}\frac{\partial \mathbf{r}_i}{\partial t}, \qquad g(q,t) = \frac{\partial \mathbf{r}_i}{\partial t}\frac{\partial \mathbf{r}_i}{\partial t}. \tag{5}$$

For the nonholonomic constraint,

$$R_k\delta q^k = 0, \tag{6}$$

where R_k is the reaction force of the constraint

$$f(q,\dot{q}) = 0, \tag{7}$$

and the variations δq^k are defined [38, 39] by

$$\frac{\partial f}{\partial \dot{q}^k}\delta q^k = 0. \tag{8}$$

Equation (8) is called Chetaev's condition [39]. Comparing equations (6) and (8), we obtain

$$R_k = \lambda\frac{\partial f}{\partial \dot{q}^k}, \tag{9}$$

where λ is the Lagrange multiplier. Chetaev's definition of variations states that the actual constrained motion should occur along a trajectory obtained by the normal projection of a force onto a constraint hypersurface. The constraint force R_k is minimum when R_k is chosen perpendicular to the constraint surface or parallel to the gradient $\partial f/\partial \dot{q}_k$.

In general, the nonholonomic system is subjected to action of the generalized force Q_k and the constraint force R_k. Then the variational equation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k - R_k\right)\delta q^k = 0. \tag{10}$$

From (9), we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k - \lambda\frac{\partial f}{\partial \dot{q}^k}\right)\delta q^k = 0. \tag{11}$$

In equation (8), we can consider $\delta\dot{q}^s$, $s = 1, 2, \ldots, n-1$, as independent variations. Then $\delta\dot{q}^n$ is not independent, and equation (8) gives

$$\delta q^n = -\left(\frac{\partial f}{\partial \dot{q}^n}\right)^{-1}\sum_{s=1}^{n-1}\frac{\partial f}{\partial \dot{q}^s}\delta q^s.$$

Suppose that λ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^n} - \frac{\partial T}{\partial q^n} - Q_n - \lambda\frac{\partial f}{\partial \dot{q}^n} = 0. \tag{12}$$

Then the term with $k = n$ in (11) is equal to zero, and equation (11) has $n-1$ terms with $k = 1, \ldots, n-1$. In equation (11), the variations with $k = 1, 2, \ldots, n-1$ are independent, and the sum is separated on $n-1$ equations. As a result, equation (11) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k + \lambda\frac{\partial f}{\partial \dot{q}^k}, \qquad (k = 1, \ldots, n). \tag{13}$$

Equations (7) and (13) form a system of $n+1$ equations with $n+1$ unknowns λ and q^k, where $k = 1, \ldots, n$. Solutions of these equations describe particles motion as a motion of the system with the nonlinear nonholonomic constraint (7).

2.2. *Nonholonomic system as a holonomic one*

In this section, we present the equations of motion with nonholonomic constraint as equations for a holonomic system.

The canonical momenta p^k are defined by

$$p_k = \frac{\partial T}{\partial \dot{q}^k} = mg_{kl}(q,t)\dot{q}^l + mg_k(q,t), \qquad (k = 1, \ldots, n). \tag{14}$$

Using (14), we can define

$$\tilde{f}(p,q,t) = f(\dot{q}(q,p,t), q, t). \tag{15}$$

Suppose that the constraint is the integral of motion. Then the total time derivative of (15) gives

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = 0, \qquad \frac{\partial \tilde{f}}{\partial p_k}\dot{p}_k + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t} = 0. \tag{16}$$

Substitution of (13) into (16) gives

$$\frac{\partial \tilde{f}}{\partial p_k}\left(\frac{\partial T}{\partial q^k} + Q_k + \lambda\frac{\partial \tilde{f}}{\partial \dot{q}^k}\right) + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t} = 0. \tag{17}$$

From equation (17), we obtain

$$\lambda = -\left(\frac{\partial \tilde{f}}{\partial p_m}\frac{\partial \tilde{f}}{\partial \dot{q}^m}\right)^{-1}\left(\frac{\partial \tilde{f}}{\partial p_\ell}\left(\frac{\partial T}{\partial q^\ell} + Q_\ell\right) + \frac{\partial \tilde{f}}{\partial q^\ell}\dot{q}^\ell + \frac{\partial \tilde{f}}{\partial t}\right). \tag{18}$$

Then the Lagrange equations (13) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k - \frac{\partial \tilde{f}}{\partial \dot{q}^k}\left(\frac{\partial \tilde{f}}{\partial p_m}\frac{\partial \tilde{f}}{\partial \dot{q}^m}\right)^{-1}\left(\frac{\partial \tilde{f}}{\partial p_\ell}\left(\frac{\partial T}{\partial q^\ell} + Q_\ell\right) + \frac{\partial \tilde{f}}{\partial q^\ell}\dot{q}^\ell + \frac{\partial \tilde{f}}{\partial t}\right). \tag{19}$$

Equations (19) describe the motion of a holonomic system with n degrees of freedom. For any trajectory of the system in the phase space, we have $\tilde{f} = 0$. If the initial values $q_k(0)$ and $\dot{q}_k(0)$ satisfy the constraint condition $f(q(0), \dot{q}(0), t_0) = 0$, then the solution of equation (19) is the motion of the nonholonomic system.

Let us define a generalized force $\Lambda_k = Q_k + R_k$, which depends on generalized velocities $\dot{q}^k$, generalized coordinates q^k and time t. If

$$\frac{\partial \Lambda_k}{\partial \dot{q}^m} + \frac{\partial \Lambda_m}{\partial \dot{q}^k} = 0, \qquad \frac{\partial \Lambda_k}{\partial q^m} + \frac{\partial \Lambda_m}{\partial q^k} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Lambda_k}{\partial \dot{q}^m} - \frac{\partial \Lambda_m}{\partial \dot{q}^k}\right),$$

known as the Helmholtz conditions, are satisfied, then a generalized potential $U = U(\dot{q}, q, t)$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}^k} - \frac{\partial U}{\partial q^k} = \Lambda_k.$$

In this case, the Hamilton variational principle has the form of the stationary action principle. To use this principle for a nonholonomic system, we should consider such trajectories that their initial conditions satisfy the constraint equation (7).

Note that the nonholonomic constraint (7) and the non-potential generalized force Q_k can be compensated such that the resulting generalized force Λ_k is a generalized potential force, and the system is a Lagrangian and non-dissipative system with holonomic constraints.

3. Constraints with fractional derivatives

3.1. Fractional derivatives

The fractional derivative has different definitions [1, 2], and exploiting any of them depends on the kind of problems, initial (boundary) conditions and the specifics of the considered physical processes. The classical definition is the so-called Riemann–Liouville derivative [1, 2]

$$\begin{aligned} {}_a\mathcal{D}_t^\alpha f(x) &= \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_a^x \frac{f(z)\,\mathrm{d}z}{(x-z)^{\alpha-m+1}}, \\ {}_t\mathcal{D}_b^\alpha f(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_x^b \frac{f(z)\,\mathrm{d}z}{(z-x)^{\alpha-m+1}}, \end{aligned} \tag{20}$$

where $m - 1 < \alpha < m$. Due to some reasons, concerning the initial conditions, it is more convenient to use the Caputo fractional derivatives [7, 42, 43]. Its main advantage is that the initial conditions take the same form as for integer-order differential equations.

Definition. *The Caputo fractional derivatives are defined by the equations*

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau, \tag{21}$$

$${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_t^b \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau, \tag{22}$$

where $m - 1 < \alpha < m$, *and* $f^{(m)}(\tau) = \mathrm{d}^m f(\tau)/\mathrm{d}\tau^m$.

Proposition 1. *The total time derivative of the Caputo fractional derivative of order α can be presented as a fractional derivative of order $\alpha + 1$ by*

$$\frac{d}{dt}\, {}_aD_t^{\alpha} f(t) = {}_aD_t^{\alpha+1} f(t) + \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} f^{(m)}(a), \tag{23}$$

where $m = [\alpha] + 1$, and $[\cdots]$ means floor function.

Proof. Definition (21) can be presented in the form

$${}_aD_t^{\alpha} f = {}_aJ_t^{m-\alpha} D_t^m f, \tag{24}$$

where ${}_aJ_t^{m-\alpha}$ is the fractional integral

$${}_aJ_t^{\varepsilon} f(t) = \frac{1}{\Gamma(\varepsilon)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\varepsilon}}\, d\tau. \tag{25}$$

The operations D_t^1 and J_t^{ε} do not commute:

$$D_t^1\, {}_aJ_t^{\varepsilon} f(t) = {}_aJ_t^{\varepsilon} D_t^1 f(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f(a). \tag{26}$$

From (24), we have

$$\frac{d}{dt}\, {}_aD_t^{\alpha} f(t) = D_t^1\, {}_aJ_t^{\varepsilon} D_t^m f(t) = D_t^1\, {}_aJ_t^{\varepsilon} f^{(m)}(t), \tag{27}$$

where ${}_aJ_t^{\varepsilon}$ is the fractional integration of order $\varepsilon = m - \alpha$. Using (26) and (27), we get

$$\begin{aligned} D_t^1 J_t^{\varepsilon} D_t^m f(t) &= {}_aJ_t^{\varepsilon} D_t^1 f^{(m)}(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a) \\ &= {}_aJ_t^{\varepsilon} f^{(m+1)}(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a) = {}_aD_t^{\alpha+1} f(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a). \end{aligned} \tag{28}$$

Substitution of (28) into (27) proves (23). □

3.2. Fractional constraints and its interpretation

To understand the physical interpretation of nonholonomic constraints with fractional derivatives, we discuss the memory effects and limiting cases widely used in physics: (1) the absence of the memory; (2) the complete memory; (3) the power-like memory.

We consider the evolution of a dynamical system in which some quantity $G(t)$ is related to another quantity $g(t)$ through a memory function $M(t)$:

$$G(t) = \int_0^t M(t-\tau) g(\tau)\, d\tau. \tag{29}$$

In mathematics, equation (29) means that the value $G(t)$ is related to $g(t)$ by the convolution operation

$$G(t) = M(t) * g(t).$$

Equation (29) is the typical non-Markovian equation obtained by studying the systems coupled to an environment, with environmental degrees of freedom being averaged. Let us consider special cases of equation (29).

(1) For a system without memory, we have the ideal Markov system, and the time dependence of the memory function is

$$M(t-\tau) = \delta(t-\tau), \tag{30}$$

where $\delta(t-\tau)$ is the Dirac delta-function. The absence of the memory means that the function $G(t)$ is defined by $g(t)$ at the only instant t. For this limiting case, the system loses all its states except for the one with infinitely high density. Using (29) and (30), we have

$$G(t) = \int_0^t \delta(t-\tau) g(\tau)\, \mathrm{d}\tau = g(t). \tag{31}$$

Expression (31) corresponds to the well-known Markov process with complete absence of memory. This process relates all subsequent states to the previous states through the single current state at each time t.

(2) If memory effects are introduced into the system the delta-function turns into some function, with the time interval during which $g(t)$ has an effect on the function $G(t)$. Let $M(t)$ be the step function

$$M(t-\tau) = \begin{cases} 1/t, & 0 < \tau < t; \\ 0, & \tau > t. \end{cases} \tag{32}$$

The factor $1/t$ is chosen to achieve normalization of the memory function to unity:

$$\int_0^t M(\tau)\, \mathrm{d}\tau = 1.$$

Then in the evolution process the system passes through all states continuously without any loss. In this case,

$$G(t) = \frac{1}{t} \int_0^t g(\tau)\, \mathrm{d}\tau,$$

and this corresponds to complete memory.

(3) The power-like memory function

$$M(t-\tau) = M_0 (t-\tau)^{\varepsilon-1} \tag{33}$$

indicates the presence of the fractional derivative or integral. Substitution of (33) into (29) gives the temporal fractional integral of order ε:

$$G(t) = \frac{\lambda}{\Gamma(\varepsilon)} \int_0^t (t-\tau)^{\varepsilon-1} g(\tau)\, \mathrm{d}\tau, \qquad 0 < \varepsilon < 1, \tag{34}$$

where $\Gamma(\varepsilon)$ is the Gamma function, and $\lambda = \Gamma(\varepsilon) M_0$. The parameter λ can be regarded as the strength of the perturbation induced by the environment of the system. If $g(t)$ is a derivative $q^{(m)}(t)$ of integer order m, then equation (34) defines the fractional Caputo derivative of $q(t)$ with respect to time

$${}_0D_t^{\alpha} q(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}} q^{(m)}(\tau)\, \mathrm{d}\tau, \tag{35}$$

where $\alpha = m - \varepsilon$. The physical interpretation of the fractional derivative is an existence of a memory effect with power-like memory function. The memory determines an interval t during which the derivative $q^{(m)}(\tau)$ has an effect on the function $G(t)$.

Equation (29) is a special case of constraint for $G(t)$ and $g(t)$, where G is directly proportional to $M * g$. In the general case, the values $G(t)$ and $g(t)$ can be related by the equation

$$f(G, M * g) = 0, \tag{36}$$

where f is the smooth function. Relation (36) defines a constraint for dynamical system. This constraint gives the memory effect. If G is a coordinate $q(t)$ or velocity $\dot{q}(t)$, and g is a derivative $q^{(m)}(t)$, then equation (36) gives the nonholonomic constraint with memory. For a power-like memory function $M(t)$, we present (36) as a constraint with fractional derivatives: $f\left(q, \dot{q}, {}_aD_t^{\alpha} q\right) = 0$ As the result, we can use the fractional calculus [1] to describe the motion of systems with the constraints (36).

3.3. Fractional equations of motion

Assume that the constraint equation has fractional derivatives:

$$f\left(q, \dot{q}, {}_aD_t^\alpha q, {}_tD_b^\alpha q\right) = 0, \tag{37}$$

i.e., it is a fractional differential equation [3]. Such a constraint can be called a fractional nonholonomic constraint. Since equation (37) has also derivatives of integer order, we can use the Chetaev definition of variation (8) and the Lagrange equations (13). For generalized potential forces

$$Q_k = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}_k} - \frac{\partial U}{\partial q_k}, \qquad (k = 1, \ldots, n),$$

and we can rewrite equation (13) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda \frac{\partial f}{\partial \dot{q}_k}, \qquad (k = 1, \ldots, n), \tag{38}$$

where $L = T(q, \dot{q}) - U(q, \dot{q})$ is the Lagrangian. To simplify our calculations, we consider

$$L = L(q, \dot{q}) = \frac{1}{2}\sum_{k=1}^{n}(\dot{q}_k)^2 - u(q), \tag{39}$$

where $u(q)$ is the potential energy of the system. Then, equation (38) becomes

$$\ddot{q}_k = -\frac{\partial u}{\partial q_k} + \lambda \frac{\partial f}{\partial \dot{q}_k}, \qquad (k = 1, \ldots, n). \tag{40}$$

Suppose that the constraint is an integral of motion, i.e., $\mathrm{d}f/\mathrm{d}t = 0$. Then

$$\frac{\partial f}{\partial \dot{q}_k}\frac{\mathrm{d}\dot{q}_k}{\mathrm{d}t} + \frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)}\frac{\mathrm{d}\left({}_aD_t^\alpha q_k\right)}{\mathrm{d}t} + \frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)}\frac{\mathrm{d}\left({}_tD_b^\alpha q_k\right)}{\mathrm{d}t} + \frac{\partial f}{\partial q_k}\frac{\mathrm{d}q_k}{\mathrm{d}t} = 0. \tag{41}$$

Equation (41) can be presented as

$$\frac{\partial f}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)} D_t^1 {}_aD_t^\alpha q_k + \frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)} D_t^1 {}_tD_b^\alpha q_k + \frac{\partial f}{\partial q_k}\dot{q}_k = 0. \tag{42}$$

Substitution of (40) into (42) gives

$$\frac{\partial f}{\partial \dot{q}_k}\left(-\frac{\partial u}{\partial q_k} + \lambda\frac{\partial f}{\partial \dot{q}_k}\right) + \frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)} D_t^1 {}_aD_t^\alpha q_k + \frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)} D_t^1 {}_tD_b^\alpha q_k + \frac{\partial f}{\partial q_k}\dot{q}_k = 0. \tag{43}$$

From this equation, one can obtain the Lagrange multiplier λ:

$$\lambda = \left(\frac{\partial f}{\partial \dot{q}_m}\frac{\partial f}{\partial \dot{q}_m}\right)^{-1}\left(\frac{\partial f}{\partial \dot{q}_l}\frac{\partial u}{\partial q_l} - \frac{\partial f}{\partial\left({}_aD_t^\alpha q_l\right)} D_t^1 {}_aD_t^\alpha q_l - \frac{\partial f}{\partial\left({}_tD_b^\alpha q_l\right)} D_t^1 {}_tD_b^\alpha q_l - \frac{\partial f}{\partial q_l}\dot{q}_l\right). \tag{44}$$

Insertion of equation (44) into equation (40) yields

$$\ddot{q}_k = -\frac{\partial u}{\partial q_k} + \frac{\partial f}{\partial \dot{q}_k}\left(\frac{\partial f}{\partial \dot{q}_m}\frac{\partial f}{\partial \dot{q}_m}\right)^{-1}$$
$$\times\left(\frac{\partial f}{\partial \dot{q}_l}\frac{\partial u}{\partial q_l} - \frac{\partial f}{\partial\left({}_aD_t^\alpha q_l\right)} D_t^1 {}_aD_t^\alpha q_l - \frac{\partial f}{\partial\left({}_tD_b^\alpha q_l\right)} D_t^1 {}_tD_b^\alpha q_l - \frac{\partial f}{\partial q_l}\dot{q}_l\right). \tag{45}$$

These equations describe the holonomic system that is equivalent to the nonholonomic one with fractional constraint. For any motion of the system, we have $f = 0$. If the initial values satisfy the constraint condition $f\left(q(0), \dot{q}(0), {}_aD_t^{\alpha}q(0), {}_tD_b^{\alpha}q(0)\right) = 0$, then the solution of equation (45) describes a motion of the system (39) with the fractional constraint (37).

3.4. Linear fractional constraint

Suppose that the constraint (37) is linear with respect to integer derivatives $\dot{q}_k$, i.e.,

$$f = a_k \dot{q}_k + \beta\left({}_aD_t^{\alpha}q, {}_tD_b^{\alpha}q, q\right). \tag{46}$$

In this case, $R_k = a_k$, and equations (45) can be presented as

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \frac{a_k}{a^2}\sum_{l=1}^{n}\left(\frac{\partial f}{\partial\left({}_aD_t^{\alpha}q_l\right)}D_t^1\, {}_aD_t^{\alpha}q_l + \frac{\partial f}{\partial\left({}_tD_b^{\alpha}q_l\right)}D_t^1\, {}_tD_b^{\alpha}q_l + \frac{\partial \beta}{\partial q_l}\dot{q}_l\right), \tag{47}$$

where $a^2 = \sum_{k=1}^{n} a_k a_k$. If

$$\beta\left({}_aD_t^{\alpha}q, {}_tD_b^{\alpha}q, q\right) = b_k\, {}_aD_t^{\alpha}q_k, \tag{48}$$

then

$$f = a_k \dot{q}_k + b_k\, {}_aD_t^{\alpha}q_k. \tag{49}$$

This constraint is linear with respect to the integer derivative $\dot{q}_k$ and fractional derivatives ${}_aD_t^{\alpha}q_k$. Then the equations of motion are

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \sum_{l=1}^{n}\frac{a_k b_l}{a^2}D_t^1\, {}_aD_t^{\alpha}q_l. \tag{50}$$

Using proposition 1, we obtain

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \sum_{l=1}^{n}\frac{a_k b_l}{a^2}\left({}_aD_t^{\alpha+1}q_l + \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)}q_l^{(m)}(a)\right), \tag{51}$$

where $q^{(m)}(a) = \left(D_t^m q(t)\right)_{t=a}$. As the result, we get the fractional equations of motion with Caputo derivative of order $\alpha+1$. The nonholonomic systems (50) with integer α are considered in [63, 64].

3.5. One-dimensional example

In the one-dimensional case ($n = 1$), equation (50) with ${}_0D_t^{\alpha}q$ has the form

$$\ddot{q} = -\frac{b_1}{a_1}D_t^1\, {}_0D_t^{\alpha}q. \tag{52}$$

Equation (52) can be presented as

$$D_t^1\left[\dot{q} + (b_1/a_1)_0D_t^{\alpha}q\right] = 0. \tag{53}$$

Then

$$\dot{q} + (b_1/a_1)_0D_t^{\alpha}q = C_0. \tag{54}$$

Supposing $\alpha > 1$, and using proposition 1, we get

$$D_t^1\left[q + (b_1/a_1)_0D_t^{\alpha-1}q\right] = \frac{b_1 t^{m-\alpha}}{a_1\Gamma(m-\alpha+1)}q(0) + C_0. \tag{55}$$

As the result, we obtain

$$ {}_0D_t^{\alpha-1}q + (a_1/b_1)q = \frac{t^{m-\alpha+1}}{\Gamma(m-\alpha+2)}q(0) + C_1 t + C_2, \tag{56} $$

where we use $x\Gamma(x) = \Gamma(x+1)$, and $C_1 = C_0 a_1/b_1$.

For $2 < \alpha < 3$, equation (56) describes the linear fractional oscillator

$$ {}_0D_t^{\alpha-1}q(t) + \omega^2 q(t) = Q(t), \tag{57} $$

where $\omega^2 = (a_1/b_1)$ is dimensionless 'frequency', and $Q(t)$ is the external force:

$$ Q(t) = \frac{t^{m-\alpha+1}}{\Gamma(m-\alpha+2)}q(0) + C_1 t + C_2. $$

The Caputo fractional derivative ${}_0D_t^{\alpha-1}$ allows us to use the regular initial conditions [3] for equation (57). The linear fractional oscillator is an object of numerous investigations [22–24, 44–50] because of different applications.

The exact solution [22, 44] of equation (57) for $2 < \alpha < 3$ is

$$ q(t) = q(0)E_{\alpha-1,1}(-\omega^2 t^{\alpha-1}) + tq'(0)E_{\alpha-1,2}(-\omega^2 t^{\alpha-1}) - \int_0^t Q(t-\tau)\dot{q}_0(\tau)\,\mathrm{d}\tau, \tag{58} $$

where

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{59} $$

is the generalized two-parameter Mittag–Leffler function [51, 52], and

$$ q_0(\tau) = E_{\alpha-1,1}(-\omega^2\tau^{\alpha-1}). $$

The decomposition of (58) is [44]:

$$ q(t) = q(0)[f_{\alpha,0}(t) + g_{\alpha,0}(t)] + t\dot{q}(0)[f_{\alpha,1}(t) + g_{\alpha,1}(t)] - \int_0^t Q(t-\tau)\dot{q}_0(\tau)\,\mathrm{d}\tau, \tag{60} $$

where

$$ \begin{aligned} f_{\alpha,k}(t) &= \frac{(-1)^k}{\pi}\int_0^\infty \mathrm{e}^{-rt}\frac{r^{\alpha-1-k}\sin(\pi\alpha)}{r^{2\alpha}+2r^\alpha\cos(\pi\alpha)+1}\,\mathrm{d}r, \\ g_{\alpha,k}(t) &= \frac{2}{\alpha}\,\mathrm{e}^{t\cos(\pi/\alpha)}\cos[t\sin(\pi/\alpha) - \pi k/\alpha], \qquad (k=0,1). \end{aligned} \tag{61} $$

For the initial conditions $q(0) = 1$, and $\dot{q}(0) = 0$:

$$ q(t) = E_\alpha(-t^\alpha) = f_{\alpha,0}(t) + g_{\alpha,0}(t) - \int_0^t Q(t-\tau)[\dot{f}_{\alpha,0}(\tau) + \dot{g}_{\alpha,0}(\tau)]\,\mathrm{d}\tau. \tag{62} $$

The first term in (62) decays in power law with time while the second term decays exponentially [22, 24, 44, 45].

3.6. Two-dimensional examples

In the two-dimensional case ($n = 2$), equation (50) has the form

$$ \ddot{q}_1 = -\frac{a_2^2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_1} + \frac{a_1a_2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_2} - \frac{a_1b_1}{a_1^2+a_2^2}D_{t\,a}^1 D_t^\alpha q_1 - \frac{a_1b_2}{a_1^2+a_2^2}D_{t\,a}^1 D_t^\alpha q_2, \tag{63} $$

$$ \ddot{q}_2 = -\frac{a_1^2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_2} + \frac{a_1a_2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_1} - \frac{a_2b_1}{a_1^2+a_2^2}D_{t\,a}^1 D_t^\alpha q_1 - \frac{a_2b_2}{a_1^2+a_2^2}D_{t\,a}^1 D_t^\alpha q_2. \tag{64} $$

Let us consider the special cases of these equations.

(1) Suppose $a_1 = 0$, then

$$\ddot{q}_1 = -\frac{\partial u}{\partial q_1}; \qquad \ddot{q}_2 = -\frac{b_1}{a_2} D^1_{t\,a} D^\alpha_t q_1 - \frac{b_2}{a_2} D^1_{t\,a} D^\alpha_t q_2. \tag{65}$$

If $a_1 = 0$, and $b_2 = 0$, then (65) are

$$\ddot{q}_1 = -\frac{\partial u}{\partial q_1}, \qquad \ddot{q}_2 = -\frac{b_1}{a_2} D^1_{t\,a} D^\alpha_t q_1. \tag{66}$$

(2) Suppose $b_1 = 0$ and $a_1 = a_2 = c$, then we have

$$\ddot{q}_1 = -\frac{1}{2}\frac{\partial u}{\partial q_1} + \frac{1}{2}\frac{\partial u}{\partial q_2} - \frac{b_2}{2c} D^1_{t\,a} D^\alpha_t q_2, \tag{67}$$

$$\ddot{q}_2 = -\frac{1}{2}\frac{\partial u}{\partial q_2} + \frac{1}{2}\frac{\partial u}{\partial q_1} - \frac{b_2}{2c} D^1_{t\,a} D^\alpha_t q_2. \tag{68}$$

Using

$$x = \frac{q_1 + q_2}{2}, \qquad y = \frac{q_1 - q_2}{2}, \qquad g = b_2/c,$$

we can rewrite equations (67) and (68) in the forms

$$\ddot{x} = -g D^1_{t\,a} D^\alpha_t x + g D^1_{t\,a} D^\alpha_t y, \qquad \ddot{y} = -\frac{\partial U}{\partial y}, \tag{69}$$

where $U(x, y) = u(q_1, q_2) = u(x + y, x - y)$. If $U = K(x)y + s(x)$, then equation (69) is

$$\ddot{x} = -g D^1_{t\,a} D^\alpha_t x + g D^1_{t\,a} D^\alpha_t y, \qquad \ddot{y} = -K(x). \tag{70}$$

Using $D^\alpha_t y = D^{\alpha-2}_t \ddot{y}$, equation (70) gives

$$\ddot{x} = -g D^1_{t\,a} D^\alpha_t x - g D^1_{t\,a} D^{\alpha-2}_t K(x). \tag{71}$$

Then

$$D^1_t \left[\dot{x} + g_a D^\alpha_t x + g_a D^{\alpha-2}_t K(x)\right] = 0.$$

For $\alpha > 2$, we can get

$$\dot{x} + g_a D^\alpha_t x + g_a D^{\alpha-2}_t K(x) + C = 0, \tag{72}$$

where C is a constant. Using ${}_a D^\varepsilon_{t\,a} J^\varepsilon_t f(t) = f(t)$, and ${}_a D^\varepsilon_{t\,a} D^\alpha_t f(t) = D^m_t f(t)$, where $\varepsilon = m - \alpha$, equation (72) can be written as

$$D^\varepsilon x + g x^{(m)} + g D^{m-2}_t K(x) = 0. \tag{73}$$

If $m = 2$ $(1 < \alpha < 2)$, then

$$g^{-1}\ddot{x} + D^\varepsilon x + g_a K(x) = 0. \tag{74}$$

This equation can be considered as an equation of nonlinear fractional oscillator [24, 45], where the fractional derivative describes the power dumping.

4. Fractional conditional extremum

4.1. Extremum for fractional constraint

Let us consider the stationary value of an action integral

$$\delta \int_a^b \mathrm{d}t\, L(q, \dot{q}) = 0,$$

for the lines that satisfy the constraint equation $f(q, \dot{q}) = 0$. Using the Lagrange multiplier $\mu = \mu(t)$, we get the variational equation

$$\delta \int_a^b \mathrm{d}t [L(q, \dot{q}) + \mu f(q, \dot{q})] = 0.$$

Then the Euler–Lagrange equations [41] are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \mu \left(\frac{\partial f}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_k}\right) - \dot{\mu}\frac{\partial f}{\partial \dot{q}_k}, \qquad (k = 1, \ldots, n). \tag{75}$$

Note that these equations consist of the derivative of Lagrange multiplier $\dot{\mu}$. The proof of equation (75) is realized in [41].

For the fractional constraint

$$f\left(q, \dot{q}, {}_aD_t^\alpha q, {}_tD_b^\alpha q\right) = 0, \tag{76}$$

we can define the Lagrangian as

$$\mathcal{L}\left(q, \dot{q}, {}_aD_t^\alpha q, {}_tD_b^\alpha q, \lambda\right) = L(q, \dot{q}) + \mu(t) f\left(q, \dot{q}, {}_aD_t^\alpha q, {}_tD_b^\alpha q\right). \tag{77}$$

Using the Agrawal variational equation [40], we obtain the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} + {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\left({}_aD_t^\alpha q_k\right)} + {}_tD_b^\alpha \frac{\partial \mathcal{L}}{\partial\left({}_tD_b^\alpha q_k\right)} = 0, \qquad (k = 1, \ldots, n). \tag{78}$$

Substitution of equation (77) into equation (78) gives

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \mu \frac{\partial f}{\partial q_k} + {}_aD_t^\alpha \left(\mu \frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)}\right) + {}_tD_b^\alpha \left(\mu \frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu \frac{\partial f}{\partial \dot{q}_k}\right) = 0. \tag{79}$$

These equations describe the fractional conditional extremum.

Let us consider applicability of the stationary action principle for mechanical systems with fractional nonholonomic constraints. The equations of motion are derived from the d'Alembert–Lagrange principle. The fractional conditional extremum can be obtained from the stationary action principle. In general, these equations are not equivalent [41]. The condition of this equivalence for fractional constraints is suggested in the proposition.

Proposition 2. *Equations (38) and (79) for nonholonomic system with fractional constraint (76) have the equivalent set of solutions if the conditions*

$$\left[\mu \frac{\partial f}{\partial q_k} + {}_aD_t^\alpha \left(\mu \frac{\partial f}{\partial_a D_t^\alpha q_k}\right) + {}_tD_b^\alpha \left(\mu \frac{\partial f}{\partial_t D_b^\alpha q_k}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu \frac{\partial f}{\partial \dot{q}_k}\right)\right] \delta q_k = 0, \qquad \frac{\partial f}{\partial \dot{q}_k}\delta q_k = 0 \tag{80}$$

are satisfied.

Proof. To prove the proposition, we multiply equations (38) and (79) on the variation δq^k and consider a sum with respect to k:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right)\delta q_k = \lambda \frac{\partial f}{\partial \dot{q}_k}\delta q_k, \tag{81}$$

$$\left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k + \left[\mu\frac{\partial f}{\partial q_k} + {}_aD_t^\alpha\left(\mu\frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)}\right) + {}_tD_b^\alpha\left(\mu\frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\partial f}{\partial \dot{q}_k}\right)\right]\delta q_k = 0. \tag{82}$$

From the definition of variations (8), equation (81) is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right)\delta q_k = 0. \tag{83}$$

Substituting equation (83) into equation (82), we obtain (80). □

It is known [41] that the stationary action principle cannot be derived from the d'Alembert–Lagrange principle for a wide class of nonholonomic and non-Hamiltonian systems. The same can be applied to the case of nonlinear fractional nonholonomic constraints.

4.2. Hamilton's approach

Using equation (78), we define the momenta

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{q}_k} + \mu\frac{\partial f}{\partial \dot{q}_k}, \tag{84}$$

and the Hamiltonian

$$\mathcal{H}(q, p) = p_k\dot{q} - \mathcal{L}, \tag{85}$$

where $\mathcal{L} = L + \mu f$. Equation (78) gives

$$\frac{\mathrm{d}p_k}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial q_k} + {}_aD_t^\alpha\left(\mu\frac{\partial f}{\partial\left({}_aD_t^\alpha q_k\right)}\right) + {}_tD_b^\alpha\left(\mu\frac{\partial f}{\partial\left({}_tD_b^\alpha q_k\right)}\right). \tag{86}$$

To simplify our calculations, we consider the Lagrangian

$$L = \tfrac{1}{2}(\dot{q})^2 - u(q),$$

and the fractional nonholonomic constraint

$$f = A_k\left(q, {}_aD_t^\alpha q\right)\dot{q}_k = 0, \qquad (\alpha \neq 1). \tag{87}$$

From equations (84) and (87), we obtain

$$p_k = \dot{q}_k + \mu A_k\left(q, {}_aD_t^\alpha q\right). \tag{88}$$

Then the Hamilton equations are

$$\dot{q}_k = p_k - \mu A_k\left(q, {}_aD_t^\alpha q\right), \tag{89}$$

$$\dot{p}_k = -\frac{\partial u(q)}{\partial q_k} + \mu(t)\dot{q}_l\frac{\partial A_l}{\partial q_k} + {}_aD_t^\alpha\left(\mu\dot{q}_l\frac{\partial A_l\left({}_aD_t^\alpha q\right)}{\partial_a D_t^\alpha q_k}\right). \tag{90}$$

To find the Lagrange multiplier $\mu = \mu(t)$, we multiply equation (88) on the functions a_k and consider the sum with respect to k:

$$A_kp_k = A_k\dot{q}_k + \mu A_kA_k = \mu A^2. \tag{91}$$

Here, we use constraint (87), and the notation $A^2 = A_k A_k$, where $A_k = A_k\big(q, {}_aD_t^\alpha q\big)$. From (91), we get

$$\mu = \frac{A_k p_k}{A^2}. \tag{92}$$

Substitution of (92) into equations (89) and (90) gives

$$\dot{q}_k = \left(\delta_{kl} - \frac{A_k A_l}{A^2}\right) p_l, \tag{93}$$

$$\dot{p}_k = -\frac{\partial u(q)}{\partial q_k} + \frac{A_m p_m}{A^2}\frac{\partial A_l}{\partial q_k}\dot{q}_l + {}_aD_t^\alpha\left(\frac{A_m p_m}{A^2}\frac{\partial A_l}{\partial_a D_t^\alpha q_k}\dot{q}_l\right). \tag{94}$$

If $a_k = 0$, then we have usual equations of motion for Hamiltonian systems. Note that we derive Hamiltonian equations from Euler–Lagrange equations without using the Legendre transformation, which is typically used.

5. Geometric method for fractional nonholonomic systems

In this section, connections with recent works on geometric methods in nonholonomic mechanics [65–67] are considered.

Suppose $L(t, q, v)$ is a Lagrangian of dynamical system, where q^k are coordinates, and v^k are the velocities. We have the additional condition $\dot{q}^k = v^k$. In this case, we introduce p^k as independent Lagrange multipliers, and all the variables q^k, v^k, p_k have to be varied. The corresponding Lagrange equations are

$$\dot{q}^k = v^k, \qquad \dot{p}_k = \frac{\partial L}{\partial q^k}, \qquad p_k = \frac{\partial L}{\partial v^k}. \tag{95}$$

In the space of variables (t, q, p, v), the extended Poincare–Cartan 1-form is

$$\omega_{h*} = p_k dq^k + [L - p_k v^k]\,\mathrm{d}t. \tag{96}$$

For the nonholonomic constraint (37), we define the Chetaev 2-form [67] by

$$C = -\lambda\frac{\partial f}{\partial \dot{q}^k}\,\mathrm{d}t \wedge \mathrm{d}q^k, \tag{97}$$

where λ is a Lagrange multiplier.

Proposition 3. *The exterior derivative of the form (96) is defined by the equation*

$$\mathrm{d}\omega_{h*} = [D_t p_k - D_{q^k} L]\,\mathrm{d}t \wedge \mathrm{d}q^k - [\mathrm{d}q^k - v^k\,\mathrm{d}t] \wedge \mathrm{d}p_k - [p_k - D_{v^k} L]\,\mathrm{d}v^k \wedge \mathrm{d}t. \tag{98}$$

Proof. The proof of this proposition is realized in [66]. □

Action principle for nonholonomic systems. *The trajectory of a dynamical system with constraint (37) can be derived by finding the path for which the form (96) is equal to the Chetaev 2-form , i.e.,*

$$\mathrm{d}\omega_{h*} = C. \tag{99}$$

From equations (98), (97) and (99), we get

$$D_t p_k - D_{q^k} L = \lambda\frac{\partial f}{\partial \dot{q}^k}, \qquad \mathrm{d}q^k - v^k\,\mathrm{d}t = 0, \qquad p_k - D_{v^k} L = 0. \tag{100}$$

It is easy to see that equation (100) can be presented as

$$D_{q^k} L - \left[\frac{\mathrm{d}}{\mathrm{d}t} D_{v^k} L \right]_{v=\dot{q}} = \lambda \frac{\partial f}{\partial \dot{q}^k}. \tag{101}$$

As the result, we obtain

$$\frac{\partial L}{\partial q^k} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^k} \right) = \lambda \frac{\partial f}{\partial \dot{q}^k}, \qquad k = 1, \ldots, n, \tag{102}$$

which are the same as the Euler–Lagrange equations (38) for the nonholonomic system.

Fractional exterior derivative and fractional differential forms are considered in [26, 66, 68]. Let us define

$$\omega_{h*\alpha} = p_k (\mathrm{d}q^k)^\alpha + [L - p_k (v^k)^\beta](\mathrm{d}t)^\alpha, \qquad \beta > 0, \tag{103}$$

which is a fractional generalization of the extended Poincare–Cartan 1-form (96).

Proposition 4. *The fractional exterior derivative of the fractional 1-form (103) is defined by*

$$\mathrm{d}^\alpha \omega_{h*\alpha} = \left[D_t^\alpha p_k - D_{q^k}^\alpha L \right] (\mathrm{d}t)^\alpha \wedge (\mathrm{d}q^k)^\alpha - \left(D_{p_k}^\alpha p_k \right) [(\mathrm{d}q^k)^\alpha - (v^k)^\beta (\mathrm{d}t)^\alpha] \wedge (\mathrm{d}p_k)^\alpha - \left[p_k D_{v^k}^\alpha (v^k)^\beta - D_{v^k}^\alpha L \right] (\mathrm{d}v^k)^\alpha \wedge (\mathrm{d}t)^\alpha. \tag{104}$$

Proof. This proposition is proved in [66]. □

Fractional generalization of the Chetaev 2-form (97) can be defined by

$$C_\alpha = -\lambda \frac{\partial f}{\partial \dot{q}^k} (\mathrm{d}t)^\alpha \wedge (\mathrm{d}q^k)^\alpha. \tag{105}$$

Fractional action principle for nonholonomic systems. *The trajectory of fractional dynamical systems with constraint (37) can be derived by finding the path for which the fractional extended Poincare–Cartan 1-form (103) is equal to the fractional Chetaev 2-form (105), i.e.,*

$$\mathrm{d}^\alpha \omega_{h*\alpha} = C_\alpha. \tag{106}$$

Using (104), (105) and (106), we have

$$\left(D_t^\alpha p_k \right) - D_{q^k}^\alpha L = \lambda \frac{\partial f}{\partial \dot{q}^k}, \qquad (\mathrm{d}q^k)^\alpha - (v^k)^\beta (\mathrm{d}t)^\alpha = 0, \qquad p_k D_{v^k}^\alpha (v^k)^\beta - D_{v^k}^\alpha L = 0. \tag{107}$$

From the relation

$$D_v^\alpha v^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} v^{\beta-\alpha}, \tag{108}$$

where $\beta > -1$, we get

$$D_t^\alpha p_k = D_{q^k}^\alpha L + \lambda \frac{\partial f}{\partial \dot{q}^k}, \qquad (v^k)^\beta = \frac{(\mathrm{d}q^k)^\alpha}{(\mathrm{d}t)^\alpha}, \qquad p_k = \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)} (v^k)^{\alpha-\beta} D_{v^k}^\alpha L. \tag{109}$$

Substitution of the third equation from (109) into the first one gives

$$D_{q^k}^\alpha L - \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)} D_t^\alpha \left[(v^k)^{\alpha-\beta} D_{v^k}^\alpha L \right]_{(v^k)^\beta = (\dot{q}^k)^\alpha} = \lambda \frac{\partial f}{\partial \dot{q}^k}. \tag{110}$$

Equation (110) has the dependence

$$(v^k)^\beta = (\dot{q}^k)^\alpha. \tag{111}$$

It is easy to see that equation (110) looks unusually even for $\beta = 1$. Therefore we consider $\beta = \alpha$ for the form (103). Using $\beta = \alpha$ in equation (103), we obtain

$$D^\alpha_{q^k} L - \Gamma(2-\alpha) D^\alpha_t \left[D^\alpha_{v^k} L \right]_{v=\dot{q}} = \lambda \frac{\partial f}{\partial \dot{q}^k} \tag{112}$$

that is *fractional Euler–Lagrange equations*. As the result, the fractional equations of motion for nonholonomic systems are presented by

$$\hat{E}^\alpha_k L(t, q, \dot{q}) = \lambda \frac{\partial f}{\partial \dot{q}^k}, \qquad i = 1, \ldots, n, \tag{113}$$

where

$$\hat{E}^\alpha_k = D^\alpha_{q^k} - \Gamma(2-\alpha) D^\alpha_t D^\alpha_{\dot{q}^i}. \tag{114}$$

For $\alpha = 1$, equations (113) are the Euler–Lagrange equations (102).

6. Conclusion

The classical mechanics of nonholonomic systems has recently been employed to study a wide variety of problems in the molecular dynamics [53]. In molecular dynamics calculations, nonholonomic systems can be exploited to generate statistical ensembles as the canonical, isothermal-isobaric and isokinetic ensembles [54–64]. Using fractional nonholonomic constraints, we can consider a fractional extension of the statistical mechanics of conservative Hamiltonian systems to a much broader class of systems. Let us point out some nonholonomic systems that can be generalized by using the nonholonomic constraint with fractional derivatives.

(1) In the papers [54–57], the constant temperature systems with minimal Gaussian constraint are considered. These systems are non-Hamiltonian ones and they are described by the non-potential forces that are proportional to the velocity, and the Gaussian nonholonomic constraint. Note that this constraint can be represented as an addition term to the non-potential force [64].

(2) In the papers [63, 64], the canonical distribution is considered as a stationary solution of the Liouville equation for a wide class of non-Hamiltonian systems. This class is defined by a very simple condition: the power of the non-potential forces must be proportional to the velocity of the Gibbs phase (elementary phase volume) change. This condition defines the general constant temperature systems. Note that the condition is a nonholonomic constraint. This constraint leads to the canonical distribution as a stationary solution of the Liouville equations. For the linear friction, we derived the constant temperature systems. A general form of the non-potential forces is derived in [64].

Fractional constraints can be used to describe the evolution of dynamical system in which some coordinates and velocities are related to other coordinates or velocities through power-law memory function. The constraints with power-like memory function are considered by using the fractional calculus. It allows us to take into account the memory effects by fractional equations.

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