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Continuous limit of discrete systems with long-range interaction

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Abstract

Discrete systems with long-range interactions are considered. Continuous medium models as continuous limit of discrete chain system are defined. Long-range interactions of chain elements that give the fractional equations for the medium model are discussed. The chain equations of motion with long-range interaction are mapped into the continuum equation with the Riesz fractional derivative. We formulate the consistent definition of continuous limit for the systems with long-range interactions. In this paper, we consider a wide class of long-range interactions that give fractional medium equations in the continuous limit. The power-law interaction is a special case of this class.

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1. Introduction

Derivatives or integrals of noninteger order [1–5] have found many applications in recent studies in mechanics and physics [6–10]. Equations which involve derivatives or integrals of noninteger order are very successful in describing anomalous kinetics and transport and continuous time random walks [11–15]. Usually, the fractional equations for dynamics or kinetics appear as some phenomenological models. Recently, a method to obtain fractional analogues of equations of motion was considered for sets of coupled particles with a long-range interaction [16–19]. Examples of systems with interacting oscillators, spins or waves are used for many applications in physics, chemistry and biology [20–34]. In the continuous limit, the equations of motion for discrete systems give the continuous media equation. The procedure has already been used to derive fractional sine-Gordon and fractional wave Hilbert equation [16, 18], to study synchronization of coupled oscillators [17], to derive fractional Ginzburg–Landau equation [17] and for chaos in discrete nonlinear Schrödinger equation [19]. In [16–19], only the power-law long-range interactions are considered. In this paper, we consider a wide class of long-range interactions that give fractional medium equations in continuous limit. The power-law interaction is a special case of this class.

Long-range interaction (LRI) has been the subject of investigations for a long time. An infinite one-dimensional Ising model with LRI was considered by Dyson [20]. The d-dimensional classical Heisenberg model with long-range interaction is described in [21, 22], and their quantum generalization can be found in [23, 24]. Solitons in a one-dimensional lattice with the long-range Lennard-Jones-type interaction were considered in [28]. Kinks in the Frenkel–Kontorova model with long-range interparticle interactions were studied in [29]. The properties of time periodic spatially localized solutions (breathers) on discrete chains in the presence of algebraically decaying interactions were described in [32, 33]. Energy and decay properties of discrete breathers in systems with LRI have also been studied in the framework of the Klein–Gordon [27] and discrete nonlinear Schrödinger equations [30]. A main property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails. The lattice models with power-like long-range interactions [18, 31–33, 35–37] have similar properties. As was shown in [17–19], the analysis of the equations with fractional derivatives can provide results for the space asymptotics of their solutions.

The goal of this paper is to formulate the consistent definition of continuous limit (transform operation) for the systems with long-range interactions (LRI). This aim is realized by propositions 1, 4 and definitions 1, 2. The power-law LRI is considered in [16–19]. The exact continuous limit results for power-law LRI were formulated in propositions 2, 3. This operation is used to consider a wide class of long-range interactions that can be called alpha-interaction. In continuous limit, the equations of motion give the medium equations with fractional derivatives. The power-law interaction is a special case of this class of α-interactions. We show how the continuous limit for the systems of oscillators with long-range interaction can be described by the corresponding fractional equation.

In section 2, the transform operation that maps the discrete equations into continuous medium equation is defined. In section 3, the Fourier series transform of the equations of a system with long-range interaction is realized. A wide class of long-range interactions that can give the fractional equations in the continuous limit is considered. In section 4, the fractional equations are obtained from three-dimensional discrete system. In section 5, the linear power-law long-range interactions with positive integer and noninteger powers are considered. The correspondent continuous medium equations are discussed. In section 6, the nonlinear long-range interactions for the discrete systems are used to derive the Burgers, Korteweg–de Vries and Boussinesq equations and their fractional generalizations in the continuous limit. The conclusion is given in section 7.

2. Transform operation

Let us consider a one-dimensional system of interacting oscillators that are described by the equations of motion,

$$\frac{\partial^s u_n}{\partial t^s} = g\hat{I}_n(u) + F(u_n), \tag{1}$$

where $s = 1, 2$ and u_n are displacements from the equilibrium. The terms $F(u_n)$ characterize an interaction of the oscillators with the external on-site force. The term $\hat{I}_n(u)$ is defined by

$$\hat{I}_n(u) \equiv \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m)W(u_n,u_m), \tag{2}$$

and it takes into account the interaction of the oscillators in the system.

For linear long-range interaction we have $W(u_n, u_m) = u_n - u_m$, and the interaction term (2) is

$$\hat{I}_n(u) \equiv \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m)[u_n - u_m]. \tag{3}$$

In this paper, we consider a wide class of interactions (3) that create a possibility of presenting the continuous medium equations with fractional derivatives. We also discuss the term (2) with $W(u_n, u_m) = f(u_n) - f(u_m)$ as nonlinear long-range interaction. As the examples, we consider $f(u) = u^2$ and $f(u) = u - gu^2$ that gives the Burgers, Korteweg–de Vries and Boussinesq equations and their fractional generalizations in the continuous limit.

Let us define the operation, which transforms equations (1) for $u_n(t)$ into continuous medium equation for $u(x,t)$. We assume that $u_n(t)$ are Fourier coefficients of some function $\hat{u}(k,t)$. Then we define the field $\hat{u}(k,t)$ on $[-K/2, K/2]$ as

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t)\,\mathrm{e}^{-\mathrm{i}kx_n} = \mathcal{F}_\Delta\{u_n(t)\}, \tag{4}$$

where $x_n = n\Delta x$, $\Delta x = 2\pi/K$ is distance between oscillators and

$$u_n(t) = \frac{1}{K}\int_{-K/2}^{+K/2} \mathrm{d}k\,\hat{u}(k,t)\,\mathrm{e}^{\mathrm{i}kx_n} = \mathcal{F}_\Delta^{-1}\{\hat{u}(k,t)\}. \tag{5}$$

These equations are the basis for the Fourier transform, which is obtained by transforming from a discrete variable to a continuous one in the limit $\Delta x \to 0$ $(K \to \infty)$. The Fourier transform can be derived from (4) and (5) in the limit as $\Delta x \to 0$. Replace the discrete $u_n(t) = (2\pi/K)u(x_n,t)$ with continuous $u(x,t)$ while letting $x_n = n\Delta x = 2\pi n/K \to x$. Then change the sum to an integral, and equations (4), (5) become

$$\tilde{u}(k,t) = \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}kx} u(x,t) = \mathcal{F}\{u(x,t)\}, \tag{6}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k\,\mathrm{e}^{\mathrm{i}kx}\tilde{u}(k,t) = \mathcal{F}^{-1}\{\tilde{u}(k,t)\}. \tag{7}$$

Here,

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \tag{8}$$

and $\mathcal{L}$ denotes the passage to the limit $\Delta x \to 0$ $(K \to \infty)$. Note that $\tilde{u}(k,t)$ is a Fourier transform of the field $u(x,t)$, and $\hat{u}(k,t)$ is a Fourier series transform of $u_n(t)$, where we can use $u_n(t) = (2\pi/K)u(n\Delta x,t)$. The function $\tilde{u}(k,t)$ can be derived from $\hat{u}(k,t)$ in the limit $\Delta x \to 0$.

The map of a discrete model into the continuous one can be defined by the transform operation.

Definition 1. *Transform operation $\hat{T}$ is a combination $\hat{T} = \mathcal{F}^{-1}\mathcal{L}\mathcal{F}_\Delta$ of the operations:*

(1) The Fourier series transform:

$$\mathcal{F}_\Delta: \quad u_n(t) \to \mathcal{F}_\Delta\{u_n(t)\} = \hat{u}(k,t); \tag{9}$$

(2) The passage to the limit $\Delta x \to 0$:

$$\mathcal{L}: \quad \hat{u}(k,t) \to \mathcal{L}\{\hat{u}(k,t)\} = \tilde{u}(k,t); \tag{10}$$

(3) The inverse Fourier transform:

$$\mathcal{F}^{-1}: \quad \tilde{u}(k,t) \to \mathcal{F}^{-1}\{\tilde{u}(k,t)\} = u(x,t). \tag{11}$$

The operation $\hat{T} = \mathcal{F}^{-1}\mathcal{L}\mathcal{F}_\Delta$ is called a transform operation, since it performs a transform of a discrete model of coupled oscillators into the continuous medium model.

3. From discrete to continuous equation

Let us consider the interparticle interaction that is described by (3), where $J(n,m)$ satisfies the condition

$$J(n,m) = J(n-m) = J(m-n), \qquad \sum_{n=1}^{\infty} |J(n)|^2 < \infty. \tag{12}$$

Note that $J(-n) = J(n)$.

Definition 2. *The interaction terms (2) and (12) in the equation of motion (1) are called α-interaction if the function*

$$\hat{J}_\alpha(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn} J(n) = 2\sum_{n=1}^{\infty} J(n)\cos(kn) \tag{13}$$

satisfies the condition

$$\lim_{k\to 0} \frac{[\hat{J}_\alpha(k) - \hat{J}_\alpha(0)]}{|k|^\alpha} = A_\alpha, \tag{14}$$

where $\alpha > 0$ and $0 < |A_\alpha| < \infty$.

Condition (14) means that $\hat{J}_\alpha(k) - \hat{J}_\alpha(0) = O(|k|^\alpha)$, i.e.

$$\hat{J}_\alpha(k) - \hat{J}_\alpha(0) = A_\alpha |k|^\alpha + R_\alpha(k), \tag{15}$$

for $k \to 0$, where

$$\lim_{k\to 0} R_\alpha(k)/|k|^\alpha = 0. \tag{16}$$

Examples of functions $J(n)$ for α-interactions can be summarized in the table of the appendix.

Proposition 1. *The transform operation $\hat{T}$ maps the discrete equations of motion*

$$\frac{\partial^s u_n(t)}{\partial t^s} = g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n,m)[u_n(t) - u_m(t)] + F(u_n(t)) \tag{17}$$

with noninteger α-interaction into the fractional continuous medium equations:

$$\frac{\partial^s}{\partial t^s} u(x,t) - G_\alpha A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) - F(u(x,t)) = 0, \tag{18}$$

where $\partial^\alpha/\partial|x|^\alpha$ is the Riesz fractional derivative and $G_\alpha = g|\Delta x|^\alpha$ is a finite parameter.

Proof. To derive the equation for the field $\hat{u}(k,t)$, we multiply equation (17) by $\exp(-\mathrm{i}kn\Delta x)$, and summing over n from $-\infty$ to $+\infty$. Then,

$$\sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} \frac{\partial^s}{\partial t^s} u_n(t) = g \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)[u_n - u_m] + \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} F(u_n). \tag{19}$$

The left-hand side of (19) gives

$$\sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} \frac{\partial^s u_n(t)}{\partial t^s} = \frac{\partial^s}{\partial t^s} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} u_n(t) = \frac{\partial^s \hat{u}(k,t)}{\partial t^s}, \tag{20}$$

where $\hat{u}(k,t)$ is defined by (4). The second term on the right-hand side of (19) is

$$\sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} F(u_n) = \mathcal{F}_\Delta\{F(u_n)\}. \tag{21}$$

The first term on the right-hand side of (19) is

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)[u_n - u_m]$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)u_n - \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)u_m. \tag{22}$$

Using (4) and (24), the first term on the rhs of (22) gives

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)u_n = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} u_n \sum_{\substack{m'=-\infty \\ m'\neq 0}}^{+\infty} J(m') = \hat{u}(k,t)\hat{J}_\alpha(0), \tag{23}$$

where we use (12) and $J(m'+n,n) = J(m')$, and

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n) = \mathcal{F}_\Delta\{J(n)\}. \tag{24}$$

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)u_m = \sum_{m=-\infty}^{+\infty} u_m \sum_{\substack{n=-\infty \\ n\neq m}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)$$

$$= \sum_{m=-\infty}^{+\infty} u_m \, \mathrm{e}^{-\mathrm{i}km\Delta x} \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn'\Delta x} J(n') = \hat{u}(k,t)\hat{J}_\alpha(k\Delta x), \tag{25}$$

where we use $J(m, n'+m) = J(n')$.

As a result, equation (19) has the form

$$\frac{\partial^s \hat{u}(k,t)}{\partial t^s} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\hat{u}(k,t) + \mathcal{F}_\Delta\{F(u_n)\}, \tag{26}$$

where $\mathcal{F}_\Delta\{F(u_n)\}$ is an operator notation for the Fourier series transform of $F(u_n)$. □

The Fourier series transform $\mathcal{F}_\Delta$ of (17) gives (26). We will be interested in the limit $\Delta x \to 0$. Using (15), equation (26) can be written as

$$\frac{\partial^s}{\partial t^s}\hat{u}(k,t) - G_\alpha \hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{27}$$

where we use finite parameter $G_\alpha = g|\Delta x|^\alpha$ and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha - R_\alpha(k\Delta x)|\Delta x|^{-\alpha}. \tag{28}$$

Note that R_α satisfies the condition

$$\lim_{\Delta x\to 0} \frac{R_\alpha(k\Delta x)}{|\Delta x|^\alpha} = 0.$$

The expression for $\hat{\mathcal{T}}_{\alpha,\Delta}(k)$ can be considered as a Fourier transform of the operator (3). Note that $g \to \infty$ for the limit $\Delta x \to 0$, if G_α is a finite parameter.

The passage to the limit $\Delta x \to 0$ for the third term of (27) gives

$$\mathcal{L}: \quad \mathcal{F}_{\Delta} F(u_n) \to \mathcal{L}\mathcal{F}_{\Delta} F(u_n). \tag{29}$$

Then,

$$\mathcal{L}\mathcal{F}_{\Delta}\{F(u_n)\} = \mathcal{F}\{\mathcal{L}F(u_n)\} = \mathcal{F}\{F(\mathcal{L}u_n)\} = \mathcal{F}\{F(u(x,t))\}, \tag{30}$$

where we use $\mathcal{L}\mathcal{F}_{\Delta} = \mathcal{F}\mathcal{L}$.

As a result, equation (27) in the limit $\Delta x \to 0$ obtains

$$\frac{\partial^s}{\partial t^s}\tilde{u}(k,t) - G_\alpha \hat{\mathcal{T}}_\alpha(k)\tilde{u}(k,t) - \mathcal{F}\{F(u(x,t))\} = 0, \tag{31}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \qquad \hat{\mathcal{T}}_\alpha(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha.$$

The inverse Fourier transform of (31) gives

$$\frac{\partial^s}{\partial t^s}u(x,t) - G_\alpha \mathcal{T}_\alpha(x)u(x,t) - F(u(x,t)) = 0, \tag{32}$$

where $\mathcal{T}_\alpha(x)$ is an operator,

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}. \tag{33}$$

Here, we have used the connection between the Riesz fractional derivative and its Fourier transform [2]: $|k|^\alpha \longleftrightarrow -\partial^\alpha/\partial|x|^\alpha$.

As a result, we obtain continuous medium equations (18).

Examples of the interaction terms $J(n)$ that give the operators (33) in continuous medium equations are summarized in the following table:

$J(n)$	$\mathcal{T}_\alpha(x)$
$\left(\frac{(-1)^n \pi^{\alpha+1}}{\alpha+1} - \frac{(-1)^n \pi^{1/2}}{(\alpha+1)\lvert n\rvert^{\alpha+1/2}} L_1(\alpha+3/2, 1/2, \pi n)\right)$	$-\partial^\alpha/\partial\lvert x\rvert^\alpha$
$\frac{(-1)^n}{n^2}$	$-(1/2)\partial^2/\partial x^2$
$\frac{1}{n^2}$	$-\mathrm{i}\pi\,\partial/\partial x$
$\lvert n\rvert^{-(\beta+1)}$, $(0 < \beta < 2, \beta \neq 1)$	$-2\Gamma(-\beta)\cos(\pi\beta/2)\partial^\beta/\partial\lvert x\rvert^\beta$
$\lvert n\rvert^{-(\beta+1)}$, $(\beta > 2, \beta \neq 3, 4, \ldots)$	$\zeta(\beta-1)\partial^2/\partial x^2$
$\frac{(-1)^n}{\Gamma(1+\alpha/2+n)\Gamma(1+\alpha/2-n)}$ $(\beta > -1/2)$	$-\frac{1}{\Gamma(\alpha+1)}\partial^\alpha/\partial\lvert x\rvert^\alpha$
$\frac{(-1)^n}{a^2-n^2}$	$-\frac{a\pi}{2\sin(\pi a)}\partial^2/\partial x^2$
$J(n) = 1/n!$	$4\mathrm{ei}\partial/\partial x$

4. Fractional three-dimensional lattice equation

The generalization of the three-dimensional case can be easily realized. Let us consider the three-dimensional lattice that is described by the equations of motion

$$\frac{\partial^s u_{\mathbf{n}}}{\partial t^s} = g \sum_{\substack{\mathbf{m}=-\infty \\ \mathbf{m}\neq\mathbf{n}}}^{+\infty} J(\mathbf{n},\mathbf{m})[u_{\mathbf{n}} - u_{\mathbf{m}}] + F(u_{\mathbf{n}}), \tag{34}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ and $J(\mathbf{n},\mathbf{m}) = J(\mathbf{n}-\mathbf{m}) = J(\mathbf{m}-\mathbf{n})$. We suppose that $u_{\mathbf{n}}(t)$ are Fourier coefficients of the function $\hat{u}(\mathbf{k},t)$:

$$\hat{u}(\mathbf{k},t) = \sum_{\mathbf{n}=-\infty}^{+\infty} u_{\mathbf{n}}(t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{r}_{\mathbf{n}}} = \mathcal{F}_\Delta\{u_{\mathbf{n}}(t)\}, \tag{35}$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and

$$\mathbf{r_n} = \sum_{i=1}^{3} n_i \mathbf{a}_i .$$

Here, $\mathbf{a}_i$ are the translational vectors of the lattice. The continuous medium model can be derived in the limit $|\mathbf{a}_i| \to 0$.

To derive the equation for $\hat{u}(\mathbf{k}, t)$, we multiply (34) by $\exp(-\mathrm{i}\mathbf{k}\mathbf{r_n})$, and summing over $\mathbf{n}$. Then, we obtain

$$\frac{\partial^s \hat{u}(\mathbf{k}, t)}{\partial t^s} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(\mathbf{ka})]\hat{u}(\mathbf{k}, t) + \mathcal{F}_\Delta\{F(u_\mathbf{n})\}, \tag{36}$$

where $\mathcal{F}_\Delta\{F(u_\mathbf{n})\}$ is an operator notation for the Fourier series transform of $F(u_\mathbf{n})$ and

$$\hat{J}_\alpha(\mathbf{ka}) = \sum_{\mathbf{n}=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{r_n}} J(\mathbf{n}). \tag{37}$$

For the three-dimensional lattice, we define the α-interaction with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, as an interaction that satisfies the conditions

$$\lim_{k\to 0} \frac{[\hat{J}_\alpha(\mathbf{k}) - \hat{J}_\alpha(0)]}{|k_i|^{\alpha_i}} = A_{\alpha_i}, \qquad (i = 1, 2, 3), \tag{38}$$

where $0 < |A_{\alpha_i}| < \infty$. Conditions (38) mean that

$$\hat{J}_\alpha(\mathbf{k}) - \hat{J}_\alpha(0) = \sum_{i=1}^{3} A_{\alpha_i} |k_i|^{\alpha_i} + \sum_{i=1}^{3} R_{\alpha_i}(\mathbf{k}), \tag{39}$$

where

$$\lim_{k_i\to 0} R_{\alpha_i}(\mathbf{k})/|k_i|^{\alpha_i} = 0. \tag{40}$$

In the continuous limit ($|\mathbf{a}_i| \to 0$), the α-interaction in the three-dimensional lattice gives the continuous medium equations with the derivatives $\partial^{\alpha_1}/\partial x^{\alpha_1}$, $\partial^{\alpha_2}/\partial y^{\alpha_2}$ and $\partial^{\alpha_3}/\partial z^{\alpha_3}$:

$$\frac{\partial^s u(\mathbf{r}, t)}{\partial t^s} = -g \sum_{i=1}^{3} A_{\alpha_i} \frac{\partial^{\alpha_i} u(\mathbf{r}, t)}{\partial |x|^{\alpha_i}} + F(u(\mathbf{r}, t)). \tag{41}$$

This equation describes multifractional properties of continuous medium.

5. Linear power-law long-range interaction

Let us consider the chain with linear long-range interaction that is defined by the equation of motion

$$\frac{\partial^s u_n}{\partial t^s} = g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J(n, m)[u_n - u_m] + F(u_n), \tag{42}$$

where $J(n, m) = J(|n - m|)$ and

$$J(n) = |n|^{-(\beta+1)} \tag{43}$$

with positive integer number β.

Proposition 2. *The power-law interaction (43) for the odd number β is α-interaction with $\alpha = 1$ for $\beta = 1$, and $\alpha = 2$ for $\beta = 3, 5, 7, \ldots$. For even numbers β, (43) is not α-interaction. For odd number β, the transform operation $\hat{T}$ maps the equations of motion with the interaction (43) into the continuous medium equation (42) with derivatives of first order for $\beta = 1$,*

$$\frac{\partial^s}{\partial t^s}u(x,t) - \mathrm{i}G_1\frac{\partial}{\partial x}u(x,t) - F(u(x,t)) = 0, \qquad (44)$$

and the second order for other odd β ($\beta = 2m-1$, $m = 2, 3, 4, \ldots$),

$$\frac{\partial^s}{\partial t^s}u(x,t) - G_2\frac{\partial^2}{\partial x^2}u(x,t) - F(u(x,t)) = 0, \qquad (45)$$

where

$$G_1 = \pi g\Delta x, \qquad G_2 = \frac{(-1)^{m-1}(2\pi)^{2m-2}}{4(2m-2)!}B_{2m-2}\,g(\Delta x)^2 \qquad (46)$$

are the finite parameters.

Proof. From (26), we get the equation for $\hat{u}(k,t)$ in the form

$$\frac{\partial^s\hat{u}(k,t)}{\partial t^s} + g[\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0)]\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \qquad (47)$$

where

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x}|n|^{-(1+\beta)}. \qquad (48)$$

The function (48) can be represented by

$$\hat{J}_\alpha(k\Delta x) = \sum_{n=1}^{+\infty}\frac{1}{n^{1+\beta}}(\mathrm{e}^{-\mathrm{i}kn\Delta x} + \mathrm{e}^{\mathrm{i}kn\Delta x}) = 2\sum_{n=1}^{+\infty}\frac{1}{n^{1+\beta}}\cos(kn\Delta x). \qquad (49)$$

Then, we can use (section 5.4.2.7 in [47]) the relations

$$\sum_{n=1}^{\infty}\frac{\cos(nk)}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2(2m)!}B_{2m}\left(\frac{k}{2\pi}\right), \qquad (0 \leqslant k \leqslant 2\pi), \qquad (50)$$

where $m = 1, 2, 3, \ldots$, and $B_{2m}(z)$ are the Bernoulli polynomials [46], which are defined by

$$B_n(k) = \sum_{m=0}^{n} C_n^m B_m k^{n-m}. \qquad (51)$$

Here, B_m are the Bernoulli numbers. Note $B_{2m-1} = 0$ for $m = 2, 3, 4, \ldots$ [46]. □

For $\beta = 1$, we have

$$\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0) = \tfrac{1}{2}(k\Delta x)^2 - \pi k\Delta x \approx -\pi k\Delta x. \qquad (52)$$

For $\beta = 2m-1$ ($m = 2, 3, \ldots$),

$$\hat{J}_\alpha(k) = \frac{(-1)^{m-1}}{(2m)!}(2\pi)^{2m}B_{2m}\left(\frac{k}{2\pi}\right), \qquad (0 \leqslant k \leqslant 2\pi). \qquad (53)$$

Then,

$$\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0) \approx \frac{(-1)^{m-1}(2\pi)^{2m-2}}{4(2m-2)!}B_{2m-2}(k\Delta x)^2. \qquad (54)$$

For $\beta = 0$, we have ([47], section 5.4.2.9) the relation

$$\sum_{n=1}^{\infty} \frac{\cos(nk)}{n} = -\ln[2\sin(k/2)]. \tag{55}$$

Then, the limit $\Delta x \to 0$ gives

$$\hat{J}_\alpha(k\Delta x) \approx -\ln(k\Delta x) \to \infty. \tag{56}$$

For even numbers β,

$$|\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0)|/|k\Delta x|^\beta \to \infty, \tag{57}$$

since the expression has the logarithmic poles.

The transition to the limit $\Delta x \to 0$ in equation (47) with $\beta = 1$ gives

$$\frac{\partial^s \tilde{u}(k,t)}{\partial t^s} - G_1 k \tilde{u}(k,t) - \mathcal{F}\{F(u(x,t))\} = 0, \tag{58}$$

where $G_1 = \pi g \Delta x$ is a finite parameter. The inverse Fourier transform of (58) leads to the continuous medium equation (44) with coordinate derivative of first order. For $s = 1$, this equation can be considered as the nonlinear Schrödinger equation.

The limit $\Delta x \to 0$ in equation (47) with $\beta = 2m - 1$ $(m = 2, 3, \ldots)$ gives

$$\frac{\partial^s \tilde{u}(k,t)}{\partial t^s} + G_2 k^2 \tilde{u}(k,t) - \mathcal{F}\{F(u(x,t))\} = 0, \tag{59}$$

where G_2 is a finite parameter (46). The inverse Fourier transform of (59) leads to the partial differential equation (45) of second order. For $s = 2$, this equation can be considered as a nonlinear diffusion equation, and for $s = 1$ as a nonlinear wave equation.

Proposition 3. *The power-law interaction (43) with noninteger β is α-interaction with $\alpha = \beta$ for $0 < \beta < 2$, and $\alpha = 2$ for $\beta > 2$. For $0 < \beta < 2$ $(\beta \neq 1)$, the transform operation $\hat{T}$ maps the discrete equations with the interaction (43) into the continuous medium equation with fractional Riesz derivatives of order α:*

$$\frac{\partial^s}{\partial t^s} u(x,t) - G_\alpha A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) = F(u(x,t)), \qquad 0 < \alpha < 2, \quad (\alpha \neq 1). \tag{60}$$

For $\alpha > 2$ $(\alpha \neq 3, 4, 5, \ldots)$, the continuous medium equation has the coordinate derivatives of second order:

$$\frac{\partial^s}{\partial t^s} u(x,t) + G_\alpha \zeta(\alpha - 1) \frac{\partial^2}{\partial |x|^2} u(x,t) = F(u(x,t)), \qquad \alpha > 2, \quad (\alpha \neq 3, 4, \ldots). \tag{61}$$

Proof. For fractional positive α, the function (48) can be represented by

$$\hat{J}_\alpha(k\Delta x) = \sum_{n=1}^{+\infty} \frac{1}{n^{1+\alpha}} (\mathrm{e}^{-\mathrm{i}kn\Delta x} + \mathrm{e}^{\mathrm{i}kn\Delta x}) = Li_{1+\alpha}(\mathrm{e}^{\mathrm{i}k\Delta x}) + Li_{1+\alpha}(\mathrm{e}^{-\mathrm{i}k\Delta x}), \tag{62}$$

where $Li_\beta(z)$ is a polylogarithm function. Using the series representation of the polylogarithm [45],

$$Li_\beta(e^z) = \Gamma(1-\beta)(-z)^{\beta-1} + \sum_{n=0}^{\infty} \frac{\zeta(\beta - n)}{n!} z^n, \qquad |z| < 2\pi, \quad \beta \neq 1, 2, 3, \ldots, \tag{63}$$

we obtain

$$\hat{J}_\alpha(k\Delta x) = A_\alpha |\Delta x|^\alpha |k|^\alpha + 2\sum_{n=0}^{\infty} \frac{\zeta(1 + \alpha - 2n)}{(2n)!} (\Delta x)^{2n} (-k^2)^n, \qquad \alpha \neq 0, 1, 2, 3, \ldots, \tag{64}$$

where $\zeta(z)$ is the Riemann zeta-function, $|k\Delta x| < 2\pi$, and

$$A_\alpha = 2\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right). \tag{65}$$

From (64), we have

$$J_\alpha(0) = 2\zeta(1+\alpha).$$

Then,

$$\hat{J}_\alpha(k\Delta x) - \hat{J}_\alpha(0) = A_\alpha|\Delta x|^\alpha|k|^\alpha + 2\sum_{n=1}^{\infty}\frac{\zeta(1+\alpha-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n, \tag{66}$$

where $\alpha \neq 0, 1, 2, 3, \ldots$, and $|k\Delta x| < 2\pi$. □

Substitution of (66) into equation (47) gives

$$\frac{\partial^s\hat{u}(k,t)}{\partial t^s} + gA_\alpha|\Delta x|^\alpha|k|^\alpha\hat{u}(k,t) + 2g\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n\hat{u}(k,t)$$
$$-\mathcal{F}_\Delta\{F(u_n(t))\} = 0. \tag{67}$$

We will be interested in the limit $\Delta x \to 0$. Then, equation (67) can be written as

$$\frac{\partial^s}{\partial t^s}\hat{u}(k,t) + G_\alpha\hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \qquad \alpha \neq 0, 1, 2, \ldots, \tag{68}$$

where we use the finite parameter

$$G_\alpha = g|\Delta x|^{\min\{\alpha;2\}}, \tag{69}$$

and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = \begin{cases} A_\alpha|k|^\alpha - |\Delta x|^{2-\alpha}\zeta(\alpha-1)k^2, & 0<\alpha<2, \quad (\alpha\neq 1); \\ |\Delta x|^{\alpha-2}A_\alpha|k|^\alpha - \zeta(\alpha-1)k^2, & \alpha > 2, \quad (\alpha \neq 3, 4, \ldots). \end{cases} \tag{70}$$

The expression for $\hat{\mathcal{T}}_{\alpha,\Delta}(k)$ can be considered as a Fourier transform of the interaction operator (2). From (69), we see that $g \to \infty$ for the limit $\Delta x \to 0$, and finite value of G_α.

The transition to the limit $\Delta x \to 0$ in equation (68) gives

$$\frac{\partial^s}{\partial t^s}\tilde{u}(k,t) + G_\alpha\hat{\mathcal{T}}_\alpha(k)\tilde{u}(k,t) - \mathcal{F}\{F(u(x,t))\} = 0, \qquad (\alpha \neq 0, 1, 2, \ldots), \tag{71}$$

where

$$\hat{\mathcal{T}}_\alpha(k) = \begin{cases} A_\alpha|k|^\alpha, & 0<\alpha<2, \quad \alpha\neq 1; \\ -\zeta(\alpha-1)k^2, & 2<\alpha, \quad \alpha\neq 3, 4, \ldots. \end{cases} \tag{72}$$

The inverse Fourier transform to (71) is

$$\frac{\partial^s}{\partial t^s}u(x,t) + G_\alpha\mathcal{T}_\alpha(x)u(x,t) - F(u(x,t)) = 0, \qquad \alpha \neq 0, 1, 2, \ldots, \tag{73}$$

where

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = \begin{cases} -A_\alpha\partial^\alpha/\partial|x|^\alpha, & (0<\alpha<2, \quad \alpha\neq 1); \\ \zeta(\alpha-1)\partial^2/\partial|x|^2, & (\alpha>2, \quad \alpha\neq 3, 4, \ldots). \end{cases}$$

As a result, we obtain the continuous medium equations (60) and (61).

For $s = 1$ and $F(u) = 0$, equation (60) is the fractional kinetic equation that describes the fractional superdiffusion [12, 15, 39]. If $F(u)$ is a sum of linear and cubic terms, then equation (60) has the form of the fractional Ginzburg–Landau equation [40–44]. A remarkable property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails.

6. Nonlinear long-range interaction

Let us consider the discrete equations with nonlinear long-range interaction:

$$\hat{I}_n(u) = \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_\alpha(n,m)[f(u_n) - f(u_m)], \tag{74}$$

where $f(u)$ is a nonlinear function of $u_n(t)$ and $J_\alpha(n,m) = J_\alpha(n-m)$ defines the α-interaction. As an example of $J_\alpha(n)$, we can use

$$J_\alpha(n) = \frac{(-1)^n}{\Gamma(1+\alpha/2+n)\Gamma(1+\alpha/2-n)}. \tag{75}$$

For $\alpha = 1, 2, 3, 4$, the interactions with $f(u) = u^2$ and $f(u) = u - gu^2$ give the Burgers, Korteweg–de Vries and Boussinesq equations in the continuous limit. For fractional α in equation (75), we can obtain the fractional generalization of these equations.

Proposition 4. *The transform operation maps the equations of motion*

$$\frac{\partial^s u_n(t)}{\partial t^s} = g \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_\alpha(n-m)[f(u_n) - f(u_m)] + F(u_n), \tag{76}$$

where F is an external on-site force, and $J_\alpha(n)$ defines the α-interaction, into the continuous medium equations

$$\frac{\partial^s u(x,t)}{\partial t^s} = G_\alpha A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} f(u(x,t)) + F(u(x,t)), \tag{77}$$

where $G_\alpha = g|\Delta x|^\alpha$ is a finite parameter.

Proof. The Fourier series transform of the interaction term (74) can be represented as

$$\sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} \hat{I}_n(u) = \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)[f(u_n) - f(u_m)]$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m) f(u_n) - \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m) f(u_m). \tag{78}$$

For the first term on the rhs of (78):

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m) f(u_n) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} f(u_n) \sum_{\substack{m'=-\infty \\ m'\neq 0}}^{+\infty} J(m') = \mathcal{F}_\Delta\{f(u_n)\}\hat{J}_\alpha(0), \tag{79}$$

where we use $J(m'+n, n) = J(m')$.

For the second term on the rhs of (78):

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m) f(u_m) = \sum_{m=-\infty}^{+\infty} f(u_m) \sum_{\substack{n=-\infty \\ n\neq m}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn\Delta x} J(n,m)$$

$$= \sum_{m=-\infty}^{+\infty} f(u_m)\, \mathrm{e}^{-\mathrm{i}km\Delta x} \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} \mathrm{e}^{-\mathrm{i}kn'\Delta x} J(n') = \mathcal{F}_\Delta\{f(u_n)\}\hat{J}_\alpha(k\Delta x), \tag{80}$$

where we use $J(m, n'+m) = J(n')$.

As a result, we obtain

$$\frac{\partial^s \hat{u}(k,t)}{\partial t^s} = g[\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\mathcal{F}_\Delta\{f(u_n)\} + \mathcal{F}_\Delta\{F(u_n)\}, \tag{81}$$

where $\hat{u}(k,t) = \mathcal{F}_\Delta\{u_n(t)\}$ and $\hat{J}_\alpha(k\Delta x) = \mathcal{F}_\Delta\{J(n)\}$.

For the limit $\Delta x \to 0$, equation (81) can be written as

$$\frac{\partial^s}{\partial t^s}\hat{u}(k,t) - G_\alpha \hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n(t))\} = 0, \tag{82}$$

where we use finite parameter $G_\alpha = g|\Delta x|^\alpha$ and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha - R_\alpha(k\Delta x)|\Delta x|^{-\alpha}. \tag{83}$$

Here, the function R_α satisfies the condition

$$\lim_{\Delta x\to 0} \frac{R_\alpha(k\Delta x)}{|\Delta x|^\alpha} = 0.$$

In the limit $\Delta x \to 0$, we get

$$\frac{\partial^s}{\partial t^s}\tilde{u}(k,t) - G_\alpha \hat{\mathcal{T}}_\alpha(k)\mathcal{F}\{f(u(x,t))\} - \mathcal{F}\{F(u(x,t))\} = 0, \tag{84}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \qquad \hat{\mathcal{T}}_\alpha(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha|k|^\alpha.$$

The inverse Fourier transform of (84) gives the continuous medium equation (77). □

Let us consider examples of quadratic-nonlinear long-range interactions:

(1) The continuous limit of the lattice equation

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} J_1(n,m)\left[u_n^2 - u_m^2\right] + g_2 \sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} J_2(n,m)[u_n - u_m], \tag{85}$$

where $J_i(n)$ $(i = 1, 2)$ define the α_i-interactions with $\alpha_1 = 1$ and $\alpha_2 = 2$, gives the Burgers equation [48] that is a nonlinear partial differential equation of second order:

$$\frac{\partial}{\partial t}u(x,t) + G_1 u(x,t)\frac{\partial}{\partial x}u(x,t) - G_2\frac{\partial^2}{\partial x^2}u(x,t) = 0. \tag{86}$$

It is used in fluid dynamics as a simplified model for turbulence, boundary layer behaviour, shock wave formation and mass transport. If we consider $J_2(n,m)$ with fractional $\alpha_2 = \alpha$, then we get the fractional Burgers equation that is suggested in [49]. In general, the fractional Burgers equation is

$$\frac{\partial}{\partial t}u(x,t) + G_{\alpha_1} u(x,t)\frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}}u(x,t) - G_{\alpha_2}\frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}}u(x,t) = 0. \tag{87}$$

(2) The continuous limit of the system of equation

$$\frac{\partial u_n(t)}{\partial t} = g_1 \sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} J_1(n,m)\left[u_n^2 - u_m^2\right] + g_3 \sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty} J_3(n,m)[u_n - u_m], \tag{88}$$

where $J_i(n)$ $(i = 1, 3)$ define the α_i-interactions with $\alpha_1 = 1$ and $\alpha_3 = 3$, gives the Korteweg–de Vries (KdV) equation

$$\frac{\partial}{\partial t}u(x,t) - G_1 u(x,t)\frac{\partial}{\partial x}u(x,t) + G_3\frac{\partial^3}{\partial x^3}u(x,t) = 0. \tag{89}$$

First formulated as a part of an analysis of shallow-water waves in canals, it has subsequently been found to be involved in a wide range of physics phenomena, especially those exhibiting shock waves, travelling waves and solitons. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behaviour and mass transport.

If we use noninteger α_i-interactions for $J_i(n)$, then we get the fractional generalization of the KdV equation [50, 51]:

$$\frac{\partial}{\partial t}u(x,t) - G_{\alpha_1}u(x,t)\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}}u(x,t) + G_{\alpha_3}\frac{\partial^{\alpha_3}}{\partial x^{\alpha_3}}u(x,t) = 0. \tag{90}$$

(3) The continuous limit of the equation

$$\frac{\partial^2 u_n(t)}{\partial t^2} = g_2 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_2(n,m)[f(u_n) - f(u_m)] + g_4 \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} J_4(n,m)[u_n - u_m], \tag{91}$$

where

$$f(u) = u - gu^2,$$

and $J_i(n)$ define the α_i-interactions with $\alpha_2 = 2$ and $\alpha_4 = 4$, gives the Boussinesq equation that is a nonlinear partial differential equation of fourth order:

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_2\frac{\partial^2}{\partial x^2}u(x,t) + gG_2\frac{\partial^2}{\partial x^2}u^2(x,t) + G_4\frac{\partial^4}{\partial x^4}u(x,t) = 0. \tag{92}$$

This equation was formulated as a part of an analysis of long waves in shallow water. It was subsequently applied to problems in the percolation of water in porous subsurface strata. It also crops up in the analysis of many other physical processes. The fractional Boussinesq equation is

$$\frac{\partial^2}{\partial t^2}u(x,t) - G_{\alpha_2}\frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}}u(x,t) + gG_{\alpha_2}\frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}}u^2(x,t) + G_{\alpha_4}\frac{\partial^{\alpha_4}}{\partial x^{\alpha_4}}u(x,t) = 0. \tag{93}$$

7. Conclusion

Discrete system of long-range interacting oscillators serves as a model for numerous applications in physics, chemistry, biology, etc. Long-range interactions are important types of interactions for complex media. An interesting situation arises when we consider a wide class of α-interactions, where α is noninteger. A remarkable feature of these interactions is the existence of a transform operation that replaces the set of coupled individual oscillator equations by the continuous medium equation with the space derivative of noninteger order α. Such a transform operation is an approximation that appears in the continuous limit. This limit allows us to consider different models in unified way by applying tools of fractional calculus.

We can assume that an asymmetric interaction term $(J(n-m) \neq J(|n-m|))$ leads to other forms of the fractional derivative [2].

Note that a fractional derivative can result from a fractional difference as interaction term, just as an nth order difference leads to an nth derivative [2]. It follows from the representation of the Riesz fractional derivative by Grunwald–Letnikov fractional derivative:

$$\frac{\partial^\alpha}{\partial|x|^\alpha}u(x,t) \simeq -\frac{1}{2\cos(\pi\alpha/2)}\frac{1}{h^\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}[u(x-nh,t)+u(x+nh,t)], \tag{94}$$

where $h = \Delta x$ is the discretization parameter.

A similar approach to fractional dynamics in the context of the diffusion equation was developed in the papers [52, 53]. In those papers, a continuum limit of (non-interacting) random particle motions leads to a fractional equation.

Appendix

$J(n)$	$\hat{J}_\alpha(k) - \hat{J}_\alpha(0)$
$\left(\frac{(-1)^n\pi^{\alpha+1}}{\alpha+1} - \frac{(-1)^n\pi^{1/2}}{(\alpha+1)\lvert n\rvert^{\alpha+1/2}}L_1(\alpha+3/2, 1/2, \pi n)\right)$	$\lvert k\rvert^\alpha$
$\frac{(-1)^n}{n^2}$	$(1/2)k^2$
$\frac{1}{n^2}$	$\frac{1}{2}[k^2 - 2\pi k]$, $(0 \leqslant k \leqslant 2\pi)$
$\lvert n\rvert^{-(\beta+1)}$, $(0 < \beta < 2, \beta \neq 1)$	$2\Gamma(-\beta)\cos(\pi\beta/2)\lvert k\rvert^\beta$
$\lvert n\rvert^{-(\beta+1)}$, $(\beta > 2, \beta \neq 3, 4, \ldots)$	$-\zeta(\alpha-1)k^2$
$\frac{(-1)^n}{\Gamma(1+\alpha/2+n)\Gamma(1+\alpha/2-n)}$, $(\beta > -1/2)$	$\frac{2^\alpha}{\Gamma(\alpha+1)}\sin^\alpha\left(\frac{k}{2}\right)$
$\frac{(-1)^n}{a^2-n^2}$	$\frac{\pi}{a\sin(\pi a)}\cos(ak) - \frac{1}{a^2}$, $(0 < k < 2\pi)$
$J(n) = 1/n!$	$e^{\cos k}\cos(\sin k)$, $\lvert k\rvert < \infty$

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