

# Fractional equations of kicked systems and discrete maps

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## Abstract

Starting from kicked equations of motion with derivatives of non-integer orders, we obtain ‘fractional’ discrete maps. These maps are generalizations of well-known universal, standard, dissipative, kicked damped rotator maps. The main property of the suggested fractional maps is a long-term memory. The memory effects in the fractional discrete maps mean that their present state evolution depends on all past states with special forms of weights. These forms are represented by combinations of power-law functions.

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## 1. Introduction

There are a number of distinct areas of physics where the basic problems can be reduced to the study of simple discrete maps. Discrete maps have been used for the study of evolution problems, possibly as a substitute of differential equations [1–4]. They lead to a much simpler formalism, which is particularly useful in simulations. The standard map is one of the most widely studied maps. In this paper, we consider fractional generalizations of discrete maps that can be used to study the evolution described by fractional differential equations [5–7].

The treatment of nonlinear dynamics in terms of discrete maps is a very important step in understanding the qualitative behavior of continuous systems described by differential equations. Note that the continuous limit of discrete systems with long-range interactions gives differential equations with derivatives of non-integer orders (see for example, [8–10]). The derivatives of non-integer orders are a natural generalization of the ordinary differentiation of integer order. Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems. The application of memory effects to discrete maps means that their present state evolution depends on all past states [11–16].

The mapping  $x_{n+1} = f(x_n)$  does not have any memory, as the value  $x_{n+1}$  only depends on  $x_n$ . The introduction of memory means that the discrete value  $x_{n+1}$  is connected with the previous values  $x_n, x_{n-1}, \dots, x_1$ . Particularly, any system, which is described by a discrete map, will have a full memory [16], if each state of the system is a simple sum of all previous states:

$$x_{n+1} = \sum_{k=1}^n f(x_k), \quad (1)$$

where  $f(x)$  is a function that defines the discrete map. In general, the expression of equation (1) can tend to infinity. Note that the full memory exists for functions that give a finite sum in equation (1). The full memory is ideal because it has the same action upon the next states as all the others in memory. The map with a long-term memory can be expressed as

$$x_{n+1} = \sum_{k=1}^n V_\alpha(n, k) f(x_k), \quad (2)$$

where the weights  $V_\alpha(n, k)$ , and the parameter  $\alpha$  characterize the non-ideal memory effects. The forms of the functions  $f(x)$  and  $V_\alpha(n, k)$  in equation (2) are obtained by the differential equation. Therefore the conditions on  $f(x)$  are not discussed. Note that linear differential (kicked) equations give the linear function  $f(x) = x$ . The discrete maps with memory are considered, for example, in the papers [11–16]. The important question is a connection of fractional equation of motion and the discrete maps with memory. It is important to derive discrete maps with memory from equation of motion with fractional derivatives.

Here, we point out the well-known notions such as the full memory and the long-term memory (LTM). For the one-dimensional case, the full memory is defined by equation (1), and the LTM is defined by equation (2). In general, the expression of equation (1) can tend to infinity. Note that the full memory exists for functions that give a finite sum. This is a basic condition on  $f(x)$  in equation (1). In this paper, we consider the LTM for fractional equations. For the one-dimensional case, the LTM is defined by equation (2). The one-dimensional case is considered for simplification. In this paper, we consider a two-dimensional case of variables  $(x_n, p_n)$ . Note that the conditions on the function  $f(x)$  of the LTM are not defined *a priori*. This function is obtained from an equation of motion. Therefore the conditions on  $f(x)$  are not discussed. The properties of the function  $f(x)$  are defined by the kicked differential equation. For example, the functions  $f(x)$ , which are derived from linear differential (kicked) equations, are linear ( $f(x) = x$ ).

It was shown [17] that perturbed by a periodic force, the nonlinear system with a fractional derivative exhibits a new type of chaotic motion called the fractional chaotic attractor. The fractional discrete maps allow us to study new types of attractors that are called pseudochaotic [17].

In section 2, a brief review of discrete maps is considered to fix notation and provide convenient references. In section 3, a fractional generalization of the universal map is obtained from kicked fractional equations of motion with order  $1 < \alpha \leq 2$ . We prove that the usual universal map is a special case of the fractional universal map. Some examples of the fractional universal map are suggested. In section 4, a fractional universal map for the case of  $\alpha > 2$  is obtained. In section 5, a fractional kicked damped rotator map is derived. Finally, a short conclusion is given in section 6.

## 2. Universal and standard map

In this section, a brief review of well-known discrete maps is considered to fix notation and provide convenient references. For details, see [1–4].

Let us consider the equations of motion

$$\ddot{x} + KG(x) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) = 0 \quad (3)$$

in which perturbation is a periodic sequence of delta-function-type pulses (kicks) following with period  $T = 2\pi/\nu$ ,  $K$  is an amplitude of the pulses, and  $G(x)$  is some function. This equation can be presented in the Hamiltonian form

$$\dot{x} = p, \quad \dot{p} + KG(x) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) = 0. \quad (4)$$

It is well known that these equations can be represented in the form of discrete map (see, for example, [4]). Between any two kicks there is a free motion

$$p = \text{const}, \quad x = pt + \text{const}. \quad (5)$$

The solution of the left side of the  $n$ th kick

$$x_n = x(t_n - 0) = \lim_{\varepsilon \rightarrow 0^+} x(nT - \varepsilon), \quad p_n = p(t_n - 0) = \lim_{\varepsilon \rightarrow 0^+} p(nT - \varepsilon), \quad t_n = nT \quad (6)$$

is connected with the solution on the right-hand side of the kick  $x(t_n + 0)$ ,  $p(t_n + 0)$  by equation (4), and the condition of continuity  $x(t_n + 0) = x(t_n - 0)$ . The integration of (4) over the interval  $(t_n - \varepsilon, t_n + \varepsilon)$  gives

$$p(t_n + 0) = p(t_n - 0) - KTG(x_n).$$

Using notation (6), and the solution (5), we can derive the iteration equations

$$x_{n+1} = x_n + p_{n+1}T, \quad p_{n+1} = p_n - KTG(x_n). \quad (7)$$

Equations (7) are called the universal map.

If  $G(x) = -x$ , then equations (7) give the Anosov-type system

$$x_{n+1} = x_n + p_{n+1}T, \quad p_{n+1} = KT x_n + p_n. \quad (8)$$

For  $G(x) = \sin(x)$ , equations (7) are

$$x_{n+1} = x_n + p_{n+1}T, \quad p_{n+1} = p_n - KT \sin(x_n). \quad (9)$$

This map is known as the standard or Chirikov–Taylor map [1].

## 3. Fractional generalization of the universal map for $1 < \alpha \leq 2$

In this section, a fractional generalization of the differential equation (3) is suggested. The discrete map that corresponds to the fractional equation of order  $1 < \alpha \leq 2$  is derived. This map can be considered as a generalization of the universal map for the case  $1 < \alpha \leq 2$ .

Let us consider a fractional generalization of (3) in the form

$${}_0D_t^\alpha x + KG(x) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) = 0 \quad (1 < \alpha \leq 2), \quad (10)$$

where  ${}_0D_t^\alpha$  is the Riemann–Liouville fractional derivative [5–7], which is defined by

$${}_0D_t^\alpha x = D_{t_0}^2 I_t^{2-\alpha} x = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{\alpha-1}} \quad (1 < \alpha \leq 2). \quad (11)$$

Here we use the notation  $D_t^2 = d^2/dt^2$ , and  ${}_0I_t^\alpha$  is a fractional integration [5–7].

**Proposition 1.** *The fractional differential equation of the kicked system (10) is equivalent to the discrete map*

$$p_{n+1} = p_n - KTG(x_n), \quad (12)$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^n p_{k+1} V_\alpha(n-k), \quad (1 < \alpha \leq 2), \quad (13)$$

where the function  $V_\alpha(z)$  is defined by

$$V_\alpha(z) = z^{\alpha-1} - (z-1)^{\alpha-1}. \quad (14)$$

**Proof.** Let us define the variable  $\xi(t)$  such that

$${}_0^C D_t^{2-\alpha} \xi = x(t), \quad (15)$$

where  ${}_0^C D_t^{2-\alpha}$  is the Caputo fractional derivative

$${}_0^C D_t^{2-\alpha} \xi = {}_0I_t^{\alpha-1} D_t^1 \xi = \frac{1}{\Gamma(\alpha-1)} \int_0^t \frac{d\tau}{(t-\tau)^{2-\alpha}} \frac{d\xi(\tau)}{d\tau} \quad (0 \leq 2-\alpha < 1). \quad (16)$$

Using Lemma 2.22 of [7], we get

$${}_0I_t^{2-\alpha} {}_0^C D_t^{2-\alpha} \xi = \xi(t) - \xi(0). \quad (17)$$

Then

$${}_0D_t^\alpha x = D_{t_0}^2 {}_0I_t^{2-\alpha} x = D_{t_0}^2 {}_0I_t^{2-\alpha} {}_0^C D_t^{2-\alpha} \xi = D_t^2 (\xi(t) - \xi(0)) = D_t^2 \xi. \quad (18)$$

Substitution of (18) and (15) into equation (10) gives

$$D_t^2 \xi + KG({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) = 0, \quad (1 < \alpha \leq 2). \quad (19)$$

This fractional equation can be presented in the Hamiltonian form

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} + KG({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) &= 0. \end{aligned} \quad (20)$$

Between any two kicks there is a free motion

$$\eta = \text{const}, \quad \xi = \eta t + \text{const}. \quad (21)$$

The integration of (20) over  $(t_n + \varepsilon, t_{n+1} - \varepsilon)$  gives

$$\xi(t_{n+1} - 0) = \xi(t_n + 0) - \eta_{n+1} T, \quad \eta(t_{n+1} - 0) = \eta(t_n + 0). \quad (22)$$

The solution of the left side of the  $n$ th kick

$$\xi_n = \xi(t_n - 0) = \lim_{\varepsilon \rightarrow 0} \xi(nT - \varepsilon), \quad \eta_n = \eta(t_n - 0) = \lim_{\varepsilon \rightarrow 0} \eta(nT - \varepsilon), \quad t_n = nT \quad (23)$$

is connected with the solution on the right-hand side of the kick  $\xi(t_n + 0), \eta(t_n + 0)$  by equation (20), and the continuity condition

$$\xi(t_n + 0) = \xi(t_n - 0). \tag{24}$$

The integration of (20) over the interval  $(t_n - \varepsilon, t_n + \varepsilon)$  gives

$$\eta(t_n + 0) = \eta(t_n - 0) - KTG({}_0^C D_{t_n}^{2-\alpha} \xi). \tag{25}$$

Using notation (23), and the solution (21), we get

$$\eta(t_n + 0) = \eta(t_{n+1} - 0) = \eta_{n+1}. \tag{26}$$

Substituting (24) and (25) into (22), we derive the iteration equations

$$\xi_{n+1} = \xi_n + \eta_{n+1}T, \tag{27}$$

$$\eta_{n+1} = \eta_n - KTG({}_0^C D_{t_n}^{2-\alpha} \xi). \tag{28}$$

To derive a map, we should express the fractional derivative

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{1}{\Gamma(\alpha - 1)} \int_0^{t_n} \frac{d\tau}{(t_n - \tau)^{2-\alpha}} \frac{d\xi(\tau)}{d\tau} \tag{29}$$

through the variables (23). Using  $\dot{\xi} = \eta$ , we have

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{1}{\Gamma(\alpha - 1)} \int_0^{t_n} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{2-\alpha}}. \tag{30}$$

It can be presented as

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{1}{\Gamma(\alpha - 1)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{2-\alpha}}, \tag{31}$$

where  $t_{k+1} = t_k + T = (k + 1)T, t_k = kT$  and  $t_0 = 0$ .

For the interval  $(t_k, t_{k+1})$ , equations (21) and (23) give

$$\eta(\tau) = \eta(t_k + 0) = \eta(t_{k+1} - 0) = \eta_{k+1}, \quad \tau \in (t_k, t_{k+1}). \tag{32}$$

Then

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{2-\alpha}} &= \eta_{k+1} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha-2} d\tau \\ &= \eta_{k+1} \int_{t_n-t_{k+1}}^{t_n-t_k} z^{\alpha-2} dz = \eta_{k+1} \frac{z^{\alpha-1}}{\alpha-1} \Big|_{t_n-t_{k+1}}^{t_n-t_k} \\ &= \frac{1}{\alpha-1} \eta_{k+1} [(t_n - t_k)^{\alpha-1} - (t_n - t_{k+1})^{\alpha-1}] \\ &= \eta_{k+1} \frac{T^{\alpha-1}}{\alpha-1} [(n-k)^{\alpha-1} - (n-k-1)^{\alpha-1}]. \end{aligned} \tag{33}$$

Using  $(\alpha - 1)\Gamma(\alpha - 1) = \Gamma(\alpha)$  and equation (33), the fractional derivative (31) can be presented as

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \eta_{k+1} V_\alpha(n-k), \quad (1 < \alpha \leq 2), \tag{34}$$

where

$$V_\alpha(z) = z^{\alpha-1} - (z-1)^{\alpha-1}. \tag{35}$$

As a result, equation (28) takes the form

$$\xi_{n+1} = \xi_n + \eta_{n+1}T, \tag{36}$$

$$\eta_{n+1} = \eta_n - KTG \left( \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \eta_{k+1} V_\alpha(n-k) \right), \quad (1 < \alpha \leq 2). \tag{37}$$

These equations define the fractional generalization of the universal map for variables  $(\xi_n, \eta_n)$ .

The fractional equation (10) in the Hamiltonian form can be presented as

$$\dot{p} + KG(x) \sum_{n=0}^{\infty} \delta \left( \frac{t}{T} - n \right) = 0, \tag{38}$$

$${}_0D_t^{\alpha-1}x = p, \quad (1 < \alpha \leq 2), \tag{39}$$

where we use  ${}_0D_t^\alpha = D_t^1{}_0D_t^{\alpha-1}$ .

Equations (15) and (34) give

$$x_n = {}_0^C D_{t_n}^{2-\alpha} \xi = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \eta_{k+1} V_\alpha(n-k). \tag{40}$$

Using equations (17) and (39), we obtain

$$p = {}_0D_t^{\alpha-1}x = D_t^1{}_0I_t^{2-\alpha}x = D_t^1{}_0I_t^{2-\alpha}{}_0^C D_t^{2-\alpha} \xi = D_t^1(\xi(t) - \xi(0)) = D_t^1 \xi. \tag{41}$$

The definition of  $\eta$  in (20) and equation (41) give

$$p = \dot{\xi} = \eta, \quad p_n = \eta_n.$$

As a result, equations (37) and (40) give

$$p_{n+1} = p_n - KTG(x_n), \tag{42}$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^n p_{k+1} V_\alpha(n-k), \quad (1 < \alpha \leq 2), \tag{43}$$

where  $V_\alpha(z)$  is defined in (35).

This ends the proof. □

Equations (42) and (43) define the fractional universal map in the phase space  $(x_n, p_n)$ . These equations are the generalization of the map (7).

Note that the form of equations (43) is defined by both equations (38) and (39). Equation (43) cannot be considered as an iteration representation of equation (39) only. If we use the other form of equation (38), then equation (43) is changed.

Let us consider some examples of fractional universal map (36) and (37).

**Example 1.** Let us prove that the fractional universal map for  $\alpha = 2$  gives the usual universal map. Substitution of equation (36) in the form

$$\eta_{k+1} = \frac{1}{T} (\xi_{k+1} - \xi_k) \tag{44}$$

into the iteration equation (37) gives

$$\eta_{n+1} = \eta_n - KTG \left( \frac{T^{\alpha-2}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} (\xi_{k+1} - \xi_k) V_\alpha(n-k) \right). \tag{45}$$

Then, the fractional map (36), (37) is defined by

$$\begin{aligned} \xi_{n+1} &= \xi_n + \eta_{n+1}T, \\ \eta_{n+1} &= \eta_n - KTG \left( \frac{T^{\alpha-2}}{\Gamma(\alpha)} \sum_{k=1}^n (\xi_k - \xi_{k-1}) V_\alpha(n-k) \right), \quad (1 < \alpha \leq 2). \end{aligned} \tag{46}$$

For  $\alpha = 2$ , we have  $V_\alpha(z) = 1$ , and

$$\sum_{k=0}^{n-1} (\xi_{k+1} - \xi_k) = \xi_n - \xi_0. \tag{47}$$

Then equation (45) gives

$$\eta_{n+1} = \eta_n - KTG (\xi_n - \xi_0). \tag{48}$$

As a result, equations (46) with  $\alpha = 2$  give the usual universal map (7) for the variables  $x_n = \xi_n - \xi_0$  and  $p_n = \eta_n$ .

**Example 2.** If  $G(x) = -x$ , then equations (36) and (37) are

$$\begin{aligned} \xi_{n+1} &= \xi_n + \eta_{n+1}T, \\ \eta_{n+1} &= \eta_n - K \frac{T^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \eta_{k+1} V_\alpha(n-k), \quad (1 < \alpha \leq 2), \end{aligned} \tag{49}$$

where  $V_\alpha(z)$  is defined in (35). For the space  $(x_n, p_n)$ , this generalization can be presented as

$$p_{n+1} = p_n - K \frac{T^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{n-1} p_{k+1} V_\alpha(n-k), \tag{50}$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^n p_{k+1} V_\alpha(n-k). \tag{51}$$

Equations (50) and (51) define a fractional generalization of the Anosov-type system.

**Example 3.** If  $G(x) = \sin(x)$ , equations (36) and (37) give

$$\begin{aligned} \xi_{n+1} &= \xi_n + \eta_{n+1}T, \\ \eta_{n+1} &= \eta_n - KT \sin \left( \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \eta_{k+1} V_\alpha(n-k) \right), \quad (1 < \alpha \leq 2). \end{aligned} \tag{52}$$

This map can be considered as a fractional generalization of the standard map. The other possible form of equation (52) is

$$\begin{aligned} \xi_{n+1} &= \xi_n + \eta_{n+1}, \\ \eta_{n+1} &= \eta_n - K \sin \left( \frac{1}{\Gamma(\alpha)} \sum_{k=1}^n (\xi_k - \xi_{k-1}) [(n-k)^{\alpha-1} - (n-k-1)^{\alpha-1}] \right), \end{aligned} \tag{53}$$

where we use equations (44), (35),  $T = 1$ , and  $1 < \alpha \leq 2$ . For  $(x_n, p_n)$ , equations (42) and (43) with  $G(x) = \sin(x)$  give

$$p_{n+1} = p_n - KT \sin(x_n), \tag{54}$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^n p_{k+1} V_\alpha(n-k), \quad (1 < \alpha \leq 2). \tag{55}$$

These equations define a fractional standard map on the phase space, which can be called the fractional Chirikov–Taylor map.

**Example 4.** The fractional generalization of the dissipative standard map [18, 19] can be defined by

$$\xi_{n+1} = \xi_n + \eta_{n+1}, \tag{56}$$

$$\eta_{n+1} = b\eta_n - K \sin \left( \frac{1}{\Gamma(\alpha)} \sum_{k=1}^n (\xi_k - \xi_{k-1}) [(n-k)^{\alpha-1} - (n-k-1)^{\alpha-1}] \right). \tag{57}$$

For  $b = 1$ , we get the fractional standard map (53).

#### 4. The fractional universal map for $\alpha > 2$

In this section, a fractional differential equation (10) is used for  $\alpha > 2$ . The discrete maps that correspond to the fractional equations are derived. These maps can be considered as a generalization of the universal map for the case  $\alpha > 2$ , i.e., the fractional universal map (36), (37) can be generalized from  $1 < \alpha \leq 2$  to  $\alpha > 2$ .

Let us consider the fractional equation

$${}_0D_t^\alpha x + KG(x) \sum_{n=0}^{\infty} \delta \left( \frac{t}{T} - n \right) = 0, \quad (m-1 < \alpha \leq m), \tag{58}$$

where  ${}_0D_t^\alpha$  is the Riemann–Liouville fractional derivative of order  $\alpha$ ,  $m-1 < \alpha \leq m$ , which is defined [5–7] by

$${}_0D_t^\alpha x = D_t^m {}_0I_t^{m-\alpha} x = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{\alpha-1}}, \quad (m-1 < \alpha \leq m). \tag{59}$$

Here we use the notation  $D_t^m = d^m/dt^m$ , and  ${}_0I_t^{m-\alpha}$  is a fractional integration [5–7].

**Proposition 2.** *The fractional differential equation of the kicked system (58) is equivalent to the discrete map*

$$x_n = \frac{1}{\Gamma(\alpha - m + 1)} \sum_{k=0}^{n-1} \left( \sum_{l=0}^{m-3} \frac{T^{\alpha+l-m+1}}{l!} \eta_n^{l+1} V_\alpha^{m,l}(n-k) + \frac{T^{\alpha-1}}{(m-2)!} \eta_{n+1}^{m-1} V_\alpha^{m,m-2}(n-k) \right), \tag{60}$$

$$p_{n+1}^s = p_n^s + \sum_{l=1}^{m-s-2} \frac{T^l}{l!} p_n^{s+l} + \frac{T^{m-s-1}}{(m-s-1)!} p_{n+1}^{m-1}, \quad (s = 1, \dots, m-2), \tag{61}$$

$$p_{n+1}^{m-1} = p_n^{m-1} - KTG(x_n), \quad (m-1 < \alpha \leq m), \tag{62}$$

where the functions  $V_\alpha^{m,l}(z)$  are defined by

$$V_\alpha^{m,l}(z) = \int_{z-1}^z (z-y)^l y^{\alpha-m} dy = \int_0^1 y^l (z-y)^{\alpha-m} dy, \quad l = 1, \dots, m-2, \tag{63}$$

and  $1 \leq l \leq m-1 < \alpha \leq m$ ,  $l, m \in \mathbb{N}$ .

**Proof.** Let us consider equation (10) with  $m - 1 < \alpha \leq m$ . Using  $\xi = \xi(t)$ , such that

$${}_0^C D_t^{m-\alpha} \xi = x(t), \quad (m - 1 < \alpha \leq m), \tag{64}$$

we obtain

$${}_0 D_t^\alpha x = {}_0 D_t^{\alpha C} D_t^{m-\alpha} \xi = D_t^m I_t^{m-\alpha C} D_t^{m-\alpha} \xi = D_t^m \xi, \quad (0 < m - \alpha < 1).$$

Equation (10) can be presented as

$$D_t^m \xi + K G({}_0^C D_t^{m-\alpha} \xi) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) = 0, \quad (m - 1 < \alpha < m). \tag{65}$$

Let us define

$$\eta^s(t) = D_t^s \xi(t), \quad (s = 0, \dots, m - 1). \tag{66}$$

Then the Hamiltonian form of equation (65) is

$$\begin{aligned} \dot{\eta}^s &= \eta^{s+1}, \quad (s = 0, \dots, m - 2), \\ \dot{\eta}^{m-1} + K T G({}_0^C D_t^{m-\alpha} \eta^0) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - n\right) &= 0. \end{aligned} \tag{67}$$

For  $t \in (t_n + 0, t_{n+1} - 0)$ , equation (65) is

$$D_t^m \xi = 0,$$

and the solution can be presented as

$$\xi(t) = \sum_{l=0}^{m-1} C_l (t - t_n)^l, \quad (m \geq 3). \tag{68}$$

Substitution of (68) into (66) gives

$$\eta^s(t) = \sum_{l=s}^{m-1} C_l l(l-1) \dots (l-s+1) (t - t_n)^{l-s}. \tag{69}$$

For  $t = t_n$ , we have

$$\eta^s(t_n + 0) = C_s s!.$$

Then

$$C_l = \frac{1}{l!} \eta^s(t_n + 0). \tag{70}$$

Using equation (70), the relations

$$\eta^s(t_n + 0) = \eta^s(t_n - 0) = \eta_n^s, \quad (s = 0, \dots, m - 2),$$

and

$$\eta^{m-1}(t_n + 0) = \eta^{m-1}(t_{n+1} - 0) = \eta_{n+1}^{m-1},$$

we present equation (69) in the form

$$\begin{aligned} \eta^s(t) &= \sum_{l=s}^{m-2} \frac{1}{(l-s)!} \eta_n^l (t - t_n)^{l-s} + \frac{1}{(m-s-1)!} \eta_{n+1}^{m-1} (t - t_n)^{m-s-1}, \\ (s &= 0, \dots, m - 2). \end{aligned} \tag{71}$$

Here, we use

$$\frac{l(l-1) \dots (l-s+1)}{l!} = \frac{1}{(l-s)!}. \tag{72}$$

Equation (71) can be rewritten as

$$\eta^s(t) = \sum_{l=0}^{m-s-2} \frac{1}{l!} \eta_n^{s+l} (t - t_n)^l + \frac{1}{(m-s-1)!} \eta_{n+1}^{m-1} (t - t_n)^{m-s-1}, \quad (s = 0, \dots, m-2). \tag{73}$$

For  $t = t_{n+1}$ , this equation gives

$$\eta_{n+1}^s = \sum_{l=0}^{m-s-2} \frac{1}{l!} \eta_n^{s+l} T^l + \frac{1}{(m-s-1)!} \eta_{n+1}^{m-1} T^{m-s-1}, \quad (s = 0, \dots, m-2). \tag{74}$$

As a result, the iteration equations are

$$\eta_{n+1}^s = \sum_{l=0}^{m-s-2} \frac{1}{l!} \eta_n^{s+l} T^l + \frac{1}{(m-s-1)!} \eta_{n+1}^{m-1} T^{m-s-1}, \quad (s = 0, \dots, m-2), \tag{75}$$

$$\eta_{n+1}^{m-1} = \eta_n^{m-1} - K T G({}_0^C D_{t_n}^{m-\alpha} \eta^0). \tag{76}$$

The Caputo fractional derivative

$${}_0^C D_{t_n}^{m-\alpha} \eta^0 = {}_0 I_{t_n}^{\alpha-m+1} D_t^1 \eta^0 = {}_0 I_{t_n}^{\alpha-m+1} \eta^1$$

can be presented as

$$x_n = {}_0^C D_{t_n}^{m-\alpha} \xi = \frac{1}{\Gamma(\alpha - m + 1)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\eta^1(\tau) d\tau}{(t_n - \tau)^{m-\alpha}}, \tag{77}$$

where  $t_k = kT$ .

For  $t \in (t_k, t_{k+1})$ , equation (73) is

$$\eta^1(t) = \sum_{l=0}^{m-3} \frac{1}{l!} \eta_n^{l+1} (t - t_k)^l + \frac{1}{(m-2)!} \eta_{n+1}^{m-1} (t - t_k)^{m-2}. \tag{78}$$

Substituting (78) into (77), and using

$$\int_{t_k}^{t_{k+1}} \frac{(t - t_k)^l}{(t_n - t)^{m-\alpha}} dt = T^{\alpha-m+l+1} \int_k^{k+1} \frac{(\tau - k)^l}{(n - \tau)^{m-\alpha}} d\tau,$$

we obtain

$$\int_{t_k}^{t_{k+1}} \frac{\eta^1(t) dt}{(t_n - t)^{m-\alpha}} = \sum_{l=0}^{m-3} \frac{T^{\alpha+l-m+1}}{l!} \eta_n^{l+1} V_\alpha^{m,l}(n-k) + \frac{T^{\alpha-1}}{(m-2)!} \eta_{n+1}^{m-1} V_\alpha^{m,m-2}(n-k), \tag{79}$$

where

$$V_\alpha^{m,l}(n-k) = \int_k^{(k+1)} \frac{(z-k)^l}{(n-z)^{m-\alpha}} dz, \quad l = 1, \dots, m-2. \tag{80}$$

These functions can be defined by

$$V_\alpha^{m,l}(z) = \int_{z-1}^z (z-y)^l y^{\alpha-m} dy = \int_0^1 y^l (z-y)^{\alpha-m} dy, \quad l = 1, \dots, m-2, \tag{81}$$

where  $1 \leq l \leq m-1 < \alpha \leq m, l, m \in \mathbb{N}$ .

Let us use equation (64) in the form

$$x_n = {}_0^C D_{t_n}^{m-\alpha} \xi, \quad p_n^s = \eta_n^s, \quad (s = 1, \dots, m-1).$$

As a result, we obtain

$$x_n = \frac{1}{\Gamma(\alpha - m + 1)} \sum_{k=0}^{n-1} \left( \sum_{l=0}^{m-3} \frac{T^{\alpha+l-m+1}}{l!} \eta_n^{l+1} V_\alpha^{m,l}(n-k) + \frac{T^{\alpha-1}}{(m-2)!} \eta_{n+1}^{m-1} V_\alpha^{m,m-2}(n-k) \right), \tag{82}$$

$$p_{n+1}^s = p_n^s + \sum_{l=1}^{m-s-2} \frac{T^l}{l!} p_n^{s+l} + \frac{T^{m-s-1}}{(m-s-1)!} p_{n+1}^{m-1}, \quad (s = 1, \dots, m-2), \tag{83}$$

$$p_{n+1}^{m-1} = p_n^{m-1} - KTG(x_n), \quad (m-1 < \alpha \leq m). \tag{84}$$

This ends the proof. □

**Remark 1.** Equations (82)–(84) define a fractional generalization of the universal map for  $\alpha > 2$ . For  $G(x_n) = x_n$ , equations (82)–(84) define the fractional Anosov-type system with  $\alpha > 2$ . For  $G(x_n) = \sin x_n$ , we have the fractional standard map for  $\alpha > 2$ .

**Remark 2.** The functions that are defined by integrals (81) can be expressed in elementary functions. For example, equation (80) with  $l = 0$  gives

$$V_\alpha^{m,0}(n-k) = \frac{1}{\alpha - m + 1} [(n-k)^{\alpha-m+1} - (n-k-1)^{\alpha-m+1}]. \tag{85}$$

This equation for  $m = 2$  can be presented as

$$V_\alpha^{2,0}(n-k) = \frac{1}{\alpha - 1} V_\alpha(n-k).$$

For  $l = 1$ , equation (80) gives

$$V_\alpha^{m,1}(n-k) = \frac{1}{(\alpha - m + 1)(\alpha - m + 2)} \times [(n-k)^{\alpha-m+2} - (n-k-1)^{\alpha-m+1}(n-k + \alpha - m + 1)]. \tag{86}$$

**Remark 3.** Note that the function (81) can be presented through the hypergeometric function (section 2.1.3 of [21])  $F(a, b, c; z)$  by the relation

$$V_\alpha^{m,l}(z) = \frac{1}{(l+1)z^{m-\alpha}} F(m-\alpha, l+1, l+2; z^{-1}), \tag{87}$$

where

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt,$$

and we use

$$\frac{\Gamma(l+1)}{\Gamma(l+2)} = \frac{l!}{(l+1)!} = \frac{1}{l+1}.$$

### 5. Fractional kicked damped rotator (FKDR) map

In this section, a fractional generalization of the differential equation for a kicked damped rotator is suggested. The discrete map that corresponds to the fractional differential equation is derived.

Let us consider a kicked damped rotator [2]. The equation of motion for this rotator is

$$\ddot{x} + q\dot{x} = KG(x) \sum_{n=0}^{\infty} \delta(t - nT). \tag{88}$$

It is well known that this equation gives [2] the two-dimensional map

$$y_{n+1} = e^{-qT} [y_n + KG(x_n)], \tag{89}$$

$$x_{n+1} = x_n + \frac{1 - e^{-qT}}{q} [y_n + KG(x_n)]. \tag{90}$$

Let us consider the fractional generalization of equation (88) in the form

$${}_0D_t^\alpha x - q_0D_t^\beta x = KG(x) \sum_{n=0}^{\infty} \delta(t - nT), \tag{91}$$

where

$$q \in \mathbb{R}, \quad 1 < \alpha \leq 2, \quad \beta = \alpha - 1,$$

and  ${}_0D_t^\alpha$  is the Riemann–Liouville fractional derivative [5–7] defined by (11). Note that we use the minus on the left-hand side of equation (91), where  $q$  can have positive and negative values.

**Proposition 3.** *The fractional differential equation of the kicked system (91) is equivalent to the discrete map*

$$p_{n+1} = e^{qT} [p_n + KG(x_n)], \tag{92}$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=0}^n p_{k+1} W_\alpha(qT, k-n), \tag{93}$$

where the functions  $W_\alpha(a, b)$  are defined by

$$W_\alpha(a, b) = a^{1-\alpha} e^{a(b+1)} [\Gamma(\alpha-1, ab) - \Gamma(\alpha-1, a(b+1))], \tag{94}$$

and  $\Gamma(a, b)$  is the incomplete Gamma function,

$$\Gamma(a, b) = \int_b^\infty y^{a-1} e^{-y} dy. \tag{95}$$

**Proof.** Let us define the variable  $\xi(t)$  such that

$${}_0^C D_t^{2-\alpha} \xi = x(t), \tag{96}$$

where  ${}_0^C D_t^{2-\alpha}$  is the Caputo fractional derivative (16). Using

$${}_0 I_t^{2-\alpha} {}_0^C D_t^{2-\alpha} \xi = \xi(t) - \xi(0), \quad (0 \leq 2 - \alpha < 1), \tag{97}$$

we obtain

$${}_0 D_t^\alpha x = D_{t_0}^2 {}_0 I_t^{2-\alpha} x = D_{t_0}^2 {}_0 I_t^{2-\alpha} {}_0^C D_t^{2-\alpha} \xi = D_t^2 (\xi(t) - \xi(0)) = D_t^2 \xi, \tag{98}$$

and

$${}_0D_t^\beta x = D_t^1 I_t^{1-\beta} x = D_t^1 I_t^{2-\alpha} x = D_t^1 I_t^{2-\alpha} {}_0^C D_t^{2-\alpha} \xi = D_t^1 (\xi(t) - \xi(0)) = D_t^1 \xi. \tag{99}$$

Substitution of (98), (99) and (96) into equation (91) gives

$$D_t^2 \xi - q D_t^1 \xi = K G({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta(t - nT), \quad (1 < \alpha \leq 2). \tag{100}$$

The fractional equation (100) can be presented in the Hamiltonian form

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} - q\eta &= K G({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta(t - nT), \quad (1 < \alpha < 2, \quad q \in \mathbb{R}). \end{aligned} \tag{101}$$

Between any two kicks

$$\dot{\eta} - q\eta = 0. \tag{102}$$

For  $t \in (t_n + 0, t_{n+1} - 0)$ , the solution of equation (102) is

$$\eta(t_{n+1} - 0) = \eta(t_n + 0) e^{qT}. \tag{103}$$

Let us use the notation

$$\xi_n = x(t_n - 0) = \lim_{\varepsilon \rightarrow 0} \xi(nT - \varepsilon), \quad \eta_n = \eta(t_n - 0) = \lim_{\varepsilon \rightarrow 0} \eta(nT - \varepsilon), \quad t_n = nT. \tag{104}$$

For  $t \in (t_n - \varepsilon, t_{n+1} - \varepsilon)$ , the general solution of (101) is

$$\eta(t) = \eta_n e^{q(t-t_n)} + K \sum_{m=0}^{\infty} G({}_0^C D_{t_m}^{2-\alpha} \xi) \int_{t_n-\varepsilon}^t d\tau e^{q(t-\tau)} \delta(\tau - mT). \tag{105}$$

Then

$$\eta_{n+1} = e^{qT} [\eta_n + K G({}_0^C D_{t_n}^{2-\alpha} \xi)]. \tag{106}$$

Using (106), the integration of the first equation of (101) gives

$$\xi_{n+1} = \xi_n - \frac{1 - e^{qT}}{q} [\eta_n + K G({}_0^C D_{t_n}^{2-\alpha} \xi)]. \tag{107}$$

Let us consider the Caputo fractional derivative from equations (106) and (107), which is defined by

$${}_0^C D_{t_n}^{2-\alpha} \xi = {}_0 I_t^{\alpha-1} D_t^1 \xi = \frac{1}{\Gamma(\alpha-1)} \int_0^{t_n} \frac{d\tau}{(t_n - \tau)^{2-\alpha}} \frac{d\xi(\tau)}{d\tau}, \quad (0 \leq 2 - \alpha < 1). \tag{108}$$

Using  $\dot{\xi} = \eta$ , we have

$${}_0^C D_{t_n}^{2-\alpha} \xi = {}_0 I_t^{\alpha-1} \eta \frac{1}{\Gamma(\alpha-1)} \int_0^{t_n} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{\alpha-1}}. \tag{109}$$

It can be presented as

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{2-\alpha}}, \tag{110}$$

where  $t_{k+1} = t_k + T = (k + 1)T$ ,  $t_k = kT$  and  $t_0 = 0$ .

For  $\tau \in (t_k, t_{k+1})$ , equations (103) and (104) give

$$\eta(\tau) = \eta(t_k + 0) e^{q(\tau-t_k)} = \eta(t_{k+1} - 0) e^{-qT} e^{q(\tau-t_k)} = \eta_{k+1} e^{q(\tau-t_k-T)} = \eta_{k+1} e^{q(\tau-t_{k+1})}. \tag{111}$$

Then

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{\alpha-2}} &= \eta_{k+1} \int_{t_k}^{t_{k+1}} e^{q(\tau-t_{k+1})} (t_n - \tau)^{\alpha-2} d\tau \\ &= \eta_{k+1} \int_{t_n-t_{k+1}}^{t_n-t_k} e^{q(t_n-t_{k+1}-z)} z^{\alpha-2} dz = \eta_{k+1} e^{q(t_n-t_{k+1})} \int_{t_n-t_{k+1}}^{t_n-t_k} z^{\alpha-2} e^{-qz} dz \\ &= \eta_{k+1} q^{1-\alpha} e^{q(n-k-1)T} \int_{q(t_n-t_{k+1})}^{q(t_n-t_k)} y^{\alpha-2} e^{-y} dy. \end{aligned} \tag{112}$$

As a result, equation (112) gives

$$\int_{t_k}^{t_{k+1}} \frac{\eta(\tau) d\tau}{(t_n - \tau)^{\alpha-2}} = \eta_{k+1} q^{1-\alpha} e^{q(n-k-1)T} [\Gamma(\alpha - 1, q(t_n - t_{k+1})) - \Gamma(\alpha - 1, q(t_n - t_k))]. \tag{113}$$

Here  $\Gamma(a, b)$  is the incomplete Gamma function [20]:

$$\Gamma(a, b) = \int_b^\infty y^{a-1} e^{-y} dy, \quad a, b \in \mathbb{C}, \tag{114}$$

which can be defined by

$$\Gamma(a, b) = \Gamma(a) - \frac{b^a}{a} {}_1F_1(1, 1 + a; -b), \tag{115}$$

where  ${}_1F_1$  is the confluent hypergeometric Kummer function [20],

$${}_1F_1(a, c; z) = \sum_{k=0}^\infty \frac{(a)_k z^k}{(c)_k k!}. \tag{116}$$

Here  $(a)_k$  is the Pochhammer symbol

$$(a)_k = a(a + 1) \cdots (a + k - 1), \quad k \in \mathbb{N}. \tag{117}$$

Using (110) and (113), we get

$${}_0^C D_{t_n}^{2-\alpha} \xi = \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \sum_{k=0}^{n-1} \eta_{k+1} W_\alpha(qT, k - n), \quad (1 < \alpha \leq 2), \tag{118}$$

where

$$W_\alpha(a, b) = a^{1-\alpha} e^{a(b+1)} [\Gamma(\alpha - 1, ab) - \Gamma(\alpha - 1, a(b + 1))]. \tag{119}$$

Substitution of (118) into (106) and (107) gives

$$\eta_{n+1} = e^{qT} \left[ \eta_n + K G \left( \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \sum_{k=0}^{n-1} \eta_{k+1} W_\alpha(qT, k - n) \right) \right], \tag{120}$$

$$\xi_{n+1} = \xi_n - \frac{1 - e^{qT}}{q} \left[ \eta_n + K G \left( \frac{T^{\alpha-1}}{\Gamma(\alpha - 1)} \sum_{k=0}^{n-1} \eta_{k+1} W_\alpha(qT, k - n) \right) \right]. \tag{121}$$

Equations (120) and (121) can be rewritten as

$$\eta_{n+1} = e^{qT} \left[ \eta_n + KG \left( \frac{T^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=0}^{n-1} \eta_{k+1} W_\alpha(qT, k-n) \right) \right], \quad (122)$$

$$\xi_{n+1} = \xi_n + \frac{1 - e^{-qT}}{q} \eta_{n+1}. \quad (123)$$

These equations can be considered as a fractional generalization of the kicked damped rotator map for  $(\xi_n, \eta_n)$ .

Equation (91) can be presented in the Hamiltonian form

$${}_0D_t^{\alpha-1} x = p, \quad \dot{p} - q_0 D_t^\beta x = KG(x) \sum_{n=0}^{\infty} \delta(t - nT). \quad (124)$$

For  $(x_n, p_n)$ , equations (96), (98) and (99) give

$$p_n = \eta_n, \quad x_n = {}_0^C D_t^{2-\alpha} \xi.$$

As a result, we have

$$p_{n+1} = e^{qT} [p_n + KG(x_n)], \quad (125)$$

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=0}^n p_{k+1} W_\alpha(qT, k-n), \quad (126)$$

where  $W_\alpha(a, b)$  is defined in (119). This ends the proof. □

These iteration equations (125) and (126) define the fractional kicked damped rotator map (88) in phase space  $(x_n, p_n)$ .

## 6. Conclusion

There are a number of distinct areas of physics where the basic problems can be reduced to the study of simple symplectic maps. In particular the special case of two-dimensional symplectic maps has been extensively studied. Under a wide range of circumstances such maps give rise to chaotic behavior. The suggested fractional maps can be considered as a fractional generalization of the symplectic map that is derived from kicked fractional differential equations. We can suppose that fractional discrete maps can be connected with some generalization of the symplectic structure. Fractional generalization of symplectic structure [22] can be defined by using fractional differential forms.

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