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Toward lattice fractional vector calculus

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Abstract

An analog of fractional vector calculus for physical lattice models is suggested. We use an approach based on the models of three-dimensional lattices with long-range inter-particle interactions. The lattice analogs of fractional partial derivatives are represented by kernels of lattice long-range interactions, where the Fourier series transformations of these kernels have a power-law form with respect to wave vector components. In the continuum limit, these lattice partial derivatives give derivatives of non-integer order with respect to coordinates. In the three-dimensional description of the non-local continuum, the fractional differential operators have the form of fractional partial derivatives of the Riesz type. As examples of the applications of the suggested lattice fractional vector calculus, we give lattice models with long-range interactions for the fractional Maxwell equations of non-local continuous media and for the fractional generalization of the Mindlin and Aifantis continuum models of gradient elasticity.

Keywords: fractional calculus, long-range interactions, vector calculus, lattice model, Maxwell equations, gradient elasticity
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1. Introduction

The most widely used approaches to describe materials are a microscopic approach based on lattice mechanics [1–4], and a macroscopic approach based on continuum mechanics [5–7]. Continuum mechanics can be considered as a phenomenological description representing the continuous limit of lattice dynamics, where the length-scales of an continuum element are much larger than the distances between the lattice particles.

Fractional calculus [8–13] as a theory of the derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Riemann, Grünwald, Letnikov and Riesz. Fractional calculus has a long history from 1695, when the derivative of order $\alpha = 0.5$ was described by Leibniz

[14–17]. The differentiations and integration of fractional orders have wide applications in mechanics and physics [18–29]. The history of fractional vector calculus is not as long; it is less than 20 years old (see [30] and references therein). Fractional vector calculus is very important to describe processes in complex media, non-local material and distributed systems in three-dimensional space. Therefore fractional vector differential operators can be used for non-local continua and distributed systems with long-range power-law interactions [27]. Synchronization of non-linear dynamical systems with long-range interactions is discussed in [31]. Non-equilibrium phase transitions in the thermodynamic limit for long-range systems are described in [32]. Stationary states for fractional dynamical systems with long-range interactions are considered in [33, 34]. Statistical mechanics and solvable models with long-range interactions are discussed in [35] and in the review [36]. Discrete systems and a lattice with long-range interactions and its continuum limit are considered in [27]. As was shown in [37, 38] (see also [27, 39, 40]), equations with fractional derivatives can be directly connected to lattice models with long-range interactions. A connection between the dynamics of a lattice system of particles with long-range interactions and the fractional continuum equations can be proved using the transform operation [37, 38]. One-dimensional lattice models for fractional non-local elasticity and the correspondent continuum equations were suggested in [46–50]. These models describe one-dimensional lattices only. In this paper, we suggest a three-dimensional lattice approach to describe the fractional non-local continuum in three-dimensional space. A general form of the lattice model with long-range interaction which gives a continuum equation with derivatives of fractional orders in the continuum limit is suggested. It should be note that a vector calculus for physical lattice models has been considered in [41–44]. In the papers [41–44], the suggested vector difference calculus is developed for models defined on a general triangulating graph using discrete field quantities and differential operators roughly analogous to differential forms and exterior differential calculus. Note that a fractional generalization of exterior differential calculus of differential forms is suggested in [27, 30, 45], where non-locality is described by the Caputo fractional derivatives. In this paper, we use a different approach based on lattice models with long-range inter-particle interactions and continuum limits that are suggested in [37–40] for a one-dimensional case. We propose a three-dimensional generalization of the models considered in [37, 38] to formulate a lattice analog of fractional vector calculus. The continuum limits of the suggested lattice fractional vector differential operators are described by fractional derivatives of the Riesz type. As examples of the applications of lattice fractional vector calculus, we consider lattice models with long-range interactions for the fractional Maxwell equations of non-local continuous media and for the fractional generalization of the Mindlin and Aifantis continuum models of gradient elasticity.

2. The model of a physical lattice with long-range interaction

The lattice is characterized by space periodicity. In an unbounded lattice we can define three non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_2$, such that displacement of the lattice by the length of any of these vectors brings it back to itself. The vectors $\mathbf{a}_i$, $i = 1, 2, 3$, are the shortest vectors by which a lattice can be displaced and be brought back into itself. As a result, all spatial lattice points (sites) can be defined by the vector $\mathbf{n} = (n_1, n_2, n_3)$, where n_i are integer numbers. If we choose the coordinate origin at one of the sites, then the position vector of an arbitrary lattice site with $\mathbf{n} = (n_1, n_2, n_3)$ is written in the form

$$\mathbf{r}(\mathbf{n}) = \sum_{i=1}^{3} n_i \mathbf{a}_i. \tag{1}$$

In a lattice the sites are numbered in the same way as the particles, so that the vector $\mathbf{n}$ is at the same time the 'number vector' of a corresponding particle.

For simplification, we consider a lattice with mutually perpendicular vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_2$. We choose directions of the axes of the Cartesian coordinate system that coincide with the vector $\mathbf{a}_i$. Then we have $\mathbf{a}_i = a_i\ \mathbf{e}_i$, where $a_i > 0$ and $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$ are the vectors of the basis of the Cartesian coordinate system.

We assume that equilibrium positions of the particles coincide with the lattice sites $\mathbf{r}(\mathbf{n})$. A lattice site coordinate $\mathbf{r}(\mathbf{n})$ differs from the coordinate of the corresponding particle when the particle is displaced relative to the equilibrium position. To define the coordinates of a particle in this case, it is necessary to indicate its displacement with respect to its equilibrium positions. We denote the displacement from its equilibrium position for a particle with vector $\mathbf{n}$ by the vector field

$$\mathbf{u}(\mathbf{n}, t) = \sum_{k=1}^{3} u_k(\mathbf{n}, t)\ \mathbf{e}_k. \tag{2}$$

Let us consider the equations of motion for a lattice $\mathbf{n}$-particle with the vector $\mathbf{n}$ in the form

$$\begin{aligned} M\frac{\mathrm{d}^2 u_i(\mathbf{n}, t)}{\mathrm{d}t^2} = &-\sum_{\alpha}\sum_{k=1}^{3}\sum_{\mathbf{m}} K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m}) u_k(\mathbf{m}, t) \\ &+F_i(\mathbf{n}, t), \quad (i = 1, 2, 3), \end{aligned} \tag{3}$$

where M is the mass of the particle. For simplicity, we assume that all lattice particles have the mass M. The italic i, $k \in \{1, 2, 3\}$ are the coordinate indices. In (3), we mean the summation over repeated index k. The functions $u_i(\mathbf{n}, t)$ are components of the displacement vector for the particle. The coefficients $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$ describe an interaction of the $\mathbf{n}$-particle with the $\mathbf{m}$-particles in the lattice. We can consider $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$ as a two-order elastic stiffness tensor kernel that characterizes the non-locality of long-range interactions of α-type [27]. The interaction kernel $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$ can be interpreted as the effective stiffness coefficients for a virtual discrete mass–spring system that corresponds to the suggested lattice model. The interaction of lattice particles is described by $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$ with $\mathbf{n} \neq \mathbf{m}$, i.e. when there is at least one n_j, $(j = 1, 2, 3)$, of the components of the vector $\mathbf{n}$ which is different from m_j. The terms with $K_{\alpha}^{ik}(0)$ can be interpreted as a measure of the self-interaction of the lattice particles. The sum $\sum_{\mathbf{m}}$ means the summations from $-\infty$ to $+\infty$ over n_1, n_2 and n_3. The sum $\sum_{\alpha}$ means a sum over the different values of α. The parameter α in the kernel is a positive real number that characterizes a decreased rate of the long-range interaction in space. This parameter can also be considered as a degree of the power law of the lattice spatial dispersion [46, 48] which is described by the non-integer power of the wave vector components.

Let us note some important properties of the kernels $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$. The internal states of the unbounded lattices must not be changed if the lattice is displaced as a whole $(u_k(\mathbf{n}, t) = u_k = \text{const})$ when there are no external forces $(F_i(\mathbf{n}, t) = 0)$. As a result, equations (3) give

$$\sum_{\mathbf{m}} K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m}) = \sum_{\mathbf{m}} K_{\alpha}^{ik}(\mathbf{m}, \mathbf{n}) = 0. \tag{4}$$

These conditions should be satisfied for any particle in the lattice, i.e. for any vector $\mathbf{n}$. Equations (4) follow from the conservation of total momentum in the lattice.

For an unbounded homogeneous lattice, due to its homogeneity the interaction kernels $K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m})$ have the form

$$K_{\alpha}^{ik}(\mathbf{n}, \mathbf{m}) = K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m}),$$

and $K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m})$ satisfy the conditions

$$\sum_{\mathbf{m}} K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m}) = \sum_{\mathbf{n}} K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m}) = \sum_{\mathbf{n}} K_{\alpha}^{ik}(\mathbf{n}) = 0. \tag{5}$$

Using (5), we can represent (3) in the form

$$M \frac{\mathrm{d}^2 u_i(\mathbf{n}, t)}{\mathrm{d}t^2} = -\sum_{\alpha} \sum_{k=1}^{3} \sum_{m_q=-\infty}^{+\infty} K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m}) \quad \left(u_k(\mathbf{m}, t) - u_k(\mathbf{n}, t)\right)$$
$$+F_i(\mathbf{n}, t), \quad (i = 1, 2, 3). \tag{6}$$

These equations of motion have invariance with respect to their displacement of the lattice as a whole in the case of the absence of external forces even if the conditions (5) are not satisfied.

The equation for the lattice $\mathbf{n}$-particle (6) allows us to consider a wider class of long-range interactions and correspondent interaction kernels that do not satisfy the conditions (5). Moreover, the form of the sum in (6) allows us to avoid divergences and non-physical infinities in the continuum limit [27].

In general, the kernels $K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m})$ of long-range particle interactions have the form

$$K_{\alpha}^{ik}(\mathbf{n} - \mathbf{m}) = C_{qpl}^{ik}\, K_{\alpha_q}(n_1 - m_1)\, K_{\alpha_p}(n_2 - m_2)\, K_{\alpha_l}(n_3 - m_3), \tag{7}$$

where C_{qpl}^{ik} are the coupling constants, $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{m} = (m_1, m_2, m_3)$. In this equation, α_q, α_p, α_l, are positive real parameters for the directions defined by the lattice vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively.

We will consider the kernels $K_{\alpha}(n_j - m_j)$, $j = 1, 2, 3$, with different α as even (+) and odd (−) functions $K_{\alpha}^{\pm}(n_j)$ such that

$$K_{\alpha}^{\pm}\left(\pm n_j\right) = \pm K_{\alpha}^{\pm}(n_j) \tag{8}$$

which have different power-law asymptotic behaviors of the Fourier series transformations

$$\hat{K}_{\alpha}^{\pm}\left(k_j\right) = \sum_{n_j=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\, k_j\, n_j} K_{\alpha}^{\pm}(n_j), \quad (j = 1, 2, 3). \tag{9}$$

We will assume that that $\hat{K}_{\alpha}^{-}(k_j)$ for odd function $K_{\alpha}^{+}(n_j)$ is asymptotically equivalent to $\mathrm{i}\ \operatorname{sgn}(k_j)|k_j|^{\alpha}$ at $|k_j| \to 0$ We also assume that $\hat{K}_{\alpha}^{+}(k_j) - \hat{K}_{\alpha}^{+}(0)$ for even function $K_{\alpha}^{+}(n_j)$ is asymptotically equivalent to $|k_j|^{\alpha}$ at $|k_j| \to 0$, where

$$\hat{K}_{\alpha}^{+}(0) = \sum_{n_j=-\infty}^{+\infty} K_{\alpha}^{+}(n_j). \tag{10}$$

In general, we have $\hat{K}_{\alpha}^{+}(0) \neq 0$, since the conditions (5) do not hold. Note that the expression $\hat{K}_{\alpha}^{+}(k_j) - \hat{K}_{\alpha}^{+}(0)$ is a result of the Fourier series transformation of the sum of equation (6), where $\hat{K}_{\alpha}^{+}(0)$ appears as a result of the transformation of the second part of the sum in (6) with the field $u_k(\mathbf{n}, t)$. This will be shown in the proof of proposition 1 of this paper.

The condition $\hat{K}^{+}_{\alpha}(0) = 0$ strongly restricts the class of possible long-range interactions for lattice models. For example, the most frequently used kernel of long-range interaction $K^{+}_{\alpha}(n) \sim 1/|n|^{\alpha+1}$ has a non-zero value $\hat{K}^{+}_{\alpha}(0)$ which is expressed in terms of the Riemann zeta-function $\zeta(\alpha + 1)$ (for details see section 8.12 in [27]). Therefore we will consider the general case with $\hat{K}^{+}_{\alpha}(0) \neq 0$.

We will use lattice operators for the lattice analog of the scalar functions

$$U = U(\mathbf{m}, \mathbf{n}) = u(\mathbf{m}, t) - u(\mathbf{n}, t) = u(m_1, m_2, m_3, t) - u(n_1, n_2, n_3, t), \tag{11}$$

and the vector functions

$$U_i = U_i(\mathbf{m}, \mathbf{n}) = u_i(\mathbf{m}, t) - u_i(\mathbf{n}, t) = u_i(m_1, m_2, m_3, t) - u_i(n_1, n_2, n_3, t), \tag{12}$$

where $u_i(\mathbf{n}, t)$ are components of the displacement vector for a lattice particle that is defined by the vector $\mathbf{n} = (n_1, n_2, n_3)$. We also assume that the fields $u_i(\mathbf{n}, t)$ belong to the Hilbert space l_2 of square-summable sequences, where

$$\sum_{n_i=-\infty}^{+\infty} \left| u_i(\mathbf{n}, t) \right|^2 < \infty \tag{13}$$

for all $t \geqslant 0$. We use this Hilbert space to apply the Fourier series transformations.

3. Lattice analogs of vector differential operators

3.1. Lattice analogs of fractional derivatives

Let us define a lattice analog of a partial derivative of non-integer order α with respect to n_i in the direction $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$.

Definition 1. The lattice fractional partial derivatives are the operator $\mathbb{D}^{\pm}\left[{\alpha \atop i}\right]$ such that

$$\mathbb{D}^{\pm}\left[\begin{matrix}\alpha \\ i\end{matrix}\right] U(\mathbf{m}, \mathbf{n}) = \frac{1}{a_i^{\alpha}} \sum_{m_i=-\infty}^{+\infty} K^{\pm}_{\alpha}(n_i - m_i)\, (u(\mathbf{m}, t) - u(\mathbf{n}, t)) \quad (i = 1, 2, 3), \tag{14}$$

where the interaction kernels $K^{\pm}_{\alpha}(n - m)$ satisfy the following conditions.

(1) The kernels $K^{\pm}_{\alpha}(n)$ are real-valued functions of the integer variable $n \in \mathbb{Z}$. The kernel $K^{+}_{\alpha}(n)$ is an even (or symmetric with respect to zero) function and $K^{-}_{\alpha}(n)$ is an odd (or antisymmetric with respect to zero) function such that

$$K^{+}_{\alpha}(-n) = +K^{+}_{\alpha}(n), \quad K^{-}_{\alpha}(-n) = -K^{-}_{\alpha}(n) \tag{15}$$

hold for all $n \in \mathbb{Z}$.

(2) The kernels $K^{\pm}_{\alpha}(n)$ belong to the Hilbert space l_2 of square-summable sequences, where

$$\sum_{n=1}^{\infty} \left| K^{\pm}_{\alpha}(n) \right|^2 < \infty. \tag{16}$$

(3) The Fourier series transforms of the kernels $K^{+}_{\alpha}(n)$ in the form

$$\hat{K}^{+}_{\alpha}(k) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn} K^{+}_{\alpha}(n) = 2\sum_{n=1}^{\infty} K^{+}_{\alpha}(n) \cos(kn) + K^{+}_{\alpha}(0), \tag{17}$$

satisfy the condition

$$\hat{K}_\alpha^+(k) - \hat{K}_\alpha^+(0) = |k|^\alpha + o\Big(|k|^\alpha\Big), \quad (k \to 0). \tag{18}$$

The Fourier series transforms of the kernels $K_\alpha^-(n)$ in the form

$$\hat{K}_\alpha^-(k) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kn} K_\alpha^-(n) = -2\,\mathrm{i} \sum_{n=1}^{\infty} K_\alpha^-(n) \sin(kn), \tag{19}$$

satisfy the condition

$$\hat{K}_\alpha^-(k) = \mathrm{i}\ \operatorname{sgn}(k)\, |k|^\alpha + o\Big(|k|^\alpha\Big), \quad (k \to 0). \tag{20}$$

The real number $\alpha > 0$ will be called the order of the operator (14).

Note that we use the minus sign in the exponents of (17) and (19) instead of plus in order to have the plus sign for plane waves and for the Fourier series.

Using that the kernel $K_\alpha^-(n)$ is odd with respect to n, we get $K_\alpha^-(0) = 0$ and $\hat{K}_\alpha^-(0) = 0$. As a result, we can always write

$$\mathbb{D}^-\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m}, \mathbf{n}) = \mathbb{D}^-\begin{bmatrix}\alpha\\ i\end{bmatrix} u(\mathbf{m}, t). \tag{21}$$

If the kernel $K_\alpha^+(n)$ satisfies the conditions (5) in the form

$$\sum_{n_i=-\infty}^{+\infty} K_\alpha^+(n_i) = 0, \tag{22}$$

i.e. $\hat{K}_\alpha^+(0) = 0$, then we can also use

$$\mathbb{D}^+\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m}, \mathbf{n}) = \mathbb{D}^+\begin{bmatrix}\alpha\\ i\end{bmatrix} u(\mathbf{m}, t). \tag{23}$$

In general, condition (22) does not hold, and we cannot use the simplification (23).

In the conditions (18) and (20) the notation lower-case o $o(|k|^\alpha)$ means the terms that include higher powers of $|k|$ than $|k|^\alpha$. The conditions (18) and (20) also mean that we can consider arbitrary functions $K_\alpha^\pm(n-m)$ for which $\hat{K}_\alpha^+(k) - \hat{K}_\alpha^+(0)$ is asymptotically equivalent to $|k|^\alpha$, and $\hat{K}_\alpha^-(k)$ is asymptotically equivalent to $\mathrm{i}\ \operatorname{sgn}(k)|k|^\alpha$ as $|k| \to 0$ respectively.

In equation (14), the values $i = 1, 2, 3$ specify the variables n_i of the lattice that is similar to the variable x_i of the continuum in the space $\mathbb{R}^3$. The operators $\mathbb{D}_L^\pm\begin{bmatrix}\alpha\\ i\end{bmatrix}$ are lattice analogous to the partial derivatives of order α with respect to coordinates x_i for the continuum model.

In the following sections we give explicit forms of the interaction kernels used in the definition (14) of the lattice fractional derivatives.

3.2. Exact expressions for the kernels of the lattice fractional derivatives

In this section, we give exact expressions for the interaction kernels $K_\alpha^\pm(n)$ that satisfy the conditions

$$\hat{K}_\alpha^+(k) = |k|^\alpha, \quad \hat{K}_\alpha^-(k) = \mathrm{i}\ \operatorname{sgn}(k)\, |k|^\alpha, \quad (k \to 0) \tag{24}$$

instead of the asymptotic conditions (18) and (20).

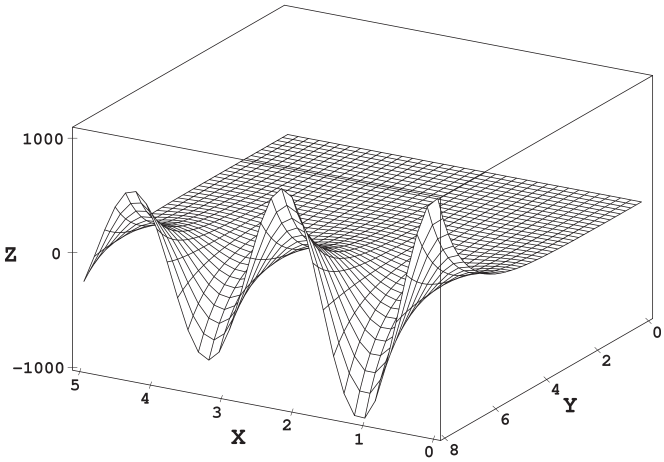

Figure 1. The plot of the function $f_+(x, y)$ (29) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 8]$.

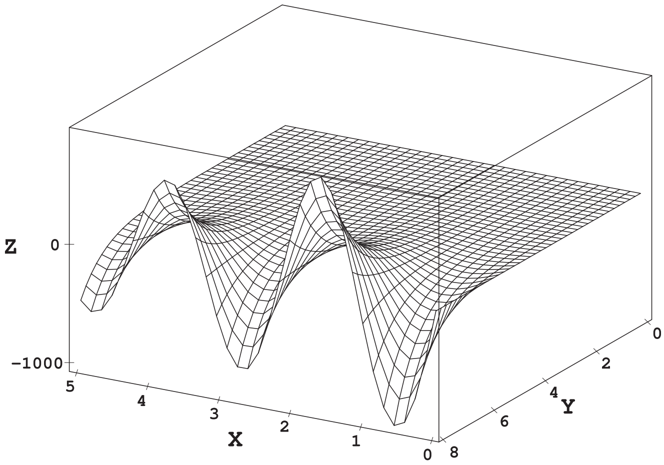

Figure 2. The plot of the function $f_-(x, y)$ (30) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 8]$.

As an example of the interaction kernel $K^+_\alpha(n)$, we consider the function

$$K^+_\alpha(n) = \frac{\pi^\alpha}{\alpha+1}\, {}_1F_2\left(\frac{\alpha+1}{2}; \frac{1}{2}, \frac{\alpha+3}{2}; -\frac{\pi^2 n^2}{4}\right), \quad (\alpha > 0), \tag{25}$$

where ${}_1F_2(a; b, c; x)$ is the Gauss hypergeometric function (see chapter 2 in [52], or section 1.6 in [9]),

$${}_1F_2(a; b, c; x) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\,\Gamma(b)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b+m)\,\Gamma(c+m)}\,\frac{x^m}{m!}. \tag{26}$$

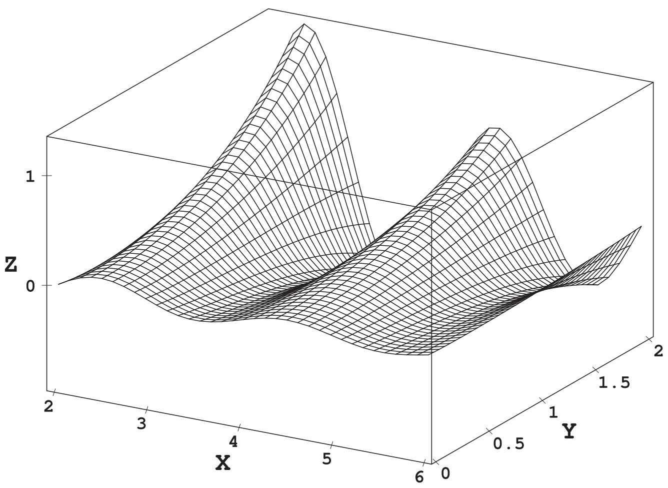

Figure 3. The plot of the function $f_+(x, y)$ (29) for the range $x \in [2, 6]$ and $y = \alpha \in [0, 2]$.

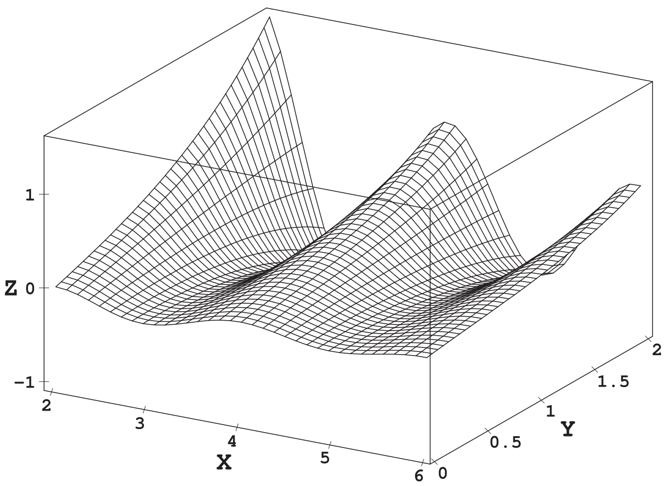

Figure 4. The plot of the function $f_-(x, y)$ (30) for the range $x \in [2, 6]$ and $y = \alpha \in [0, 2]$.

We use an inverse relation for (17) with $\hat{K}^+_\alpha(k) = |k|^\alpha$ in the form

$$K^+_\alpha(n) = \frac{1}{\pi}\int_0^\pi k^\alpha \cos(n\, k)\, \mathrm{d}k, \quad (\alpha \in \mathbb{R}, \quad \alpha > 0)$$

to obtain the expression (25) for the interaction kernel $K^+_\alpha(n)$.

As an example of the interaction kernel $K^-_\alpha(n)$, we consider the function

$$K^-_\alpha(n) = -\frac{\pi^{\alpha+1}\, n}{\alpha+2}\, {}_1F_2\left(\frac{\alpha+2}{2}; \frac{3}{2}, \frac{\alpha+4}{2}; -\frac{\pi^2 n^2}{4}\right), \quad (\alpha > 0). \tag{27}$$

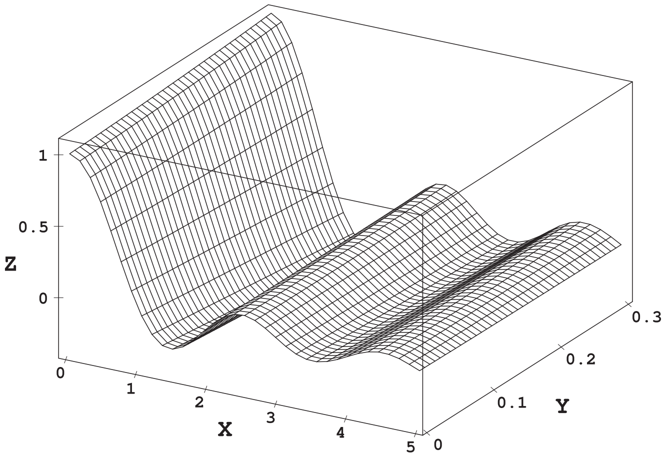

Figure 5. The plot of the function $f_+(x, y)$ (29) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 0.3]$.

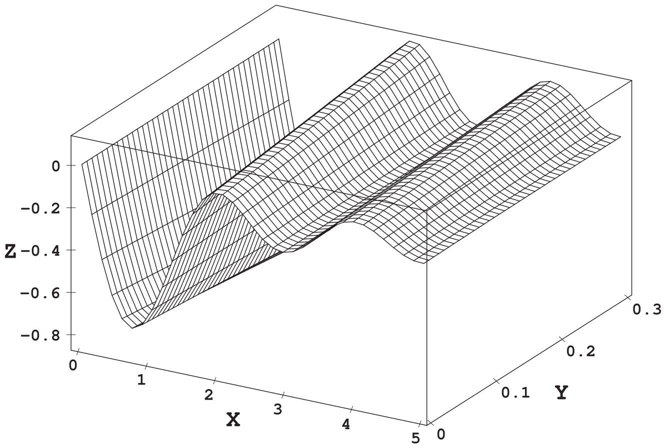

Figure 6. The plot of the function $f_-(x, y)$ (30) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 0.3]$.

Here we use an inverse relation for (19) with $\hat{K}^-_\alpha(k) = \mathrm{i}\ \mathrm{sgn}(k)\ |k|^\alpha$ in the form

$$K^-_\alpha(n) = -\frac{1}{\pi}\int_0^\pi k^\alpha\ \sin(n\,k)\,\mathrm{d}k, \quad (\alpha \in \mathbb{R}, \quad \alpha > 0) \tag{28}$$

to obtain the expression (27) for the interaction kernel $K^-_\alpha(n)$. Note that

$$K^+_\alpha(0) = \frac{\pi^\alpha}{\alpha + 1}, \quad K^-_\alpha(0) = 0$$

for all $\alpha \in \mathbb{N}$.

Note that the interaction kernels (25) and (27) for the integer and non-integer orders α describe the long-range interactions of the n-particle with all other particles ($m \in \mathbb{N}$).

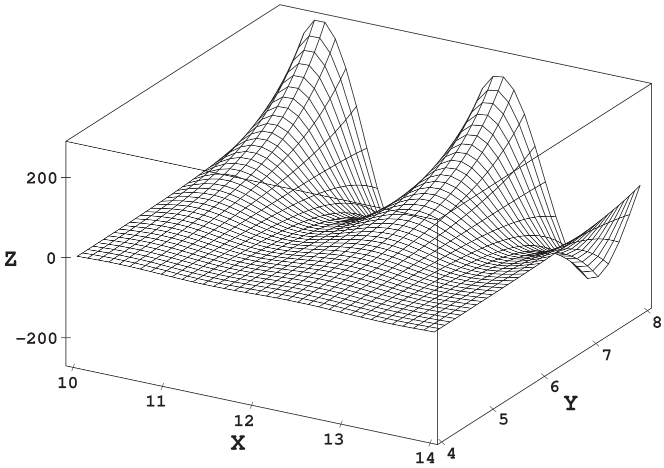

Figure 7. The plot of the function $f_+(x, y)$ (29) for the range $x \in [10, 14]$ and $y = \alpha \in [4, 8]$.

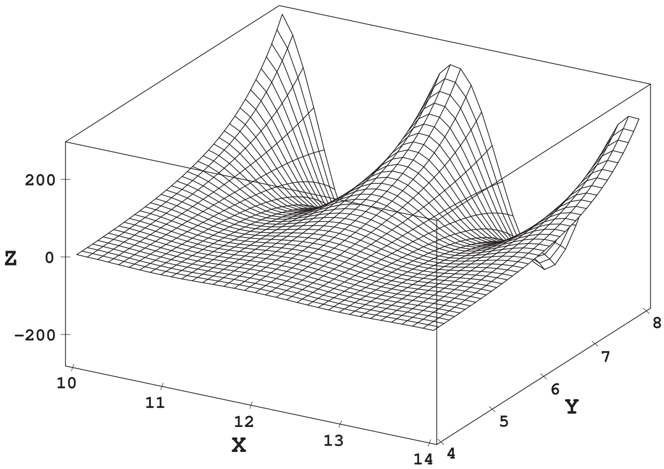

Figure 8. The plot of the function $f_-(x, y)$ (30) for the range $x \in [10, 14]$ and $y = \alpha \in [4, 8]$.

The exact expressions of the interaction kernels $K^{\pm}_{\alpha}(n)$ for integer values of α are suggested in the appendix.

To demonstrate the properties of (25) and (27), we can visualize the functions

$$f_+(x, y) = \frac{\pi^y}{y+1}\, {}_1F_2\left(\frac{y+1}{2}; \frac{1}{2}, \frac{y+3}{2}; -\frac{\pi^2 x^2}{4}\right), \tag{29}$$

$$f_-(x, y) = -\frac{\pi^{y+1} x}{y+2}\, {}_1F_2\left(\frac{y+2}{2}; \frac{3}{2}, \frac{y+4}{2}; -\frac{\pi^2 x^2}{4}\right) \tag{30}$$

of two continuous variables x and $y > 0$. The plots of the function (29) are presented in figures 1, 3, 5 and 7 for different ranges of x and y. The plots of the function (30) are given in figures 2, 4, 6 and 8.

Let us note that the kernel $K_\alpha^+(n)$ can be defined by (25) for $\alpha \in (-1, 0)$, and $K_\alpha^-(n)$ can be defined by (27) with $\alpha \in (-2, 0)$. This allows us to define the lattice fractional integrations by the same equations as the lattice fractional derivatives, but with negative α.

3.3. Asymptotic expressions for the kernels of the lattice fractional derivatives

Let us give examples of interaction kernels that satisfy the asymptotic conditions (18) and (20) of the form

$$\hat{K}_\alpha^+(k) - \hat{K}_\alpha^+(0) = |k|^\alpha + o\left(|k|^\alpha\right), \quad \hat{K}_\alpha^-(k) = \mathrm{i}\ \mathrm{sgn}\,(k)\,|k|^\alpha + o\left(|k|^\alpha\right), \quad (k \to 0). \tag{31}$$

To derive an asymptotic relation for the interactions kernels, we can use the equations 5.4.8.12 and 5.4.8.13 in [51].

Let us derive an example of the interaction kernel $K_\alpha^+(n)$ by using the series with the number 5.4.8.12 of [51]. Equation (5.4.8.12) from [51] has the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(\nu+1+n)\Gamma(\nu+1-n)} \cos(n\,k) = \frac{2^{2\nu-1}}{\Gamma(2\nu+1)} \sin^{2\nu}\left(\frac{k}{2}\right) - \frac{1}{2\Gamma^2(\nu+1)}, \tag{32}$$

where $\nu > -1/2$ and $0 < k < 2\pi$. Using that $\sin(k/2) = k/2 + o(k)$, and $\alpha = 2\nu$, equation (32) can be represented in the form

$$2\sum_{n=1}^{\infty} \frac{(-1)^n\,\Gamma(\alpha+1)}{\Gamma(\alpha/2+1+n)\Gamma(\alpha/2+1-n)} \cos(n\,k) = -\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)} + k^\alpha + o\left(k^\alpha\right), \quad (k \to 0), \tag{33}$$

where $\alpha > -1$. Comparing this equation with equations (17) and (18), we get

$$K_\alpha^+(n) = \frac{(-1)^n\,\Gamma(\alpha+1)}{\Gamma(\alpha/2+1+n)\Gamma(\alpha/2+1-n)}, \tag{34}$$

and

$$K_\alpha^+(0) = -\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}. \tag{35}$$

We can see that $K_\alpha^+(0)$ is not equal to zero for the interaction kernel (34). It can be directly verified that the kernel (34) is the even function, $K_\alpha^+(-n) = K_\alpha^+(n)$.

As a result, we have an example of the interaction kernel $K_\alpha^+(n)$ in the form

$$K_\alpha^+(n) = \frac{(-1)^n\,\Gamma(\alpha+1)}{\Gamma(\alpha/2+1+n)\,\Gamma(\alpha/2+1-n)}. \tag{36}$$

This kernel has been suggested in [37, 38] to describe long-range interactions of the lattice particles for non-integer values of α. The term $K_\alpha^+(0)$ characterizes a self-interaction of the lattice particles. The interaction of different particles is described by $K_\alpha^+(n-m)$ with $n \neq m$, i.e. $|n-m| \neq 0$. For integer values of $\alpha \in \mathbb{N}$, the kernel $K_\alpha^+(n-m) = 0$ for $|n-m| \geqslant \alpha/2 + 1$. For $\alpha = 2j$, we have $K_\alpha^+(n-m) = 0$ for all $|n-m| \geqslant j+1$. The function $K_\alpha^+(n-m)$ with an even value of $\alpha = 2j$ describes an interaction of the n-particle with $2\,j$ particles with numbers $n \pm 1 \ldots n \pm j$. To demonstrate the properties of (36), we can visualize the function

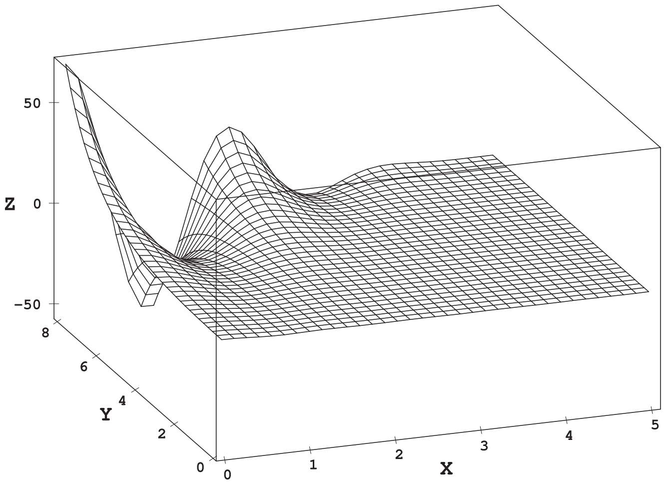

Figure 9. The plot of the function $g_+(x, y)$ (37) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 8]$.

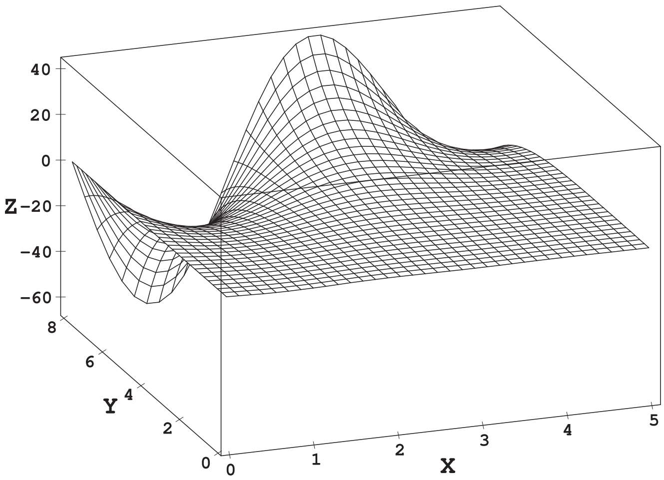

Figure 10. The plot of the function $g_-(x, y)$ (45) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 8]$.

$$g_+(x, y) = \mathrm{Re}\left[K_y^+(x)\right] = \frac{\mathrm{Re}\left[(-1)^x\right]\Gamma(y+1)}{\Gamma(y/2+1+x)\,\Gamma(y/2+1-x)} \tag{37}$$

of two continuous variables x and $y > 0$. Note that $\mathrm{Re}\,[(-1)^x] = (-1)^x$ for integer $x = n - m$. The plots of the function (37) are shown in figures 9, 11, 13 and 15 for different ranges of x and y. This function decays rapidly with growth of x and y. The function (37) defines the interaction terms $K_\alpha^+(n-m)$ by the equation $K_\alpha^+(n-m) = g_+(n-m, \alpha)$. The interaction kernels (25) and (36) can be used for integer and non-integer values of α. It is easy to see that expression (25) is more complicated than (36).

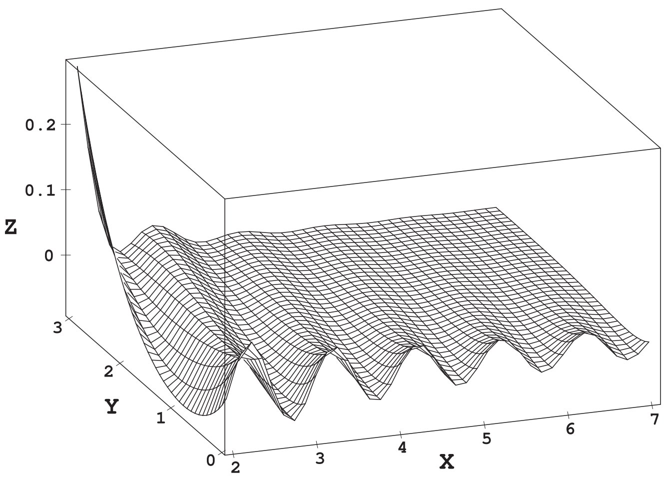

Figure 11. The plot of the function $g_{+}(x, y)$ (37) for the range $x \in [2, 7]$ and $y = \alpha \in [0, 3]$.

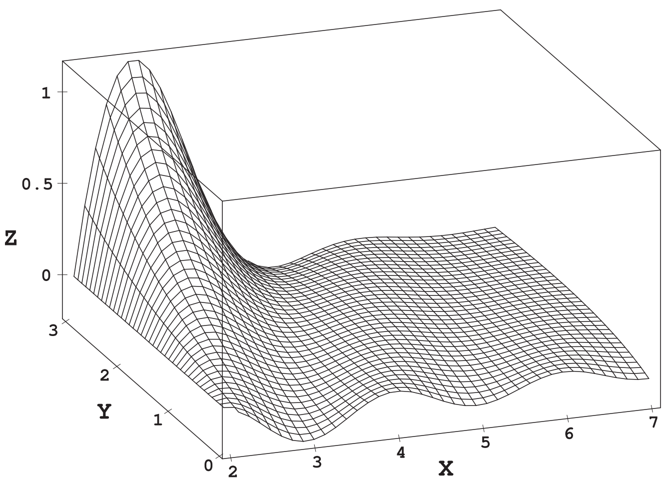

Figure 12. The plot of the function $g_{-}(x, y)$ (45) for the range $x \in [2, 7]$ and $y = \alpha \in [0, 3]$.

Let us derive an example of the interaction kernel $K_{\alpha}^{-}(n)$ using the series with the number 5.4.8.13 from [51]. Equation (5.4.8.13) of [51] has the form

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(\nu + 3/2 + m)\Gamma(\nu + 1/2 - m)} \sin((2m+1)k) = = \frac{2^{2\nu-1}}{\Gamma(2\nu+1)} \sin^{2\nu}(k), \qquad (38)$$

where $\nu > -1/2$ and $0 < k < \pi$. Shifting the variable m by unity, and using $\alpha = 2\nu$ and $\sin(k/2) = k/2 + o(k)$, equation (38) gives

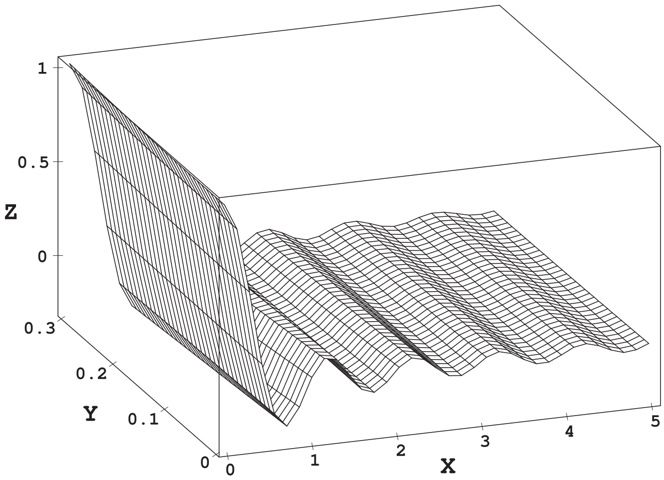

Figure 13. The plot of the function $g_+(x, y)$ (37) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 0.3]$.

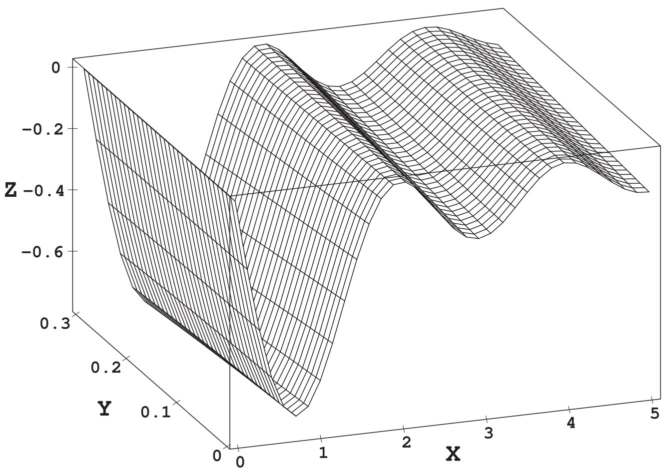

Figure 14. The plot of the function $g_-(x, y)$ (45) for the range $x \in [0, 5]$ and $y = \alpha \in [0, 0.3]$.

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\Gamma(\nu + 1/2 + m)\Gamma(\nu + 3/2 - m)} \sin((2m-1)k) = \frac{2^{\alpha-1}}{\Gamma(\alpha+1)} \sin^{\alpha}(k). \tag{39}$$

Adding a zero term of the form $0 \cdot \sin(2m\, k)$, equation (39) can be represented as

$$\sum_{m=1}^{\infty} \Bigg(\frac{(-1)^{m+1}\Gamma(\alpha+1)}{2^{\alpha-1}\Gamma(\alpha/2 + 1/2 + m)\Gamma(\alpha/2 + 3/2 - m)} \sin((2m-1)k) \\ +0 \cdot \sin(2m\, k)\Bigg) = k^{\alpha} + o\big(k^{\alpha}\big), \quad (k \to 0), \tag{40}$$

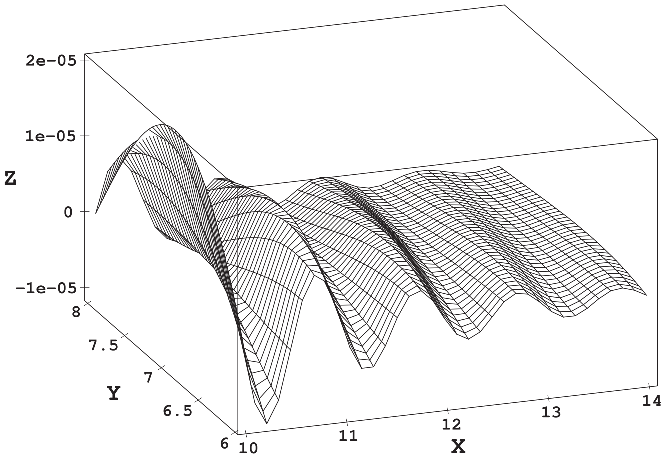

Figure 15. The plot of the function $g_+(x, y)$ (37) for the range $x \in [10, 14]$ and $y = \alpha \in [6, 8]$.

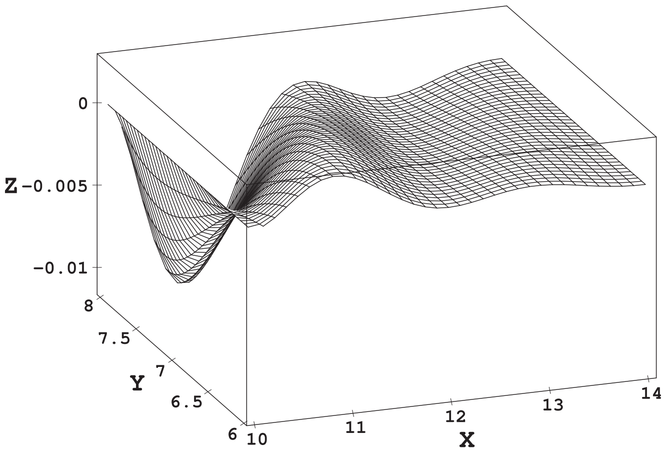

Figure 16. The plot of the function $g_-(x, y)$ (45) for the range $x \in [10, 14]$ and $y = \alpha \in [6, 8]$.

$$\begin{tikzcd} u_i(\mathbf{n},t) \arrow[r, Rightarrow, "\mathcal{F}_\Delta"] \arrow[rrr, Rightarrow, bend right, "\mathcal{F}^{-1} \circ \mathrm{Lim} \circ \mathcal{F}_\Delta"'] & \hat{u}_i(\mathbf{k},t) \arrow[r, Rightarrow, "\mathrm{Lim}"] & \tilde{u}_i(\mathbf{k},t) \arrow[r, Rightarrow, "\mathcal{F}^{-1}"] & u_i(\mathbf{r},t) \end{tikzcd}$$

Figure 17. Diagrams of sets of operations for fields.

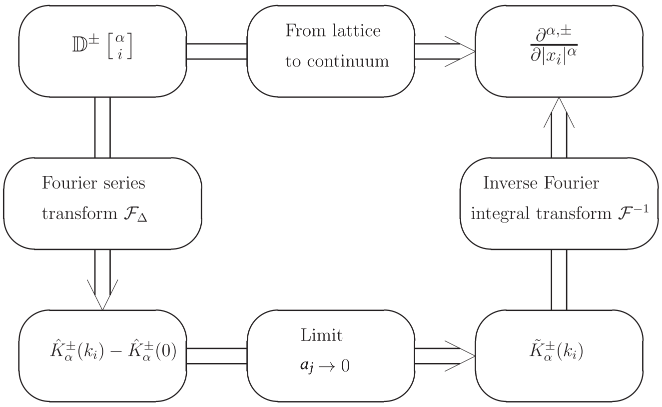

Figure 18. Diagrams of sets of operations for differential operators.

where $\alpha > -1$, and $0 < k < \pi$. Equation (40) can be rewritten in the form

$$-2\,\mathrm{i} \sum_{m=1}^{\infty} \left(K^-_{\alpha}(2m-1)\ \sin\left((2m-1)k\right) + K^-_{\alpha}(2m)\ \sin\left(2m\,k\right)\right)$$
$$= \mathrm{i}\,k^{\alpha} + o\left(k^{\alpha}\right), \quad (k \to 0), \tag{41}$$

where $k > 0$, and

$$K^-_{\alpha}(n) = \begin{cases} \dfrac{(-1)^{m+1}\,\Gamma(\alpha+1)}{2^{\alpha}\,\Gamma(\alpha/2+1/2+m)\Gamma(\alpha/2+3/2-m)}, & n = 2m-1, \quad m \in \mathbb{N}, \\ 0, & n = 2m, \quad m \in \mathbb{N}. \end{cases} \tag{42}$$

Then using equations (19) and (20), we derive the kernel $K^-_{\alpha}(n)$ for $n \in \mathbb{Z}$ and $\alpha > -1$. As a result, we have the kernels in the form of the function (42) which can be represented by

$$K^-_{\alpha}(n) = \frac{(-1)^{((n+1)/2}\left(2\left[(n+1)/2\right] - n\right)\,\Gamma(\alpha+1)}{2^{\alpha}\,\Gamma((\alpha+n)/2+1)\Gamma((\alpha-n)/2+1)}, \tag{43}$$

where the brackets [ ] mean the integral part, i.e. the floor function that maps a real number to the largest previous integer number. The expression $(2[(n+1)/2] - n)$ is equal to zero for even $n = 2m$, and it is equal to one for odd $n = 2m - 1$. This allows us to combine two cases of (42) for even and odd values of $n \in \mathbb{N}$ into the single equation (43). Note that the kernel (43) is a real-valued function since we have zero when the expression $(-1)^{((n+1)/2}$ becomes a complex number. It is easy to see that we can use equation (43) for all integer values $n \in \mathbb{Z}$. The kernel $K^-_{\alpha}(n)$ is the odd function such that

$$K^-_{\alpha}(-n) = -K^-_{\alpha}(n), \quad K^-_{\alpha}(0) = 0. \tag{44}$$

As a result, we have the interaction kernel $K^-_{\alpha}(n)$ which satisfies the asymptotic condition (20). For non-integer values of α, this kernel describes a long-range interaction of the lattice particles. To demonstrate the properties of (43), we can visualize the function

$$g_-(x,y) = \mathrm{Re}\left[K^-_y(x)\right] = \frac{\mathrm{Re}\left[(-1)^{(x+1)/2}\right]\Gamma(y+1)}{\Gamma\left((y+x)/2+1\right)\,\Gamma\left((y-x)/2+1\right)} \tag{45}$$

of two continuous variables x and $y > 0$. The plots of the function (37) are shown in figures 10, 12, 14 and 16 for different ranges of x and y. This function decays rapidly with growth of x and y. The function (37) defines the interaction terms $K_\alpha^-(n-m)$ by the equation $K_\alpha^-(n-m) = g_-(n-m,\alpha)(2[(n-m+1)/2]-(n-m))$. The interaction kernels (25) and (43) can be used for integer and non-integer values of α. It is easy to see that expression (25) is also more complicated than (43).

Some other examples of the interaction kernels with the property (31) are given in section 8 of the book [27]. For example, the most frequently used kernel of the long-range interaction

$$K_\alpha^+(n) = \frac{A(\alpha)}{|n|^{\alpha+1}}, \tag{46}$$

where we use the multiplier

$$A(\alpha) = \frac{1}{2\,\Gamma(-\alpha)\ \cos(\pi\alpha/2)}, \tag{47}$$

has the asymptotic behavior

$$\hat{K}_\alpha^+(k) = \hat{K}_\alpha^+(0) + |k|^\alpha + o\left(|k|^\alpha\right), \quad (k \to 0), \tag{48}$$

for the cases $0 < \alpha < 2$ and $\alpha \neq 1$, with the non-zero term

$$\hat{K}_\alpha^+(0) = \frac{\zeta(\alpha+1)}{\Gamma(-\alpha)\ \cos(\pi\alpha/2)}, \tag{49}$$

where $\zeta(z)$ is the Riemann zeta-function. To take into account such long-range interactions, we use the asymptotic condition for $\hat{K}_\alpha^+(k)$ in the form (18) which includes $\hat{K}_\alpha^+(0)$.

We should note that the long-range interaction with the kernel (36) is similar to the fractional central differences of type 1 suggested by Ortigueira in [55, 56]. At the same time, the interaction with kernel (43) is not directly connected with the fractional central differences of type 2 suggested in [55, 56] since the kernel of these central differences contains integer values of n instead of $n/2$ in (43). In addition, the difference of type 2 corresponds to interaction of the lattice particles with virtual particles with half-integer numbers which do not exist in the physical lattices. Therefore the fractional central differences of types 1 and 2 can be considered as the basis of a discrete analog of fractional vector calculus, which is not associated directly with the physical lattices. A discrete fractional vector calculus based on the fractional-order central differences suggested by Ortigueira in [55, 56] is considered in section 5.1 of this paper.

3.4. The properties of the lattice partial derivatives

The lattice fractional derivatives (14) are linear operators on the Hilbert space l_2 of square-summable sequences $u(\mathbf{n})$.

In general, the operators for the same values of the subscript i do not commute

$$\mathbb{D}^\pm\begin{bmatrix}\alpha_1\\ i\end{bmatrix}\mathbb{D}^\pm\begin{bmatrix}\alpha_2\\ i\end{bmatrix} \neq \mathbb{D}^\pm\begin{bmatrix}\alpha_2\\ i\end{bmatrix}\mathbb{D}^\pm\begin{bmatrix}\alpha_1\\ i\end{bmatrix}, \quad \left(\alpha_1 \neq \alpha_2\right). \tag{50}$$

The operators with different subscripts i and j commute

$$\mathbb{D}^\pm\begin{bmatrix}\alpha_1\\ i\end{bmatrix}\mathbb{D}^\pm\begin{bmatrix}\alpha_2\\ j\end{bmatrix} = \mathbb{D}^\pm\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\mathbb{D}^\pm\begin{bmatrix}\alpha_1\\ i\end{bmatrix}, \quad (i \neq j). \tag{51}$$

The semigroup property is not satisfied

$$\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\\ i\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_2\\ i\end{array}\right]\neq\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1+\alpha_2\\ i\end{array}\right],\quad\left(\alpha_1,\,\alpha_2>0\right). \tag{52}$$

An action of two repeated fractional derivatives of order α_1 is not equivalent to the action of a fractional derivative of double order $2\alpha_1$,

$$\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\\ i\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\\ i\end{array}\right]\neq\mathbb{D}^{\pm}\left[\begin{array}{c}2\,\alpha_1\\ i\end{array}\right],\quad\left(\alpha_1>0\right). \tag{53}$$

Note that these properties are similar to non-integer order derivatives [9].

It should be noted that the Leibniz rule for a lattice fractional derivative of order $s\neq 1$ does not satisfy

$$\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ i\end{array}\right](U\,V)\neq V\,\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ i\end{array}\right]U+U\,\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ i\end{array}\right]V,\quad(\alpha>0,\quad s\neq 1), \tag{54}$$

and this is a characteristic property of fractional differentiation. This property is similar to fractional derivatives with respect to coordinates [54].

We assume that the lattice derivative with the value $\alpha=0$ is the unit operator

$$\mathbb{D}^{\pm}\left[\begin{array}{c}0\\ i\end{array}\right]U=U. \tag{55}$$

The commutation relation (51) with $\alpha_1=\alpha_2=1$ is

$$\mathbb{D}^{\pm}\left[\begin{array}{c}1\\ i\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}1\\ j\end{array}\right]=\mathbb{D}^{\pm}\left[\begin{array}{c}1\\ j\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}1\\ i\end{array}\right]\quad(i\neq j). \tag{56}$$

The continuum analog of the commutation relation (56) has the form

$$\frac{\partial^2 u(\mathbf{r})}{\partial x_i\partial x_j}=\frac{\partial^2 u(\mathbf{r})}{\partial x_j\partial x_i}. \tag{57}$$

It is well known that the commutation relation (57) may be broken for discontinuous functions $u(\mathbf{r})$ and if the derivatives are not continuous. We can assume that relation (56) may be broken if we have a lattice with dislocations and disclinations. However, the exact conditions for violation of this relationship remains an open question and we consider lattices without dislocations and disclinations.

3.5. Lattice analogs of mixed partial derivatives

Let us define the lattice analogs of the mixed partial derivatives

$$\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\;\alpha_2\\ i\;j\end{array}\right]=\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\\ i\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_2\\ j\end{array}\right]\quad(i\neq j), \tag{58}$$

$$\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\;\alpha_2\;\alpha_3\\ i\;j\;k\end{array}\right]=\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_1\\ i\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_2\\ j\end{array}\right]\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha_3\\ k\end{array}\right],\quad(i\neq j\neq k\neq i), \tag{59}$$

where i, j and k take different values from $\{1;\,2;\,3\}$ and the values of i, j, k cannot coincide. The order of the operators (58) and (59) are equal to $\alpha=\alpha_1+\alpha_2$ and $\alpha=\alpha_1+\alpha_2+\alpha_3$ respectively. It should be noted that the operators (58) and (59) are not operators of second and third orders in general. If $\alpha_1=2$ and $\alpha_2=2$, then (58) is an operator of fourth order, and if $\alpha_1=2$ and $\alpha_2=\alpha_3=1/2$, then (59) is an fractional operator of third order.

Using (14), the mixed partial lattice derivatives (58) and (59) are represented by

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ \alpha_2 \\ i \ j \end{bmatrix} = \frac{1}{a_i^{\alpha_1} \ a_j^{\alpha_2}} \sum_{m_i=-\infty}^{+\infty} \ \sum_{m_j=-\infty}^{+\infty} K_{\alpha_1}^{\pm}(n_i - m_i) \ K_{\alpha_2}^{\pm}(n_j - m_j), \tag{60}$$

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ \alpha_2 \ \alpha_3 \\ i \ j \ k \end{bmatrix} = \frac{1}{a_i^{\alpha_1} \ a_j^{\alpha_2} \ a_k^{\alpha_3}} \sum_{m_i=-\infty}^{+\infty} \ \sum_{m_j=-\infty}^{+\infty} \ \sum_{m_k=-\infty}^{+\infty} K_{\alpha_1}^{\pm}(n_i - m_i) \ K_{\alpha_2}^{\pm}(n_j - m_j) \ K_{\alpha_3}^{\pm}(n_k - m_k). \tag{61}$$

If the parameter $\alpha_k = 0$, then the operator (59) is the operator of the type (58),

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ \alpha_2 \ 0 \\ i \ j \ 0 \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ \alpha_2 \\ i \ j \end{bmatrix}, \tag{62}$$

and similarly we have

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ 0 \ 0 \\ i \ 0 \ 0 \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ 0 \\ i \ 0 \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \\ i \end{bmatrix}. \tag{63}$$

Using (59) and the property (51), we can rearrange any pair of columns

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \ \alpha_2 \ \alpha_3 \\ i \ j \ k \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_2 \ \alpha_3 \ \alpha_1 \\ j \ k \ i \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_3 \ \alpha_1 \ \alpha_2 \\ k \ i \ j \end{bmatrix} = \ldots \tag{64}$$

We can define the mixed derivatives

$$\mathbb{D}^{\pm\mp}\begin{bmatrix} \alpha_1 \ \alpha_2 \\ i \ j \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}^{\mp}\begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \quad (i \neq j), \tag{65}$$

$$\mathbb{D}^{\pm\pm\mp}\begin{bmatrix} \alpha_1 \ \alpha_2 \ \alpha_3 \\ i \ j \ k \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}^{\pm}\begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \mathbb{D}^{\mp}\begin{bmatrix} \alpha_3 \\ k \end{bmatrix}, \quad (i \neq j \neq k \neq i), \tag{66}$$

$$\mathbb{D}^{\pm\pm\mp}\begin{bmatrix} \alpha_1 \ \alpha_2 \ \alpha_3 \\ i \ j \ k \end{bmatrix} = \mathbb{D}^{\pm}\begin{bmatrix} \alpha_1 \\ i \end{bmatrix} \mathbb{D}^{\mp}\begin{bmatrix} \alpha_2 \\ j \end{bmatrix} \mathbb{D}^{\mp}\begin{bmatrix} \alpha_3 \\ k \end{bmatrix}, \quad (i \neq j \neq k \neq i). \tag{67}$$

The suggested lattice fractional partial derivatives allow us to obtain lattice analogs of the fractional vector differential operators.

3.6. Lattice fractional vector differential operators

Let us define a lattice nabla operator for the lattice with the primitive vectors $\mathbf{a}_i$, $i = 1, 2, 3$, by the equation

$$\nabla_L^{\alpha,\pm} = \sum_{i=1}^{3} \frac{\mathbf{a}_i}{|\mathbf{a}_i|} \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix}. \tag{68}$$

For simplification, we consider the case $\mathbf{a}_i = a_i \ \mathbf{e}_i$, where $a_i = |\mathbf{a}_i|$ and $\mathbf{e}_i$ are the vectors of the basis of the Cartesian coordinate system. Therefore this simplification case means that the lattice is a primitive orthorhombic Bravais lattice with long-range interactions.

The lattice analogs of the vector differential operators can be defined by the following equations.

The lattice gradient for the scalar lattice field $U = U(\mathbf{m}, \mathbf{n})$ is

$$\mathrm{Grad}_L^{\alpha,\pm} U = \sum_{i=1}^{3} \mathbf{e}_i \, \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} U = \sum_{i=1}^{3} \frac{1}{a_i^{\alpha}} \mathbf{e}_i \sum_{m_i=-\infty}^{+\infty} K_{\alpha}^{\pm}(n_i - m_i) \, U(\mathbf{m}, \mathbf{n}). \quad (69)$$

The lattice divergence for the vector lattice field $\mathbf{U} = \sum_{i=1}^{3} \mathbf{e}_i \, U_i(\mathbf{m}, \mathbf{n})$ is

$$\mathrm{Div}_L^{\alpha,\pm} \mathbf{U} = \sum_{i=1}^{3} \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} U_i = \sum_{i=1}^{3} \frac{1}{a_i^{\alpha}} \sum_{m_i=-\infty}^{+\infty} K_{\alpha}^{\pm}(n_i - m_i) \, U_i(\mathbf{m}, \mathbf{n}). \quad (70)$$

The lattice curl operator for the vector lattice field $\mathbf{U} = \sum_{i=1}^{3} \mathbf{e}_i \, U_i(\mathbf{m}, \mathbf{n})$ is

$$\mathrm{Curl}_L^{\alpha,\pm} \mathbf{U} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, \mathbf{e}_i \, \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ j \end{bmatrix} U_k(\mathbf{m}, \mathbf{n}), \quad (71)$$

where ϵ_{ijk} denotes the Levi-Civita symbol.

The lattice scalar Laplacian for the scalar lattice field $U = U(\mathbf{m}, \mathbf{n})$ can be defined by two different equations with the repeated lattice derivative of orders α,

$$\Delta_L^{\alpha,\alpha,\pm} U = \mathrm{Div}_L^{\alpha,\pm} \, \mathrm{Grad}_L^{\alpha,\pm} U = \sum_{i=1}^{3} \left(\mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} \right)^2 U(\mathbf{m}, \mathbf{n}), \quad (72)$$

and by the derivative of the doubled order $2\,\alpha$,

$$\Delta_L^{2\alpha,\pm} U = \sum_{i=1}^{3} \mathbb{D}^{\pm}\begin{bmatrix} 2\,\alpha \\ i \end{bmatrix} U(\mathbf{m}, \mathbf{n}). \quad (73)$$

The violation of the semigroup property (53) leads to the fact that operators (72) and (73) do not coincide in general.

Relations for lattice fractional differential vector operations are the same as for the fractional vector analysis of non-integer order with respect to the coordinates (see section 5.3 in [30]).

4. The continuum limit for lattice fractional derivatives and lattice fractional vector differential operators

4.1. Transform of the fields on the lattice into fields on the continuum

In order to define the operation that transforms a lattice field $u_i(\mathbf{n}, t)$ into a field $u_i(\mathbf{r}, t)$ of the continuum, we use the methods suggested in [37, 38]. The transformations of components of the lattice field $u_i(\mathbf{n}, t)$ into components of the field $u_i(\mathbf{r}, t)$ of the continuum are as follows. We consider $u_i(\mathbf{n}, t)$ as Fourier series coefficients of some function $\hat{u}_i(\mathbf{k}, t)$ on $k_j \in [-k_{j0}/2, k_{j0}/2]$, then we use the continuous limit $\mathbf{k}_0 \to \infty$ to obtain $\tilde{u}_i(\mathbf{k}, t)$, and finally we apply the inverse Fourier integral transformation to obtain $u_i(\mathbf{r}, t)$. Diagrammatically, the set of operations for transformation of the field can be represented by figure 17.

The transformation operation that maps a lattice field into a continuum field is a sequence of the following three operations (for details see [37, 38]).

1. The Fourier series transform $\mathcal{F}_{\Delta}$: $u_i(\mathbf{n}, t) \to \mathcal{F}_{\Delta}\{u_i(\mathbf{n}, t)\} = \hat{u}_i(\mathbf{k}, t)$ which is defined by

$$\hat{u}_i(\mathbf{k},\,t) = \sum_{n_1,n_2,n_3=-\infty}^{+\infty} u_i(\mathbf{n},\,t)\ \mathrm{e}^{-\mathrm{i}(\mathbf{k},\mathbf{r}(\mathbf{n}))} = \mathcal{F}_\Delta\left\{u_i(\mathbf{n},\,t)\right\}, \tag{74}$$

$$u_i(\mathbf{n},\,t) = \prod_{j=1}^{3}\frac{1}{k_{j0}}\int_{-k_{j0}/2}^{+k_{j0}/2}\mathrm{d}k_j\ \hat{u}_i(\mathbf{k},\,t)\ \mathrm{e}^{\mathrm{i}(\mathbf{k},\mathbf{r}(\mathbf{n}))} = \mathcal{F}_\Delta^{-1}\left\{\hat{u}_i(\mathbf{k},\,t)\right\}, \tag{75}$$

where $\mathbf{r}(\mathbf{n}) = n_j\,\mathbf{a}_j$, and $a_j = 2\pi/k_{j0}$ is the inter-particle distance in the direction $\mathbf{a}_j$. For simplicity we assume that all lattice particles have the same inter-particle distance a_j in the direction $\mathbf{a}_j$.

2. The passage to the limit $a_j \to 0$ $(k_{j0} \to \infty)$ is denoted by Lim: $\hat{u}_i(\mathbf{k},\,t) \to \mathrm{Lim}\,\{\hat{u}_i(\mathbf{k},\,t)\} = \tilde{u}_i(\mathbf{k},\,t)$. The function $\tilde{u}_i(\mathbf{k},\,t)$ can be derived from $\hat{u}_i(\mathbf{k},\,t)$ in the limit $a_i \to 0$. Note that $\tilde{u}_i(\mathbf{k},\,t)$ is a Fourier integral transform of the field $u_i(\mathbf{r},\,t)$, and $\hat{u}(k,\,t)$ is a Fourier series transform of $u_i(\mathbf{n},\,t)$, where we use

$$u_i(\mathbf{n},\,t) = \frac{2\pi}{k_{i0}}u_i(\mathbf{r}(\mathbf{n}),\,t)$$

considering $\mathbf{r}(\mathbf{n}) = n_j a_j = 2\pi n_j/k_{j0} \to \mathbf{r}$.

3. The inverse Fourier integral transform $\mathcal{F}^{-1}$: $\tilde{u}_i(\mathbf{k},\,t) \to \mathcal{F}^{-1}\{\tilde{u}_i(\mathbf{k},\,t)\} = u_i(\mathbf{r},\,t)$ is defined by

$$\tilde{u}_i(\mathbf{k},\,t) = \prod_{j=1}^{3}\int_{-\infty}^{+\infty}\mathrm{d}x_j\ \mathrm{e}^{-\mathrm{i}k_jx_j}u_i(\mathbf{r},\,t) = \mathcal{F}\left\{u_j(\mathbf{r},\,t)\right\}, \tag{76}$$

$$u_i(\mathbf{r},\,t) = \frac{1}{(2\pi)^3}\prod_{j=1}^{3}\int_{-\infty}^{+\infty}\mathrm{d}k_j\ \mathrm{e}^{\mathrm{i}k_jx_j}\tilde{u}_i(\mathbf{k},\,t) = \mathcal{F}^{-1}\left\{\tilde{u}_i(\mathbf{k},\,t)\right\}. \tag{77}$$

Note that equations (74) and (75) in the limit $a_j \to 0$ $(k_{j0} \to \infty)$ are used to obtain the Fourier integral transform equations (76) and (77), where the sum is changed by the integral.

Using the suggested notation we can represent these transformations by the following diagram.

$$\begin{array}{ccc} u_i(\mathbf{n},t) & \xrightarrow{\textit{From Lattice to Continuum}} & u_i(\mathbf{r},t) \\ \mathcal{F}_\Delta \big\downarrow & & \big\uparrow \mathcal{F}^{-1} \\ \hat{u}_i(\mathbf{k},t) & \xrightarrow[\mathrm{Lim}\quad a_j\to 0]{} & \tilde{u}_i(\mathbf{k},t)\ . \end{array} \tag{78}$$

The combination of these three operations $\mathcal{F}^{-1}$, Lim and $\mathcal{F}_\Delta$ allows us to realize the transformation of the field of the lattice into the field of the continuum [37, 38].

4.2. The continuum limit of the lattice partial derivatives

Let us consider a transformation of a lattice fractional derivative into the fractional derivative with respect to coordinates by the combination of the operations $\mathcal{F}^{-1} \bigcirc \mathrm{Limit} \bigcirc \mathcal{F}_\Delta$. We performed transformations $\mathcal{F}^{-1} \bigcirc \mathrm{Limit} \bigcirc \mathcal{F}_\Delta$ for differential operators to map the lattice fractional derivative into the fractional derivative for the continuum. We can represent these sets of transformations from lattice operators to operators for the continuum in the form of the diagrams shown in figure 18.

The function $\hat{K}^{\pm}_{\alpha}(k_i)$ is the Fourier series transform $\mathcal{F}_{\Delta}$ of the kernels of the lattice fractional derivative $\mathbb{D}^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$. The functions $\tilde{K}^{\pm}_{\alpha}(k_i)$ are the Fourier integral transform $\mathcal{F}$ of the correspondent fractional derivative $\partial^{\alpha\pm}/\partial|x_i|^{\alpha}$ of the Riesz type.

In general, the order of the partial derivative $\partial^{\alpha\pm}/\partial|x_i|^{\alpha}$ is defined by the order of the lattice derivative $\mathbb{D}^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$ and it can be integer and non-integer. Let us give a definition of the fractional derivatives $\partial^{\alpha\pm}/\partial|x_i|^{\alpha}$.

4.3. The Riesz fractional derivative

The Riesz derivative of the order α is defined [8, 9] by the equation

$$\frac{\partial^{\alpha+}u(\mathbf{r})}{\partial|x_i|^{\alpha}} = \frac{1}{d_1(m,\alpha)}\int_{\mathbb{R}}\frac{1}{|z_i|^{\alpha+1}}\left(\Delta_i^m u\right)(\mathbf{z}_i)\,\mathrm{d}z_i, \quad (m>\alpha>0), \tag{79}$$

where $(\Delta_i^m u)(\mathbf{z}_i)$ is a finite difference of order m of a function $u(\mathbf{r})$ with the vector step $\mathbf{z}_i = x_i\,\mathbf{e}_i \in \mathbb{R}^3$ for the point $\mathbf{r}\in\mathbb{R}^3$. The non-centered difference is

$$\left(\Delta_i^m u\right)(\mathbf{z}_i) = \sum_{k=0}^{m}(-1)^k\frac{m!}{k!\,(m-k)!}\,u\left(\mathbf{r}-k\,\mathbf{z}_i\right), \tag{80}$$

and the centered difference

$$\left(\Delta_i^m u\right)(\mathbf{z}_i) = \sum_{k=0}^{m}(-1)^k\frac{m!}{k!\,(m-k)!}\,u\left(\mathbf{r}-(m/2-k)\,\mathbf{z}_i\right). \tag{81}$$

The constant $d_1(m,\alpha)$ is defined by

$$d_1(m,\alpha) = \frac{\pi^{3/2}A_m(\alpha)}{2^{\alpha}\Gamma(1+\alpha/2)\Gamma((1+\alpha)/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = 2\sum_{j=0}^{m}(-1)^{j-1}\frac{m!}{j!(m-j)!}\,j^{\alpha}$$

in the case of the non-centered difference (80), and

$$A_m(\alpha) = 2\sum_{j=0}^{[m/2]}(-1)^{j-1}\frac{m!}{j!(m-j)!}\,(m/2-j)^{\alpha}$$

in the case of the centered difference (81). The constant $d_1(m,\alpha)$ is different from zero for all $\alpha>0$ in the case of an even m and centered difference $(\Delta_i^m u)$ (see theorem 26.1 in [8]). In the case of a non-centered difference the constant $d_1(m,\alpha)$ vanishes if and only if $\alpha = 1, 3, 5, \ldots, 2[m/2]-1$. Note that the integral (79) does not depend on the choice of $m>\alpha$.

Using that $(-i)^{2j} = (-1)^j$, the Riesz fractional derivatives for even $\alpha = 2j$ are

$$\frac{\partial^{2j,+}u(\mathbf{r})}{\partial|x_i|^{2j}} = (-1)^j\,\frac{\partial^{2j}u(\mathbf{r})}{\partial x_i^{2j}}. \tag{82}$$

For $\alpha=2$ the Riesz derivative is the Laplace operator. The fractional derivatives $\partial^{\alpha,+}/\partial|x_i|^{\alpha}$ for even orders α are local operators. Note that the Riesz derivative $\partial^{1,+}/\partial|x_i|^{1}$ cannot be considered as a derivative of first order with respect to $|x_i|$. The Riesz derivatives for odd orders $\alpha = 2j + 1$ are non-local operators that cannot be considered as usual derivatives $\partial^{2j+1}/\partial x^{2j+1}$. For $\alpha = 1$ it is 'the square root of the Laplacian'.

The Fourier transform $\mathcal{F}$ of the Riesz fractional derivative is given by

$$\mathcal{F}\left(\frac{\partial^{\alpha} u(\mathbf{r})}{\partial \left| x_i \right|^{\alpha}}\right)(\mathbf{k}) = \left| k_i \right|^{\alpha} (\mathcal{F}u)(\mathbf{k}). \tag{83}$$

Equation (83) is valid for the Lizorkin space [8] and the space $C^{\infty}(\mathbb{R}^1)$ of infinitely differentiable functions on $\mathbb{R}^1$ with compact support. Using (83), we have

$$\frac{\partial^{\alpha} u(\mathbf{r})}{\partial \left| x_i \right|^{\alpha}} = \mathcal{F}^{-1}\left(\left| k_i \right|^{\alpha} (\mathcal{F}u)(\mathbf{k}) \right)(\mathbf{r}). \tag{84}$$

Equation (84) can be considered as a definition of the Riesz fractional derivative of order α.

4.4. The Riesz fractional integral

Riesz fractional integration is defined by

$$\mathbf{I}^{\alpha}_{\mathbf{r}} u(\mathbf{r}) = \mathcal{F}^{-1}\left(|\mathbf{k}|^{-\alpha} (\mathcal{F}u)(\mathbf{k}) \right). \tag{85}$$

The fractional integration (85) can be realized in the form of the Riesz potential defined as the Fourier's convolution of the form

$$\mathbf{I}^{\alpha}_{\mathbf{r}} u(\mathbf{r}) = \int_{\mathbb{R}^n} R_{\alpha}(\mathbf{r} - \mathbf{z}) u(\mathbf{z}) d\mathbf{z}, \quad (\alpha > 0), \tag{86}$$

where the function $R_{\alpha}(\mathbf{r})$ is the Riesz kernel. If $\alpha > 0$, the function $R_{\alpha}(\mathbf{r})$ is defined by

$$R_{\alpha}(\mathbf{r}) = \begin{cases} \gamma_n^{-1}(\alpha) |\mathbf{r}|^{\alpha - n} & \alpha \neq n + 2k, \\ -\gamma_n^{-1}(\alpha) |\mathbf{r}|^{\alpha - n} \ln |\mathbf{r}| & \alpha = n + 2k, \end{cases} \tag{87}$$

where $n \in \mathbb{N}$, and the constant $\gamma_n(\alpha)$ has the form

$$\gamma_n(\alpha) = \begin{cases} 2^{\alpha} \pi^{n/2} \Gamma(\alpha/2) \Big/ \Gamma\left(\dfrac{n - \alpha}{2}\right) & \alpha \neq n + 2k, \\ (-1)^{(n-\alpha)/2} 2^{\alpha - 1} \pi^{n/2}\ \Gamma(\alpha/2)\ \Gamma_{(}1 + [\alpha - n]/2_{)} & \alpha = n + 2k. \end{cases} \tag{88}$$

The Fourier transform of the Riesz fractional integration is given by

$$\mathcal{F}\left(\mathbf{I}^{\alpha}_{\mathbf{r}} u(\mathbf{r})\right) = |\mathbf{k}|^{-\alpha} (\mathcal{F}f)(\mathbf{k}). \tag{89}$$

Equation (89) holds for (86) if the function $u(\mathbf{r})$ belongs to Lizorkin space. The Lizorkin space of test functions on $\mathbb{R}^n$ is a linear space of all complex-valued infinitely differentiable functions $u(\mathbf{r})$ whose derivatives vanish at the origin

$$\Psi = \left\{ u(\mathbf{r}) \colon u(\mathbf{r}) \in S\left(\mathbb{R}^n\right), \quad \left(D^{\mathbf{n}}_{\mathbf{r}} u\right)(0) = 0, \quad |\mathbf{n}| \in \mathbb{N} \right\}, \tag{90}$$

where $S(\mathbb{R}^n)$ is the Schwartz test-function space. The Lizorkin space is invariant with respect to the Riesz fractional integration. Moreover, if $u(\mathbf{r})$ belongs to the Lizorkin space, then

$$\mathbf{I}^{\alpha}_{\mathbf{r}}\, \mathbf{I}^{\beta}_{\mathbf{r}} u(\mathbf{r}) = \mathbf{I}^{\alpha + \beta}_{\mathbf{r}} u(\mathbf{r}), \tag{91}$$

where $\alpha > 0$ and $\beta > 0$.

The Riesz fractional derivative yields an operator inverse to the Riesz fractional integration for a special space of functions

$$\frac{\partial^{\alpha,+}}{\partial|\mathbf{r}|^{\alpha}} \mathbf{I}^{\alpha}_{\mathbf{r}} u(\mathbf{r}) = u(\mathbf{r}), \quad (\alpha > 0). \tag{92}$$

Equation (92) holds for $u(\mathbf{r})$ belonging to the Lizorkin space. Moreover, this property is also valid for the Riesz fractional integration in the frame of L_p-spaces: $u(\mathbf{r}) \in L_p(\mathbb{R}^n)$ for $1 \leqslant p < n/\alpha$ (see theorem 26.3 in [8]).

4.5. Generalized conjugate Riesz derivative

We also define the new fractional derivatives $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ by the equation

$$\frac{\partial^{\alpha,-} u(\mathbf{r})}{\partial\left|x_i\right|^{\alpha}} = \mathcal{F}^{-1}\Big(\mathrm{i}\ \operatorname{sgn}\left(k_i\right) \left|k_i\right|^{\alpha} (\mathcal{F}u)(\mathbf{k})\Big)(\mathbf{r}). \tag{93}$$

Using $\mathrm{i}\, k_i\, |k_i|^{\alpha-1} = \mathrm{i}\ \operatorname{sgn}(k)\, |k_i|^{\alpha}$, and the Fourier transform of the Riesz fractional derivatives (84), the Riesz potential $\mathbf{I}^{1-\alpha}_i$ for $x_i \in \mathbb{R}$

$$\mathcal{F}\left(\mathbf{I}^{1-\alpha}_i u(\mathbf{r})\right)(\mathbf{k}) = \left|k_i\right|^{\alpha-1} (\mathcal{F}u)(\mathbf{k}), \tag{94}$$

and the usual first order derivative,

$$\mathcal{F}\left(\frac{\partial u(\mathbf{r})}{\partial x_i}\right)(\mathbf{k}) = \mathrm{i}\, k_i\ (\mathcal{F}u)(\mathbf{k}). \tag{95}$$

We can define the fractional operator (93) as a combination of the operators in the form

$$\frac{\partial^{\alpha,-}}{\partial\left|x_i\right|^{\alpha}} = \begin{cases} \dfrac{\partial}{\partial x_i} \dfrac{\partial^{\alpha-1,+}}{\partial\left|x_i\right|^{\alpha-1}} & \alpha > 1 \\[2ex] \dfrac{\partial}{\partial x_i} & \alpha = 1 \\[2ex] \dfrac{\partial}{\partial x_i}\, \mathbf{I}^{1-\alpha}_i & 0 < \alpha < 1, \end{cases} \tag{96}$$

where $\partial/\partial x_i$ is the usual derivative of first order with respect to coordinate x_i and $\mathbf{I}^{1-\alpha}_i$ is the Riesz potential of order $(1-\alpha)$ with respect to x_i,

$$\mathbf{I}^{1-\alpha}_i u(\mathbf{r}) = \int_{\mathbb{R}^1} R_{1-\alpha}(x_i - z_i)\, u\big(\mathbf{r} + (z_i - x_i)\mathbf{e}_i\big)\, \mathrm{d}z_i, \quad (\alpha < 1), \tag{97}$$

where $\mathbf{e}_i$ is the basis of the Cartesian coordinate system. For $0 < \alpha < 1$ the operator $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ is called the conjugate Riesz derivative [11]. Therefore, we call the operator $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ for all $\alpha > 0$ the generalized conjugate Riesz derivative.

The Fourier transform $\mathcal{F}$ of the fractional derivative (96) is given by

$$\mathcal{F}\left(\frac{\partial^{\alpha,-} u(\mathbf{r})}{\partial\left|x_i\right|^{\alpha}}\right)(\mathbf{k}) = \mathrm{i}\, k_i\, \left|k_i\right|^{\alpha-1} (\mathcal{F}u)(\mathbf{k}) = \mathrm{i}\ \operatorname{sgn}\left(k_i\right) \left|k_i\right|^{\alpha} (\mathcal{F}u)(\mathbf{k}). \tag{98}$$

Using (91), (92) and (96), it is easy to prove the equation

$$\frac{\partial^{\alpha,-}}{\partial \left| x_i \right|^{\alpha}} \mathbf{I}_i^{\alpha} u(\mathbf{r}) = \frac{\partial}{\partial x_i} \mathbf{I}_i^{1} u(\mathbf{r}) = u(\mathbf{r}), \quad (\alpha > 0). \tag{99}$$

Using (82) and (96), we get

$$\frac{\partial^{2j+1,-} u(\mathbf{r})}{\partial \left| x_i \right|^{2j+1}} = (-1)^j \frac{\partial^{2j+1} u(\mathbf{r})}{\partial x_i^{2j+1}}. \tag{100}$$

The fractional derivatives $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ for odd orders $\alpha = 2j + 1$ are local operators. Note that the generalized conjugate Riesz derivative $\partial^{2,-}/\partial|x_i|^2$ cannot be considered as a local derivative of second order with respect to $|x_i|$. The derivatives $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ for even orders $\alpha = 2j$ are non-local operators that cannot be considered as the usual derivatives $\partial^{2j}/\partial x^{2j}$. For $\alpha = 2$ the generalized conjugate Riesz derivative is not the Laplacian.

Equations (82) and (100) allow us to state that the usual partial derivatives of integer orders are obtained from fractional derivatives $\partial^{\alpha,\pm}/\partial|x_i|^{\alpha}$ in the following two cases. (1). For odd values $\alpha = 2j + 1 > 0$ we should use $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ only. (2). For even values $\alpha = 2j > 0$ we should use $\partial^{\alpha,+}/\partial|x_i|^{\alpha}$ only. Therefore we can formulate the following 'fractional correspondence principle': fractional generalization of the partial differential equation gives the correspondent differential equation with partial derivatives of integer orders if the fractional equation contains the fractional derivatives of the type $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ instead of the partial derivative of odd order, and $\partial^{\alpha,+}/\partial|x_i|^{\alpha}$ instead of the partial derivative of even order.

4.6. The continuum limit for lattice fractional derivatives

Let us formulate and prove a proposition about the connection between the lattice fractional derivative and the fractional derivatives of non-integer orders with respect to coordinates.

Proposition 1. *The lattice derivatives*

$$\mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} U(\mathbf{m}, \mathbf{n}) = \frac{1}{a_i^{\alpha}} \sum_{m_i=-\infty}^{+\infty} K_{\alpha}^{\pm}(n_i - m_i)\ (u(\mathbf{m}) - u(\mathbf{n})), \tag{101}$$

where $K_{\alpha}^{+}(n - m)$ is defined by (25) or (43), and $K_{\alpha}^{-}(n - m)$ is defined by (27) or (43), are transformed by the combination $\mathcal{F}^{-1} \bigcirc$ Lim $\bigcirc\, \mathcal{F}_{\Delta}$ into the fractional derivatives of order α with respect to coordinate x_i in the form

$$\mathcal{F}^{-1} \bigcirc \mathrm{Lim} \bigcirc \mathcal{F}_{\Delta}\left(\mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} \right) = \frac{\partial^{\alpha\pm}}{\partial \left| x_i \right|^{\alpha}}, \tag{102}$$

where $\partial^{\alpha,+}/\partial|x_i|^{\alpha}$ is the Riesz fractional derivative of order $\alpha > 0$ and $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ is the generalized conjugate Riesz derivative of order $\alpha > 0$.

Proof. Let us multiply equation (101) by $\exp(-\mathrm{i}k_i\, n_i\, a_i)$, and summing over n_i from $-\infty$ to $+\infty$. Then

$$\sum_{n_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix} U(\mathbf{m}, \mathbf{n})$$

$$=\frac{1}{a_i}\sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i}\, K_\alpha^{\pm}(n_i - m_i)\ (u(\mathbf{m}) - u(\mathbf{n})). \tag{103}$$

The interaction term on the right-hand side of (103) is

$$\sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} K_\alpha^{\pm}(n_i - m_i)(u(\mathbf{m}) - u(\mathbf{n}))$$

$$= \sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} K_\alpha^{\pm}(n_i - m_i)\ u(\mathbf{m})$$

$$- \sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k\, n\, a} K_\alpha^{\pm}(n_i - m_i)\ u(\mathbf{n}). \tag{104}$$

The first term on the right-hand side of (104) gives

$$\sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i}\ K_\alpha^{\pm}(n_i - m_i)\ u(\mathbf{m})$$

$$= \sum_{n_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} K_\alpha^{\pm}(n_i - m_i)\ \sum_{m_i=-\infty}^{+\infty} u(\mathbf{m})$$

$$= \sum_{n_i'=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i'\, a_i} K_\alpha^{\pm}\left(n_i'\right)\ \sum_{m_i=-\infty}^{+\infty} u(\mathbf{m})\mathrm{e}^{-\mathrm{i}k_i\, m_i\, a_i} = \hat{K}_\alpha^{\pm}\left(k_i\, a_i\right)\ \hat{u}(\mathbf{k}), \tag{105}$$

where $n_i' = n_i - m_i$, and

$$\hat{K}_\alpha^{\pm}\left(k_i\, a_i\right) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i}\ K_\alpha^{\pm}(n_i) = \mathcal{F}_\Delta\left\{K_\alpha^{\pm}(n_i)\right\}. \tag{106}$$

Using (74) and (16), the second term on the right-hand side of (104) gives

$$\sum_{n_i=-\infty}^{+\infty}\sum_{m_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} K_\alpha^{\pm}(n_i - m_i)\ u(\mathbf{n})$$

$$= \sum_{n_i=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}k_i\, n_i\, a_i} u(\mathbf{n})\ \sum_{m_i'=-\infty}^{+\infty} K_\alpha^{\pm}\left(m_i'\right) = \hat{u}(k,\, t)\hat{K}_\alpha^{\pm}(0),$$

where $m_i' = n_i - m_i$,

As a result, equation (103) has the form

$$\mathcal{F}_\Delta\left(\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ i\end{array}\right]U(\mathbf{m},\, \mathbf{n})\right) = \frac{1}{a_i^\alpha}\left(\hat{K}_\alpha^{\pm}\left(k_i\, a_i\right) - \hat{K}_\alpha^{\pm}(0)\right)\ \hat{u}(\mathbf{k}), \tag{107}$$

where $\mathcal{F}_\Delta$ is an operator notation for the Fourier series transform and $\hat{K}_\alpha^{-}(0) = 0$.

Using that

$$\hat{K}_\alpha^{+}\left(a_i\, k_i\right) - \hat{K}_\alpha^{+}(0) = \left|a_i\, k_i\right|^\alpha + o\left(\left|a_i k_i\right|^\alpha\right), \tag{108}$$

$$\hat{K}_\alpha^-\left(a_i\, k_i\right) = \mathrm{i}\ \ \mathrm{sgn}\left(k_i\right) \left|a_i\, k_i\right|^{\alpha} + o\left(\left|a_i k_i\right|^{\alpha}\right), \tag{109}$$

we get

$$\frac{1}{a_i^\alpha}\left(\hat{K}_\alpha^+\left(k_i\, a_i\right) - \hat{K}_\alpha^+(0)\right) = \left|k_i\right|^{\alpha} + \frac{1}{a_i^\alpha}\, o\left(\left|a_i k_i\right|^{\alpha}\right), \tag{110}$$

$$\frac{1}{a_i^\alpha}\, \hat{K}_\alpha^-\left(k_i\, a_i\right) = \mathrm{i}\ \ \mathrm{sgn}\left(k_i\right) \left|k_i\right|^{\alpha} + \frac{1}{a_i^\alpha}\, o\left(\left|a_i k_i\right|^{\alpha}\right). \tag{111}$$

In the limit $a_i \to 0$, we have

$$\tilde{K}_\alpha^+\left(k_i\right) = \lim_{a_i \to 0} \frac{1}{a_i^\alpha}\left(\hat{K}_\alpha^+\left(k_i\, a_i\right) - \hat{K}_\alpha^+(0)\right) = \left|k_i\right|^{\alpha}, \tag{112}$$

$$\tilde{K}_\alpha^-\left(k_i\right) = \lim_{a_i \to 0} \frac{1}{a_i^\alpha}\, \hat{K}_\alpha^-\left(k_i\, a_i\right) = \mathrm{i}\, k_i\, \left|k_i\right|^{\alpha-1}. \tag{113}$$

As a result, equation (107) in the limit $a_i \to 0$ gives

$$\mathrm{Lim}\ \circ \mathcal{F}_\Delta\left(\mathbb{D}^{\pm}\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m},\, \mathbf{n})\right) = \tilde{K}_\alpha^{\pm}\left(k_i\right)\, \tilde{u}(\mathbf{k}), \tag{114}$$

where

$$\tilde{K}_\alpha^+\left(k_i\right) = \left|k_i\right|^{\alpha}, \quad \tilde{K}_\alpha^-\left(k_i\right) = \mathrm{i}\, k_i\, \left|k_i\right|^{\alpha-1},$$

and $\tilde{u}(\mathbf{k}) = \mathrm{Lim}\ \hat{u}(\mathbf{k})$. The inverse Fourier integral transform of (114) is

$$\mathcal{F}^{-1} \circ\ \mathrm{Lim}\ \circ \mathcal{F}_\Delta\left(\mathbb{D}^{+}\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m},\, \mathbf{n})\right) = \frac{\partial^{\alpha,+} u(\mathbf{r})}{\partial\left|x_i\right|^{\alpha}}, \quad (\alpha > 0), \tag{115}$$

$$\mathcal{F}^{-1} \circ\ \mathrm{Lim}\ \circ \mathcal{F}_\Delta\left(\mathbb{D}^{-}\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m},\, \mathbf{n})\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial^{\alpha-1,+} u(\mathbf{r})}{\partial\left|x_i\right|^{\alpha-1}}\right), \quad (\alpha > 1), \tag{116}$$

$$\mathcal{F}^{-1} \circ\ \mathrm{Lim}\ \circ \mathcal{F}_\Delta\left(\mathbb{D}^{-}\begin{bmatrix}\alpha\\ i\end{bmatrix} U(\mathbf{m},\, \mathbf{n})\right) = \frac{\partial}{\partial x_i} I_i^{1-\alpha} u(\mathbf{r}), \quad (0 < \alpha < 1), \tag{117}$$

where the fractional derivative and fractional integral are

$$\frac{\partial^{\alpha,+}}{\partial\left|x_i\right|^{\alpha}} u(\mathbf{r}) = \mathcal{F}^{-1}\left\{\left|k_i\right|^{\alpha} \tilde{u}(\mathbf{k})\right\}, \quad I_i^{\alpha} u(\mathbf{r}) = \mathcal{F}^{-1}\left\{\left|k_i\right|^{-\alpha} \tilde{u}(\mathbf{k})\right\}. \tag{118}$$

Here we have used the connection (83) between the Riesz derivative and integral of the order α and their Fourier transforms.

As a result, we obtain relation (102). This ends the proof.

Using the independence of the position vectors of lattice site $\mathbf{n}_1 = (n_1, 0, 0)$, $\mathbf{n}_2 = (0, n_2, 0)$, $\mathbf{n}_3 = (0, 0, n_3)$ and the statement (102), we can prove that the continuum limits for the lattice mixed partial derivatives (58) and (59) has the form

$$\mathcal{F}^{-1} \circ\ \mathrm{Lim}\ \circ \mathcal{F}_\Delta\left(\mathbb{D}^{\pm,\pm}\begin{bmatrix}\alpha_1\ \alpha_2\\ i\ j\end{bmatrix}\right) = \frac{\partial^{\alpha_1,\pm}}{\partial\left|x_i\right|^{\alpha_1}} \frac{\partial^{\alpha_2,\pm}}{\partial\left|x_j\right|^{\alpha_2}} \quad (i \neq j), \tag{119}$$

$$\mathcal{F}^{-1} \bigcirc \text{ Lim } \bigcirc \mathcal{F}_{\Delta} \left(\mathbb{D}^{\pm,\pm,\pm} \left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ i & j & k \end{smallmatrix} \right] \right) = \frac{\partial^{\alpha_1,\pm}}{\partial \left| x_i \right|^{\alpha_1}} \frac{\partial^{\alpha_2,\pm}}{\partial \left| x_j \right|^{\alpha_2}} \frac{\partial^{\alpha_3,\pm}}{\partial \left| x_k \right|^{\alpha_3}},$$
$$(i \neq j \neq k \neq i), \tag{120}$$

and similarly for the other mixed lattice fractional derivatives. As a result, we obtain continuum limits for the lattice fractional derivatives in the form of the Riesz fractional derivatives with respect to coordinates.

4.7. The continuum limit for lattice vector differential operators

The continuum limit of the lattice vector differential operators gives the following differential operators of fractional vector calculus. These operators are defined by the Riesz fractional derivatives.

The continuum limit of the lattice gradient is

$$\text{Grad}_C^{\alpha,\pm} u(\mathbf{r}) = \mathcal{F}^{-1} \bigcirc \text{Limit} \bigcirc \mathcal{F}_{\Delta} \left\{ \text{Grad}_L^{\alpha,\pm} U \right\} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial^{\alpha,\pm} u(\mathbf{r})}{\partial \left| x_i \right|^{\alpha}}. \tag{121}$$

The continuum limit of the lattice divergence is

$$\text{Div}_C^{\alpha,\pm} \mathbf{u}(\mathbf{r}) = \mathcal{F}^{-1} \bigcirc \text{Limit} \bigcirc \mathcal{F}_{\Delta} \left\{ \text{Div}_L^{\alpha,\pm} \mathbf{U} \right\} = \sum_{i=1}^{3} \frac{\partial^{\alpha,\pm} u_i(\mathbf{r})}{\partial \left| x_i \right|^{\alpha}}. \tag{122}$$

The continuum limit of the lattice curl operator is

$$\text{Curl}_C^{\alpha,\pm} \mathbf{u}(\mathbf{r}) = \mathcal{F}^{-1} \bigcirc \text{Limit} \bigcirc \mathcal{F}_{\Delta} \left\{ \text{Curl}_L^{\alpha,\pm} \mathbf{U} \right\} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, \mathbf{e}_i \frac{\partial^{\alpha,\pm} u_k(\mathbf{r})}{\partial \left| x_j \right|^{\alpha}}, \tag{123}$$

where ϵ_{ijk} denotes the Levi-Civita symbol.

The scalar Laplacian for the scalar field can be defined by the repeated derivative of orders α in the form

$$\Delta_C^{\alpha,\alpha,\pm} u(\mathbf{r}) = \text{Div}_C^{\alpha,\pm} \text{ Grad}_C^{\alpha,\pm} u(\mathbf{r}) = \sum_{i=1}^{3} \frac{\partial^{\alpha,\pm}}{\partial \left| x_i \right|^{\alpha}} \frac{\partial^{\alpha,\pm} u(\mathbf{r})}{\partial \left| x_i \right|^{\alpha}}, \tag{124}$$

and by the derivative of the doubled order 2α,

$$\Delta_C^{2\alpha,\pm} u(\mathbf{r}) = \sum_{i=1}^{3} \frac{\partial^{2\alpha,\pm} u(\mathbf{r})}{\partial \left| x_i \right|^{2\alpha}}. \tag{125}$$

In general, the fractional derivatives (124) and (125) do not coincide [9].

The Riesz fractional derivatives $\partial^{\alpha,+}/\partial |x_i|^{\alpha}$ for $\alpha = 1$ are non-local operators that cannot be considered as the usual local derivatives $\partial/\partial x_i$, i.e. $\partial^{1,+}/\partial |x_i|^1 \neq \partial/\partial x_i$. Therefore the fractional differential vector operators (121–124) that correspond to the even (symmetric) kernels with $\alpha = 1$ are also non-local operators. At the same time, (121–124) for the odd (antisymmetric) kernels with $\alpha = 1$ give the well-known expressions for the vector differential operators. Note that the operator (125) for $\alpha = 1$ is the usual Laplacian with a minus sign, i.e. $\Delta_C^{2\alpha,+} = -\Delta$.

We can assume that the integral vector operations for the continuum can be defined using the fractional analog of Green's formula for a domain and the semigroup property for fractional integrals and derivatives suggested by Riesz (see sections 7, 10 and 11 in [53]).

5. Possible forms of lattice fractional calculus

In this paper we mainly pay attention to the lattice fractional vector operators that give the fractional derivatives of the Riesz type in a continuous limit. Let us note other possible types of lattice fractional vector calculus.

5.1. Fractional vector calculus based on the central differences of non-integer orders

The fractional-order central differences was suggested by Ortigueira in [55, 56]. In this section we generalize these differences to consider a three-dimensional case. The central difference $\Delta_j^{n,+}$ of positive even integer order n can be defined by

$$\Delta_j^{n,+} u(\mathbf{r}) = \sum_{m=-n/2}^{n/2} \frac{(-1)^{n/2+m}\, \Gamma(n+1)}{\Gamma(n/2-m+1)\, \Gamma(n/2+m+1)}\, u(\mathbf{r} - m\, \mathbf{a}_j). \quad (126)$$

The central difference $\Delta_j^{n,-}$ for positive odd integer order n is defined by

$$\Delta_j^{n,-} u(\mathbf{r}) = \sum_{m=-(n-1)/2}^{(n+1)/2} \frac{(-1)^{(n+1)/2+m}\, \Gamma(n+1)}{\Gamma((n+1)/2-m+1)\, \Gamma((n-1)/2+m+1)} \times u\Big(\mathbf{r} - (m-1/2)\, \mathbf{a}_j\Big). \quad (127)$$

These central differences allow us to define the corresponding partial derivatives defined by

$$D_j^{n,+} u(\mathbf{r}) = \lim_{a_j \to 0} \frac{\Delta_j^{n,+} u(\mathbf{r})}{\left|\mathbf{a}_j\right|^n}, \quad D_j^{n,-} u(\mathbf{r}) = \lim_{a_j \to 0} \frac{\Delta_j^{n,-} u(\mathbf{r})}{\left|\mathbf{a}_j\right|^n}. \quad (128)$$

Both derivatives (128) coincide with the usual partial derivative of even and odd integer orders with respect to x_j.

The fractional central differences of types 1 and 2 in the direction of the vector $\mathbf{a}_j$ are defined by the equations

$${}^{c}\Delta_j^{\alpha,+} u(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} \frac{(-1)^m\, \Gamma(\alpha+1)}{\Gamma(\alpha/2-m+1)\, \Gamma(\alpha/2+m+1)}\, u(\mathbf{r} - m\, \mathbf{a}_j). \quad (129)$$

$${}^{c}\Delta_j^{\alpha,-} u(\mathbf{r}) = \sum_{m=-\infty}^{+\infty} \frac{(-1)^m\, \Gamma(\alpha+1)}{\Gamma((\alpha+1)/2-m+1)\, \Gamma((\alpha-1)/2+m+1)} \times u\Big(\mathbf{r} - (m-1/2)\, \mathbf{a}_j\Big). \quad (130)$$

The fractional central differences (129) and (130) allow us to define the corresponding fractional central partial derivatives by the equations

$${}^{c}D_j^{\alpha,+} u(\mathbf{r}) = \lim_{a_j \to 0} \frac{{}^{c}\Delta_j^{\alpha,+} u(\mathbf{r})}{\left|\mathbf{a}_j\right|^\alpha}, \quad {}^{c}D_j^{\alpha,-} u(\mathbf{r}) = \lim_{a_j \to 0} \frac{{}^{c}\Delta_j^{\alpha,-} u(\mathbf{r})}{\left|\mathbf{a}_j\right|^\alpha}, \quad (131)$$

where $\alpha > -1$. These operators are called the fractional centered derivatives in [55], and the fractional central derivatives in [56]. In addition, the derivatives (131) are the fractional derivatives of the Grünwald–Letnikov type [55, 56].

We propose to call the operators (131) the fractional derivatives of the Grünwald–Letnikov–Ortigueira type in order to distinguish the operators (131) from the fractional derivatives of the Grünwald–Letnikov–Riesz type [8], which are used in lattice models in [47].

The properties of these fractional partial derivatives are describe in [55, 56]. Let us note the following properties,

$$ {}^{c}D_j^{\alpha,+}\, {}^{c}D_j^{\beta,+} u(\mathbf{r}) = {}^{c}D_j^{\alpha+\beta,+} u(\mathbf{r}), \tag{132} $$

$$ {}^{c}D_j^{\alpha,-}\, {}^{c}D_j^{\beta,-} u(\mathbf{r}) = -\, {}^{c}D_j^{\alpha+\beta,+} u(\mathbf{r}), \tag{133} $$

$$ {}^{c}D_j^{\alpha,-}\, {}^{c}D_j^{\beta,+} u(\mathbf{r}) = -\, {}^{c}D_j^{\alpha+\beta,-} u(\mathbf{r}), \tag{134} $$

where α, β, $\alpha + \beta > -1$ and $u(\mathbf{r})$ is a 'sufficiently good function'. These properties allow us to consider the fractional derivatives ${}^{c}D_j^{\alpha,\pm}$ as analogs of the continuum fractional derivatives $\partial^{\alpha,\pm}/\partial|x_j|^{\alpha}$ of the Riesz type.

Note that the expression in the definition (14) of the lattice fractional derivative $\mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ i \end{bmatrix}$ can be rewritten in the form

$$ \mathbb{D}^{\pm}\begin{bmatrix} \alpha \\ j \end{bmatrix} U(\mathbf{m}, \mathbf{n}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} K_{\alpha}^{\pm}(m_j)\, U(\mathbf{n} - m_j \mathbf{e}_j, \mathbf{n}), \quad (j = 1, 2, 3), \tag{135} $$

where we use the special case of the lattice vectors $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{m} = (m_1, m_2, m_3)$ in the form

$$ \mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)j \tag{136} $$

and the equality

$$ \sum_{m_j=-\infty}^{+\infty} K_{\alpha}^{\pm}(n_j - m_j)\, u(\mathbf{m}, t) = \sum_{m_j=-\infty}^{+\infty} K_{\alpha}^{\pm}(m_j)\, u\Big(\mathbf{n} - m_j\, \mathbf{e}_j, t\Big). \tag{137} $$

Equation (137) allows us to have an equivalent representation of the lattice fractional derivatives (14). The form (135) of derivative (14) can be generalized to give a definition of the lattice fractional derivatives based on the fractional central differences suggested by Ortigueira in [55, 56].

Let us define a lattice fractional partial derivative of the central type with respect to n_j in the direction $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$.

Definition 2. A lattice fractional partial derivative ${}^{c}\mathbb{D}^{+}\begin{bmatrix} \alpha \\ j \end{bmatrix}$ of the central type 1 is the operator

$$ {}^{c}\mathbb{D}^{+}\begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{n}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{c}K_{\alpha}^{+}(m_j)\, u(\mathbf{n} - m_j\, \mathbf{e}_j) \quad (j = 1, 2, 3), \tag{138} $$

where the interaction kernel ${}^{c}K_{\alpha}^{+}(m_j)$ is defined by the equation

$$ {}^{c}K_{\alpha}^{+}(m_j) = \frac{(-1)^{m_j}\, \Gamma(\alpha + 1)}{\Gamma\Big(\alpha/2 - m_j + 1\Big)\, \Gamma\Big(\alpha/2 + m_j + 1\Big)}. \tag{139} $$

It is easy to see that the kernels ${}^{c}K_{\alpha}^{+}(m)$ are even functions, ${}^{c}K_{\alpha}^{+}(-m) = {}^{c}K_{\alpha}^{+}(m)$.

The expression (138) with (143) for the lattice fractional derivative is based on the fractional central differences ${}^{c}\Delta_j^{\alpha,+}$ of type 1.

It should be noted that lattice models with long-range interaction of the form (143) and correspondent fractional non-local continuum models were suggested in [37, 38] (see also

[27, 50]). The motivation of this type of interaction is the power-law asymptotic behavior in the form (18). The kernel (143) describes one of the examples of a wide class of α-interactions suggested in [37, 38], where other examples of α-interactions for physical lattices have also been proposed.

Note that the kernel ${}^{c}K^{+}_{\alpha}(m)$ defined by (143) is equal to the kernel $K^{+}_{\alpha}(m)$ defined by (43) of the lattice derivative (14), i.e. we have

$$ {}^{c}K^{+}_{\alpha}(m_j) = K^{+}_{\alpha}(m_j). \tag{140}$$

Therefore the lattice fractional derivatives ${}^{c}\mathbb{D}^{+}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ and $\mathbb{D}^{+}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ are equal to each other,

$$ {}^{c}\mathbb{D}^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix} = \mathbb{D}^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix}. \tag{141}$$

These fractional derivatives are well defined for physical lattices.

Definition 3. A lattice fractional partial derivative ${}^{c}\mathbb{D}^{-}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ of the central type 2 is the operator

$$ {}^{c}\mathbb{D}^{-}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{n}, t) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{c}K^{-}_{\alpha}(m_j)\, u\Big(\mathbf{n} - \big(m_j - 1/2\big)\, \mathbf{e}_j\Big) \quad (j = 1, 2, 3), \tag{142}$$

where the interaction kernel ${}^{c}K^{\pm}_{\alpha}(m)$ is defined by the equation

$$ {}^{c}K^{-}_{\alpha}(m_j) = \frac{(-1)^{m_j}\, \Gamma(\alpha + 1)}{\Gamma\Big((\alpha+1)/2 - m_j + 1\Big)\, \Gamma\Big((\alpha-1)/2 + m_j + 1\Big)}. \tag{143}$$

This form of the lattice fractional derivative is based on the fractional difference ${}^{c}\Delta^{\alpha,-}_{j}$ of type 2. Although the kernels ${}^{c}K^{+}_{\alpha}(m)$ and $K^{+}_{\alpha}(m)$ defined by (143) and (43) are the same, it is easy to see that the kernels ${}^{c}K^{-}_{\alpha}(m)$ and $K^{-}_{\alpha}(m)$ defined by (143) and (43) are different.

Let us note some differences between ${}^{c}\mathbb{D}^{-}[{}^{\alpha}_{j}]$ and $\mathbb{D}^{-}[{}^{\alpha}_{j}]$. It is easy to see that the central kernels ${}^{c}K^{-}_{\alpha}(n)$ defined by (143) cannot be considered as odd functions, since

$$ {}^{c}K^{-}_{\alpha}(-n) \neq -\ {}^{c}K^{-}_{\alpha}(n), $$

whereas the kernel (43) is the odd function ($K^{-}_{\alpha}(-n) = -K^{-}_{\alpha}(n)$).

In addition, the lattice derivatives (142) do not have a clear physical interpretation related to the physical lattice since the operator ${}^{c}\mathbb{D}^{-}[{}^{\alpha}_{j}]$ describes an interaction of the lattice particles with an empty place between the particles. The half-integer numbers $\mathbf{n} - (m_j - 1/2)\,\mathbf{e}_j$ in (142) do not correspond to any of the lattice particles. In contrast to this, the lattice fractional derivatives (14) (or (135)) with kernel (43) describe the interaction of the lattice particle with all the other real particles of the lattice. This allows us to have a direct physical interpretation for lattice fractional derivatives (14) with kernels (27) and (43) as long-range interactions with all the particles of the physical lattice.

Let us note that the fractional central differences ${}^{c}\Delta^{\alpha,\pm}_{j}$ and the fractional derivatives ${}^{c}D^{\alpha,\pm}_{j}$ of the Grünwald–Letnikov–Ortigueira type (131) can be defined for $-1\alpha < 0$ [55, 56]. This allows us to define the lattice fractional integrals of central types 1 and 2 by equations (138) and (142) with negative $\alpha \in (-1, 0)$.

Proposition 2. *The lattice fractional derivatives* ${}^{c}\mathbb{D}^{\pm}[{}^{\alpha}_{j}]$ *of central types 1 and 2 defined by (138) and (142) are transformed by the continuous limit operation* Lim *into the fractional* *partial derivatives of order α with respect to coordinate x_j in the form*

$$\mathrm{Lim}\left({}^{\mathrm{c}}\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ j\end{array}\right] u(\mathbf{n})\right) = {}^{c}D_j^{\alpha,\pm}\, u(\mathbf{r}), \tag{144}$$

where ${}^{c}D_j^{\alpha,\pm}$ are the fractional derivatives of the Grünwald–Letnikov–Ortigueira type (131).

This proposition is based on the definition (131) of these fractional derivatives. The proof can be realized by analogy with the proof suggested in [47] for lattice models with Grünwald–Letnikov–Riesz type long-range interactions.

In proposition 2 the Fourier series transform $\mathcal{F}_{\Delta}$ and the Fourier integral transforms $\mathcal{F}^{-1}$ are not used for the transition to the continuum case. The correspondent fractional derivatives for the continuum are derived by the operation Lim of the continuous limit only.

Note that the transformation of the lattice derivative (14) with the kernel (43) by the combination of operations $\mathcal{F}^{-1} \bigcirc \mathrm{Lim} \bigcirc \mathcal{F}_{\Delta}$ gives the Riesz fractional derivative in the continuum limit. This statement is based on the asymptotic property (18) of the Fourier series transform $\hat{K}_{\alpha}^{+}(k)$ of the kernel $K_{\alpha}^{+}(n)$ in the form (17). The proof of this statement is given in [37, 38] and [27, 50]. Therefore the fractional derivative ${}^{c}D_j^{\alpha,+}$ and $\partial^{\alpha,\pm}/\partial|x_j|^{\alpha}$ should be connected for some classes of functions $u(\mathbf{r})$, since the kernels of the correspondent lattice derivatives are equal to each other (140). In addition, Ortigueira [56] proves an equivalence of the central fractional derivatives of type 1 and the one-dimensional Riesz potential, when α is not an even integer.

Also proved in the paper [56] is an equivalence of the fractional derivatives ${}^{c}D_j^{\alpha,-}$ of type 2 and the one-dimensional modified Riesz potential, when α is not an odd integer. This does not mean that the derivatives (142) with (143) are equivalent to the lattice derivatives (14) with (43), because the kernels (143) and (43) are not equivalent. Here the situation is similar to the case with the lattice derivatives (14) with two different kernels, where the exact kernel (27) and the asymptotic kernel (43) lead to an identical continuum fractional derivative. An equivalence of the continuum fractional derivatives does not mean an equivalence of correspondent lattice derivatives.

The derivatives (142) based on the fractional central differences of type 2 correspond to interaction of lattice particles with virtual particles with half-integer numbers which do not exist in the physical lattices. Therefore the suggested partial fractional central differences of types 1 and 2 and the partial fractional derivatives (138), (142) are more correctly considered as operators of a discrete analog of the fractional vector calculus which is not associated directly with the physical lattices. For the formulation of physical lattice models and for application in lattice mechanics, the lattice fractional derivatives (14) with the kernels (27), (43) and (27), (43) are more appropriate than operators based on the central differences.

5.2. Fractional vector calculus for physical lattices with long-range interaction of the Grünwald–Letnikov type

In this section, we consider a fractional vector calculus for models of a lattice with long-range interaction of the Grünwald–Letnikov type.

The difference of a fractional order $\alpha > 0$ and the correspondent fractional derivatives were introduced by Grünwald in 1867 and independently by Letnikov in 1868. The definition of the difference of non-integer orders is based on a generalization of the usual difference of integer orders. The difference of positive real order $\alpha \in \mathbb{R}_{+}$ is defined by the infinite series (see section 20 in [8]) in the form

$$\nabla^{\alpha}_{a,\pm} u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} u(x \mp na), \tag{145}$$

where $a > 0$. The difference $\nabla^{\alpha}_{a,+}$ is called a left-sided fractional difference, and $\nabla^{\alpha}_{a,-}$ is called a right-sided fractional difference. We note that the series in (145) converges absolutely and uniformly for every bounded function $u(x)$ and $\alpha > 0$. For the fractional difference, the semigroup property

$$\nabla^{\alpha}_{a} \nabla^{\beta}_{a} u(x) = \nabla^{\alpha+\beta}_{a} u(x), \quad (\alpha > 0, \quad \beta > 0) \tag{146}$$

is valid for any bounded function $u(x)$ (see property 2.29 in [9]). The Fourier transform of the fractional difference is given by

$$\mathcal{F}\left\{\nabla^{\alpha}_{a} u(x)\right\}(k) = \left(1 - \exp\left\{ika\right\}\right)^{\alpha} \mathcal{F}\left\{u(x)\right\}(k)$$

for any function $u(x) \in L_1(\mathbb{R})$ (see property 2.30 in [9]).

For integer values of $\alpha = m \in \mathbb{N}$ the differences $\nabla^{\alpha}_{a,\pm}$ are

$$\nabla^{m}_{a,\pm} u(x) = \sum_{n=1}^{m} \frac{(-1)^n \, m!}{n! \, (m-n)!} u(x \mp n\, a), \quad \left(a \in \mathbb{R}_{+}\right). \tag{147}$$

The left- and right-sided partial Grünwald–Letnikov derivatives of order $\alpha > 0$ are defined by

$${}^{GL}D^{\alpha}_{x_j,\pm} u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{\nabla^{\alpha}_{a_j,\pm} u(\mathbf{r})}{\left|a_j\right|^{\alpha}}. \tag{148}$$

Substitution of (145) into (148) gives

$${}^{GL}D^{\alpha}_{x_j,\pm} u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{\left|a_j\right|^{\alpha}} \sum_{n_j=0}^{\infty} \frac{(-1)^{n_j} \Gamma(\alpha+1)}{\Gamma\left(n_j+1\right)\Gamma\left(\alpha-n_j+1\right)} u\left(\mathbf{r} \mp n_j \mathbf{a}_j\right). \tag{149}$$

Note that these Grünwald–Letnikov derivatives for integer orders $\alpha = n \in \mathbb{N}$ are the usual partial derivatives

$${}^{GL}D^{n}_{x_j,\pm} u(\mathbf{r}) = (\pm 1)^n \frac{\partial^n u(\mathbf{r})}{\partial x_j^n}. \tag{150}$$

The fact that the differences of fractional order satisfy the semigroup property (146) allows us to prove [58] the semi-group property for the fractional derivatives in the form

$${}^{GL}D^{\alpha}_{x_j,\pm} \, {}^{GL}D^{\beta}_{x_j,\pm} = {}^{GL}D^{\alpha+\beta}_{x_j,\pm}, \quad (\alpha > 0, \quad \beta > 0). \tag{151}$$

This property leads to the commutative and associative properties of the Grünwald–Letnikov derivatives [58]. In addition, the Grünwald–Letnikov fractional derivatives coincide with the Marchaud fractional derivatives for $u(\mathbf{r}) \in L_p(\mathbb{R}^3)$, where $1 \leqslant p < \infty$ (see theorem 20.4 in [8]). The properties of the Grünwald–Letnikov fractional derivatives are described in section 20 of the book [8].

Let us define a lattice fractional partial derivative of the Grünwald–Letnikov type with respect to n_i in the direction $\mathbf{e}_i = \mathbf{a}_i/|\mathbf{a}_i|$.

Definition 4. The lattice fractional partial derivatives ${}^{GL}\mathbb{D}^{\pm}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ of the Grünwald–Letnikov type are the operators

$$^{GL}\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ j\end{array}\right] u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m_j=-\infty}^{+\infty} {}^{GL}K_{\alpha}^{\pm}(n_j - m_j)\, u(\mathbf{m}) \quad (j = 1, 2, 3), \tag{152}$$

where the interaction kernels ${}^{GL}K_{\alpha}^{\pm}(n)$ are defined by the equations

$$^{GL}K_{\alpha}^{\pm}(n) = \frac{(-1)^n\, \Gamma(1+\alpha)\, (H[n] \pm H[-n])}{2\, \Gamma(|n|+1)\, \Gamma(1+\alpha-|n|)}, \quad (\alpha > 0), \tag{153}$$

and α is the order of these derivatives, $H[n]$ is the Heaviside step function of a discrete variable n such that $H[n] = 1$ for $n \geqslant 1$, and $H[n] = 0$ for $n < 0$.

It is easy to see that the kernels ${}^{GL}K_{\alpha}^{\pm}(n)$ are even and odd functions,

$$^{GL}K_{\alpha}^{\pm}(-n) = \pm\, {}^{GL}K_{\alpha}^{\pm}(n).$$

The form of these lattice fractional derivatives is defined by the addition and subtraction of the fractional differences of the Grünwald–Letnikov type $\nabla_{a,\pm}^{\alpha}$ defined by (145).

It should be noted that lattice models with long-range interaction of the form ${}^{GL}K_{\alpha}^{+}(n)$ and correspondent fractional non-local continuum models were suggested in [47] (see also [27]).

Proposition 3. *The lattice fractional derivatives* ${}^{GL}\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ j\end{array}\right]$ *defined by (152) are transformed by the continuous limit operation* Lim *into the fractional partial derivatives of the Grünwald–Letnikov type of order* α *with respect to coordinate* x_j *in the form*

$$\mathrm{Lim}\left({}^{GL}\mathbb{D}^{\pm}\left[\begin{array}{c}\alpha\\ j\end{array}\right] u(\mathbf{m}) \right) = {}^{GL}\mathcal{D}_j^{\alpha,\pm}\, u(\mathbf{r}), \tag{154}$$

where ${}^{GL}D_j^{\alpha,\pm}$ *are the fractional derivatives of the Grünwald–Letnikov type*

$$^{GL}\mathcal{D}_j^{\alpha,\pm} = \frac{1}{2}\left({}^{GL}D_{x_j,+}^{\alpha} \pm {}^{GL}D_{x_j,-}^{\alpha} \right), \tag{155}$$

which contain the Grünwald–Letnikov fractional derivatives ${}^{GL}D_{x_j,\pm}^{\alpha}$ *defined by (149).*

This proposition can be proved by analogy with the proof for a lattice model with long-range interaction of the Grünwald–Letnikov–Riesz type suggested in [47].

Using (150), we can note that the derivatives (155) for integer orders $\alpha = n \in \mathbb{N}$ have the forms

$$^{GL}\mathcal{D}_j^{n,+} = \frac{1}{2}\left(\frac{\partial^n}{\partial x_j^n} + (-1)^n \frac{\partial^n}{\partial x_j^n} \right), \tag{156}$$

$$^{GL}\mathcal{D}_j^{n,-} = \frac{1}{2}\left(\frac{\partial^n}{\partial x_j^n} - (-1)^n \frac{\partial^n}{\partial x_j^n} \right). \tag{157}$$

These equations can be rewritten as

$$^{GL}\mathcal{D}_j^{n,+} = \begin{cases} 0, & n = 2m-1, \quad m \in \mathbb{N}, \\ \dfrac{\partial^n}{\partial x_j^n}, & n = 2m, \quad m \in \mathbb{N}, \end{cases} \tag{158}$$

$$
{}^{GL}\mathcal{D}_j^{n,-} = \begin{cases} \dfrac{\partial^n}{\partial x_j^n}, & n = 2m - 1, \quad m \in \mathbb{N}, \\ 0, & n = 2m, \quad m \in \mathbb{N}. \end{cases} \tag{159}
$$

Therefore ${}^{GL}\mathcal{D}_j^{n,+}$ is the usual derivative of integer order n for even values α only, and ${}^{GL}\mathcal{D}_j^{n,-}$ is the derivative of integer order n for odd values α only.

We assume that the lattice fractional integral operations can be defined by using (152) for $\alpha < 0$. This possibility is based on the fact that the series (145) can be used for $\alpha < 0$ (see section 20 in [8]). Equation (149) defines the Grünwald–Letnikov fractional integral if

$$
|u(x)| < c(1 + |x|)^{-\mu}, \quad \mu > |\alpha|. \tag{160}
$$

The existence of the Grünwald–Letnikov fractional integral means that we have the possibility of defining a lattice fractional integration.

The suggested lattice fractional vector calculus can be extended for bounded lattice models using the Grünwald–Letnikov fractional differences on finite intervals (see section 20.4 in [8]).

Definition 5. *The lattice fractional partial derivatives* ${}^{GL}_{B}\mathbb{D}^{\pm}\left[\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right]$ *of the Grünwald–Letnikov type for a bounded lattice with* m_j: $m_j^1 \leqslant m_j \leqslant m_j^2$ *are the operators*

$$
{}^{GL}_{B}\mathbb{D}^{\pm}\begin{bmatrix}\alpha \\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j^{\alpha}} \sum_{m_j = M_j^1}^{M_j^2} {}^{GL}K_{\alpha}^{\pm}(n_j - m_j)\, u(\mathbf{m}) \quad (j = 1, 2, 3), \tag{161}
$$

where the interaction kernels ${}^{GL}K_{\alpha}^{\pm}(n)$ *are defined by the equations (153).*

The lattice fractional derivatives ${}^{GL}_{B}\mathbb{D}^{\pm}\left[\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right]$ defined by (161) are transformed by the continuous limit operation Lim into the fractional partial derivatives of the Grünwald–Letnikov type

$$
{}^{GL}_{B}\mathcal{D}_j^{\alpha,\pm} = \frac{1}{2}\left({}^{GL}_{x_j^1}D^{\alpha}_{x_j,+} \pm {}^{GL}_{x_j^2}D^{\alpha}_{x_j,-} \right), \tag{162}
$$

which contain the Grünwald–Letnikov fractional derivatives [8, 9] defined on the finite interval $[x_j^1, x_j^2]$, where m_j^1, m_j^2 and m_j are defined by the equations $m_j^1 = \left[\frac{x_j^1}{a_j}\right]$, $m_j^2 = \left[\frac{x_j^2}{a_j}\right]$, $m_j = \left[\frac{x_j}{a_j}\right]$, in the form

$$
{}^{GL}_{B}D^{\alpha}_{x_j,\pm} u(\mathbf{r}) = \lim_{a_j \to 0+} \frac{1}{|a_j|^{\alpha}} \sum_{n_j=0}^{N_j^{\pm}} \frac{(-1)^{n_j}\,\Gamma(\alpha + 1)}{\Gamma(n_j + 1)\Gamma(\alpha - n_j + 1)}\, u(\mathbf{r} \mp n_j \mathbf{a}_j), \tag{163}
$$

where

$$
N_j^{+} = \left[\frac{x_j - x_j^1}{a_j}\right], \quad N_j^{-} = \left[\frac{x_j^2 - x_j}{a_j}\right]. \tag{164}
$$

Here the brackets [ ] mean the floor function that maps a real number to the largest previous integer number. The suggested form of fractional vector calculus for bounded lattice models is based on the Grünwald–Letnikov fractional differences on finite intervals (see section 20.4 in [8]). We assume that these calculi for bounded lattices can be developed using the proposed lattice fractional partial derivatives of the Grünwald–Letnikov type. Consistent formulations of the boundary conditions, extensivity and additivity for bounded lattice systems with long-range interactions and the correspondent continuum limits are open questions at this time.

5.3. About lattice vector calculus based on O(N)-models

We assume that it is possible to formulate a lattice fractional vector calculus based on the lattice $O(N)$-models with spin–spin long-range interactions. The classical $O(N)$-model (also called the N-vector model) is an N-dimensional lattice model suggested by Stanley in [59]. The most famous of the $O(N)$-models are the Ising model for $N = 1$, the XY-model for $N = 2$ and the Heisenberg model for $N = 3$. Lattice models of classical spins with long-range interactions were first suggested by Dyson in [60–62], where an infinite one-dimensional Ising model with long-range interactions is considered. A fractional dynamical approach for describing lattice models with long-range interaction of spin variables and the correspondent continuum models based on equations with fractional derivatives was suggested in [33, 39]. The correspondent equations for non-local continua contain derivatives of non-integer orders.

The lattice of the $O(N)$-model is a set of N-dimensional vector 'classical spins' $\mathbf{s}(\mathbf{n})$ of the unit length ($\mathbf{s}(\mathbf{n}) \in \mathbb{R}^N$, $|\mathbf{s}(\mathbf{n})| = 1$) which are placed on the $\mathbf{n}$-vertex of this N-dimensional lattice. The symbol of the orthogonal group $O(N)$ of dimension N is used in the name of the model. The orthogonal group $O(N)$ is the group of distance-preserving transformations of Euclidean space $\mathbb{R}^N$ that preserve a fixed point. An important subgroup of $O(N)$ is the special orthogonal group $SO(N)$ of the orthogonal matrices of determinant 1. The group $SO(N)$ is also called the rotation group, because the elements are the usual rotations around a point for dimension $N = 2$ and rotations around an axis for dimension $N = 3$.

Let us consider the classical lattice $O(2)$-model (also called the XY-model or the rotator model). In this lattice model, for each lattice site $\mathbf{n}$ there is a two-dimensional, unit-length vector $\mathbf{s}(\mathbf{n}) = (\cos\theta(\mathbf{n}), \sin\theta(\mathbf{n}))$. The classical spin configuration is an assignment of the angle $-\pi < \theta(\mathbf{n}) \leqslant \pi$ for each $\mathbf{n}$. For translation-invariant long-range interaction described by kernel $K(\mathbf{n} - \mathbf{m})$, and a point dependent external field $\mathbf{h}(\mathbf{n}) = (h(\mathbf{n}), 0)$, the Hamiltonian is defined in the form

$$\begin{aligned} H &= -\sum_{\mathbf{n}\neq\mathbf{m}} K(\mathbf{n}-\mathbf{m})\, (\mathbf{s}(\mathbf{n})\cdot\mathbf{s}(\mathbf{m})) - \sum_{\mathbf{n}} (\mathbf{h}(\mathbf{n})\cdot\mathbf{s}(\mathbf{n})) \\ &= -\sum_{\mathbf{n}\neq\mathbf{m}} K(\mathbf{n}-\mathbf{m})\, \cos(\theta(\mathbf{n})-\theta(\mathbf{m})) - \sum_{\mathbf{n}} h(\mathbf{n}) \cos(\theta(\mathbf{n})), \end{aligned} \tag{165}$$

where the sum runs over all pairs of spins $(\mathbf{n}, \mathbf{m})$ and the point $\cdot$ denotes the standard Euclidean scalar product for $\mathbb{R}^2$. In the Hamiltonian (165), the interaction is described by the periodic (trigonometrical) functions.

A fractional dynamical approach for describing one-dimensional lattice models with long-range interaction of spin variables and the correspondent fractional non-local continuum models is suggested in section V of [39]. The continuum equations which correspond to equations of a lattice with long-range interacting spins contain fractional derivatives of the Riesz type.

In general, for $O(N)$ lattice models, we should take into account the symmetries of these lattice systems. We should have the correspondent symmetry for the fractional non-local continuum if the continuous limit is formulated correctly. In the general case, the Riesz fractional derivatives and integrals on a circle cannot be defined in a consistent way. It is natural that the operations of fractional integration and differentiation are to be defined in such a way that they transform periodic functions into periodic ones. The Riesz fractional integration and differentiation do not have this property. Therefore a three-dimensional generalization of the approach suggested in [39] does not allows us to take into account $O(N)$ symmetry since this approach uses the Riesz fractional derivatives. For periodic functions, another type of fractional integro-differentiation should be used instead of the Riesz type. We assume that the Weyl fractional derivatives (see section 19 in [8]) or generalizations of the Grünwald–Letnikov fractional derivatives for the periodic case (see section 20.2 in [8]) should be used to preserve $O(N)$ symmetry in the continuum limit and to map periodic functions into periodic.

The Weyl fractional integral of order α is defined by

$$ {}^{W}I_x^{\alpha,\pm}\varphi(x) = \frac{1}{2\pi}\int_0^{2\pi} \Psi^{\alpha,\pm}(z)\, \varphi(x-z)\, dz, \quad (\alpha > 0) \tag{166} $$

where $z \in (0, 2\pi)$, $x \in \mathbb{R}^1$, and the function $\varphi(x)$ is the 2π-periodic function with zero mean value. The kernels $\Psi^{\alpha\pm}(z)$ of these integrals are

$$ \Psi^{\alpha,\pm}(z) = 2\sum_{n=1}^{\infty} \frac{\cos(n\, z \mp \alpha\, \pi/2)}{n^{\alpha}}. \tag{167} $$

The kernels can be expressed in terms of generalized Riemann zeta-functions

$$ \Psi^{\alpha,\pm}(z) = \frac{(2\pi)^{\alpha}\zeta(1-\alpha, \pm z/2\pi)}{\Gamma(\alpha)}, \quad (0 < z < 2\pi). \tag{168} $$

In the case of a positive integer $\alpha = m \in \mathbb{N}$, the kernels may be represented by

$$ \Psi^{m,\pm}(z) = -\frac{(\pm 2\pi)^m}{m!}\, B_m(z/2\pi), \tag{169} $$

where $B_m(z)$ is the mth Bernoulli polynomial. In the case of positive integer $\alpha = m \in \mathbb{N}$, the Weyl integration correspond to the usual integration.

The Weyl fractional derivative of order $n-1 < \alpha < n$ can be defined by the equation

$$ \mathcal{D}^{\alpha,\pm}\varphi(x) = (\pm 1)^n \frac{d^n}{dx^n}\, {}^{W}I_x^{n-\alpha,\pm}\varphi(x). \tag{170} $$

This operator is called the Weyl–Liouville derivative [8].

For the 2π-periodic function, we have the Fourier series

$$ \varphi(x) \sim \sum_{n=-\infty}^{+\infty} \varphi_n\, e^{inx}, \quad \varphi_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\varphi(x)\, dx, \tag{171} $$

the Weyl fractional integration is

$$ {}^{W}I_x^{\alpha,\pm}\varphi(x) \sim \sum_{n=-\infty}^{+\infty} \frac{\varphi_n}{(\pm in)^{\alpha}}\, e^{inx}, \tag{172} $$

and the Weyl fractional differentiation is

$$ {}^{W}D_x^{\alpha,\pm}\varphi(x) \sim \sum_{n=-\infty}^{+\infty} (\pm in)^{\alpha}\, \varphi_n\, e^{inx}, \tag{173} $$

where $n \in \mathbb{Z}$, and

$$ (\pm in)^{\alpha} = |n|^{\alpha}\, \exp\{\pm \mathrm{sgn}(n)\, i\, \pi\, \alpha/2\}. $$

Relations (172) and (173) allow us to hold the requirement that fractional integrals and derivatives of the 2π-periodic function are again a 2π-periodic function.

The lattice fractional vector calculus based on the classical $O(N)$ lattice models with spin–spin long-range interactions may be associated not only with the Weyl fractional derivatives (see section 9 in [8]), but also with the Grünwald–Letnikov fractional differences for the periodic case (see section 20.2 in [8]). Note that the existence of the Grünwald–Letnikov derivative for the periodic case is equivalent to this function being represented by the Weyl fractional integral (see section 20.2 in [8]) up to a constant term.

It should be noted that a fractional dynamical approach to discrete models, which describes the long-range coupled evolution of N rotators, populating the unitary circle and interacting via a cosine-like potential, and the correspondent continuum limit are considered in [33].

Unfortunately, a consistent formulation of lattice fractional calculus for $O(N)$-models of lattices with long-range interaction of classical spins remains an open question at this time. We can only assume that this calculus may be associated with the Weyl fractional derivatives (see section 9 in [8]), or the Grünwald–Letnikov fractional derivatives for the periodic case (see section 20.2 in [8]).

6. Examples of three-dimensional lattice models

In this section, we give some examples of the application of the suggested lattice fractional vector calculus. The three-dimensional lattice models with long-range interactions and the correspondent fractional non-local continuum models are suggested for the fractional Maxwell equations of non-local continuous media, and for the fractional generalization of the Mindlin and Aifantis continuum models of gradient elasticity.

6.1. A three-dimensional lattice analog of Maxwell equations

The well-known Maxwell equations for the electrodynamics of continuous media [63, 64] have the form

$$\operatorname{div} \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t), \tag{174}$$

$$\operatorname{div} \mathbf{B}(\mathbf{r}, t) = 0, \tag{175}$$

$$\operatorname{curl} \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \tag{176}$$

$$\operatorname{curl} \mathbf{H}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}, \tag{177}$$

where $\mathbf{E}$ is the electric field strength, $\mathbf{D}$ is the electric displacement field, $\mathbf{B}$ is the magnetic induction (the magnetic flux density), $\mathbf{H}$ is the magnetic field strength, ρ is the electric charge density and $\mathbf{j}$ is the electric current density.

Let us define the electric and magnetic fields on the three-dimensional lattice by equation (12). The electric field strength for the lattice is

$$\mathbf{E} = \sum_{i=1}^{3} \mathbf{e}_i \, E_i(\mathbf{m}, \mathbf{n}, t) = \sum_{i=1}^{3} \mathbf{e}_i \, \big(E_i(\mathbf{m}, t) - E_i(\mathbf{n}, t)\big), \tag{178}$$

where $E_i(\mathbf{n}, t)$ can be considered as components of the electric field for a lattice site that is defined by the spatial lattice points with the vector $\mathbf{n} = (n_1, n_2, n_3)$. The other fields $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{j}$ and ρ for the three-dimensional lattice with long-range interaction are defined analogously.

Using the lattice operators (70) and (71), we can write the equations

$$\mathrm{Div}_L^{\alpha,\pm}\, \mathbf{D}(\mathbf{m}, \mathbf{n}, t) = \rho(\mathbf{m}, \mathbf{n}, t), \tag{179}$$

$$\mathrm{Div}_L^{\alpha,\pm}\, \mathbf{B}(\mathbf{m}, \mathbf{n}, t) = 0, \tag{180}$$

$$\mathrm{Curl}_L^{\alpha,\pm}\, \mathbf{E}(\mathbf{m}, \mathbf{n}, t) = -\frac{\partial \mathbf{B}(\mathbf{m}, \mathbf{n}, t)}{\partial t}, \tag{181}$$

$$\mathrm{Curl}_L^{\alpha,\pm}\, \mathbf{H}(\mathbf{m}, \mathbf{n}, t) = \mathbf{j}(\mathbf{m}, \mathbf{n}, t) + \frac{\partial \mathbf{D}(\mathbf{m}, \mathbf{n}, t)}{\partial t}. \tag{182}$$

These equations can be considered as the Maxwell equations for the lattice with long-range interaction of the α-type. Lattice equations (183)–(186) with $\mathrm{Div}_L^{\alpha,+}$ and $\mathrm{Curl}_L^{\alpha,+}$ for $\alpha = 1$ give continuum equations with non-local operators of first order in the continuous limit. For this case the correspondence principle does not hold.

It is obvious that we would like to have a fractional generalization of partial differential equations which would enable us to obtain the original equations in the limit case when the order's generalized derivatives become equal to the initial integer values. This correspondence principle and the fact that only the fractional derivatives $\partial^{\alpha,-}/\partial |x_j|^{\alpha}$ for $\alpha = 1$ are the usual local derivatives of first order, allow us to consider equations (183)–(186) with $\mathrm{Div}_L^{\alpha,-}$ and $\mathrm{Curl}_L^{\alpha,-}$ as basic lattice equations. In addition, we can use (21). Then these basic lattice fractional Maxwell equations are

$$\mathrm{Div}_L^{\alpha,-}\, \mathbf{D}(\mathbf{m}, t) = \rho(\mathbf{m}, t), \tag{183}$$

$$\mathrm{Div}_L^{\alpha,-}\, \mathbf{B}(\mathbf{m}, t) = 0, \tag{184}$$

$$\mathrm{Curl}_L^{\alpha,-}\, \mathbf{E}(\mathbf{m}, t) = -\frac{\partial \mathbf{B}(\mathbf{m}, t)}{\partial t}, \tag{185}$$

$$\mathrm{Curl}_L^{\alpha,-}\, \mathbf{H}(\mathbf{m}, t) = \mathbf{j}(\mathbf{m}, t) + \frac{\partial \mathbf{D}(\mathbf{m}, t)}{\partial t}. \tag{186}$$

For $\alpha = 1$, equations (183)–(186) give equations (174)–(177) in the continuous limit.

The continuum limit of the lattice equations (183)–(186) gives the fractional Maxwell equations for the electrodynamics of non-local continuous media

$$\mathrm{Div}_C^{\alpha,\pm}\, \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t), \tag{187}$$

$$\mathrm{Div}_C^{\alpha,\pm}\, \mathbf{B}(\mathbf{r}, t) = 0, \tag{188}$$

$$\mathrm{Curl}_C^{\alpha,\pm}\, \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \tag{189}$$

$$\mathrm{Curl}_C^{\alpha,\pm}\, \mathbf{H}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}, \tag{190}$$

where $\mathrm{Div}_C^{\alpha,\pm}$ and $\mathrm{Curl}_C^{\alpha,\pm}$ are differential vector operators of order $\alpha > 0$ defined by equations (122) and (123).

For components the fractional Maxwell equations (187)–(190) can be represented as

$$\sum_{i=1}^{3} \frac{\partial^{\alpha,\pm} D_i(\mathbf{r},\, t)}{\partial \left| x_i \right|^{\alpha}} = \rho(\mathbf{r},\, t), \tag{191}$$

$$\sum_{i=1}^{3} \frac{\partial^{\alpha,\pm} B_i(\mathbf{r},\, t)}{\partial \left| x_i \right|^{\alpha}} = 0, \tag{192}$$

$$\sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial^{\alpha,\pm} E_k(\mathbf{r},\, t)}{\partial \left| x_j \right|^{\alpha}} = -\frac{\partial B_i(\mathbf{r},\, t)}{\partial t}, \tag{193}$$

$$\sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial^{\alpha,\pm} H_k(\mathbf{r},\, t)}{\partial \left| x_j \right|^{\alpha}} = j_i(\mathbf{r},\, t) + \frac{\partial D_i(\mathbf{r},\, t)}{\partial t}, \tag{194}$$

where $\partial^{\alpha\pm}/\partial|x_i|^{\alpha}$ are the fractional derivative of order $\alpha > 0$. For $\alpha = 1$, equations (191)–(194) with the derivatives $\partial^{\alpha,+}/\partial|x_i|^{\alpha}$ cannot be considered as the local Maxwell differential equations (174)–(177) since the Riesz derivatives with $\alpha = 1$ are non-local operators. In this case the Maxwell differential equations (191)–(194) with $\alpha = 1$ describe non-local media. The fractional Maxwell equations (191)–(194) with the generalized conjugate Riesz derivatives $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ of order $\alpha = 1$ are the usual Maxwell equations (174)–(177).

The fractional Maxwell equations (191)–(194) with the derivatives $\partial^{\alpha,-}/\partial|x_i|^{\alpha}$ of non-integer orders $\alpha > 0$ can be considered as the main equations of fractional non-local electrodynamics, and these equations correspond to the lattice model described by equations (183)–(186).

Note that the fractional Maxwell equations (191)–(194) with fractional derivatives of the Riesz type differ from the fractional Maxwell equations proposed in [27, 30], where the fractional Caputo derivatives are used.

6.2. Three-dimensional lattice models for fractional generalization of Mindlin's gradient elasticity

Mindlin [65] presented a theory of elasticity with microstructure, where different quantities are used for the microscale and for the macroscale. In Mindlin's theory of elasticity [65–67], the kinetic energy density and the deformation energy density are written in terms of quantities for the microscale and the macroscale. Gradient elasticity models are simplified versions of the elasticity theory with microstructure, in which the deformation energy density is only represented in terms of the macroscopic displacements. These versions differ in the assumed relation between the microscopic deformation and the macroscopic displacement. At the same time, despite the theoretical differences between these models, the equations for displacements of these models are identical [65–67].

The equations for Mindlin's gradient elasticity model can be obtained [65–67] using the following expressions for the kinetic and deformation energy densities. The deformation energy densities is

$$\begin{aligned} U = {} & \frac{1}{2}\lambda\ \varepsilon_{ii}\varepsilon_{jj} + \mu\ \varepsilon_{ij}\varepsilon_{ij} + \lambda_1\ \varepsilon_{ik,i}\varepsilon_{jj,k} + \lambda_2\ \varepsilon_{kk,i}\varepsilon_{jj,i} \\ & + \lambda_3\ \varepsilon_{ik,i}\varepsilon_{jk,j} + \lambda_4\ \varepsilon_{jk,i}\varepsilon_{jk,i} + \lambda_5\ \varepsilon_{jk,i}\varepsilon_{ij,k}, \end{aligned} \tag{195}$$

where ε_{ij} is the strain, λ and μ are the usual Lame constants and the various λ_i ($i = 1 ,..., 5$) are five additional constitutive coefficients. We also can use a simplification of the kinetic energy density [65] in the form

$$T = \frac{1}{2}\rho\, \partial_t u_i\, \partial_t u_i + \frac{1}{2}\rho\, l_1^2\, \dot{u}_{i,j}\, \dot{u}_{i,j}, \tag{196}$$

where ρ is the mass density, u_k is the displacement, $\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$. Using these expressions for the kinetic and deformation energy densities, we obtain the equations for displacements. Mindlin's equations for displacements have the form

$$\rho\, \ddot{u}_i - \rho\, l_1^2 \sum_{j=1}^{3} \partial_j^2 \ddot{u}_i = (\lambda + \mu) \sum_{j=1}^{3} \partial_i\, \partial_j u_j + \mu \sum_{j=1}^{3} \partial_j^2 u_i$$
$$- (\lambda + \mu)\, l_2^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \partial_k^2\, \partial_i\, \partial_j u_j - \mu\, l_3^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \partial_k^2\, \partial_j^2 u_i + f_i, \tag{197}$$

where f_i are the components of the body force, $u_i = u_i(\mathbf{r}, t)$ are the components of the displacement field for the continuum and

$$l_2^2 = \frac{4\lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 + 3\lambda_5}{2\,(\lambda + \mu)}, \qquad l_3^2 = \frac{\lambda_3 + 2\lambda_4 + \lambda_5}{2\,\mu}. \tag{198}$$

As a result, continuum equations (197) have two Lame constants and three additional parameters l_1, l_2 and l_3. All additional parameters have the dimension of length and can be linked to the underlying lattice microstructure.

In order to derive a fractional generalization of Mindlin's equations (197) and a correspondent three-dimensional lattice model, we assume that the lattice is characterized by the mutually perpendicular vectors $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3$ with equal length $a_1 = a_2 = a_3 = a$. For a primitive cubic Bravais lattice [7], we have three coupling constants and three gradient coupling constants.

Let us consider the lattice equation in the form

$$M\, \ddot{u}_i(\mathbf{n}, t) = A_0(\alpha) \sum_{j=1}^{3} \mathbb{D}^{+}\begin{bmatrix} 2\alpha \\ j \end{bmatrix} \ddot{U}_i(\mathbf{m}, \mathbf{n}, t) - A_1(\alpha) \sum_{j:\, j\neq i} \mathbb{D}^{-,-}\begin{bmatrix} \alpha & \alpha \\ j & i \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t)$$
$$-A_2(\alpha)\, \mathbb{D}^{+}\begin{bmatrix} 2\alpha \\ i \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t) - A_3(\alpha) \sum_{j\neq i} \mathbb{D}^{+}\begin{bmatrix} 2\alpha \\ j \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t)$$
$$-B_1(\alpha) \sum_{j:\, j\neq i} \left(\mathbb{D}^{-,-}\begin{bmatrix} 3\alpha & \alpha \\ j & i \end{bmatrix} + \mathbb{D}^{-,-}\begin{bmatrix} \alpha & 3\alpha \\ j & i \end{bmatrix} \right) U_j(\mathbf{m}, \mathbf{n}, t)$$
$$-B_2(\alpha) \sum_{j:\, j\neq i} \mathbb{D}^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ j & i \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t) - B_3(\alpha)\, \mathbb{D}^{+}\begin{bmatrix} 4\alpha \\ i \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t)$$
$$-B_4(\alpha) \sum_{\substack{k,j \\ k\neq j; k\neq i; j\neq i}} \mathbb{D}^{-,-,+}\begin{bmatrix} \alpha & \alpha & 2\alpha \\ j & i & k \end{bmatrix} U_j(\mathbf{m}, \mathbf{n}, t) - B_5(\alpha) \sum_{\substack{k,j \\ k\neq j}} \mathbb{D}^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ j & k \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t)$$
$$-B_6(\alpha) \sum_{j=1}^{3} \mathbb{D}^{+}\begin{bmatrix} 4\alpha \\ j \end{bmatrix} U_i(\mathbf{m}, \mathbf{n}, t) + F_i(\mathbf{n}, t), \tag{199}$$

where $U_i(\mathbf{m}, \mathbf{n}, t) = u_i(\mathbf{m}, t) - u_i(\mathbf{n}, t)$, and $A_1(\alpha)$, $A_2(\alpha)$, $A_3(\alpha)$, and $B_1(\alpha)$, ..., $B_6(\alpha)$ are the corresponding coupling constants of the lattice long-range interactions. If we consider the lattice with the interaction kernels $K_\alpha^+(n)$ that satisfy the conditions (22), then $K_\alpha^+(0) = 0$, and we can use $u(\mathbf{m}, t)$ instead of $U(\mathbf{m}, \mathbf{n})$ in equation (199).

This three-dimensional lattice model in the continuum limit gives a fractional generalization of Mindlin's model of the first gradient elasticity, if the Lame constants λ and μ are defined by the coupling constants

$$\frac{\mu_\alpha}{\rho} = \frac{A_3(\alpha)}{M}, \quad \frac{\lambda_\alpha}{\rho} = \frac{1}{M}\left(A_1(\alpha) - A_3(\alpha)\right). \tag{200}$$

The three additional parameters l_1, l_2 and l_3 of Mindlin's model are

$$l_1^2(\alpha) = \frac{A_0(\alpha)}{M}, \quad l_2^2(\alpha) = \frac{B_1(\alpha)}{A_1(\alpha)}, \quad l_3^2(\alpha) = \frac{B_5(\alpha)}{A_3(\alpha)}, \tag{201}$$

where the coupling constants are not independent

$$A_2(\alpha) = A_1(\alpha) + A_3(\alpha), \quad B_1(\alpha) = B_2(\alpha) = B_3(\alpha) = B_4(\alpha), \quad B_5(\alpha) = B_6(\alpha). \tag{202}$$

In the continuum limit ($a \to 0$), we obtain the equations for the fractional non-local continuum model which is a generalization of Mindlin's first gradient elasticity. These equations have the form

$$\begin{aligned}
\rho\, \ddot{u}_i = {} & \rho\, l_1^2(\alpha) \sum_{j=1}^{3} \frac{\partial^{2\alpha,+} \ddot{u}_i}{\partial |x_i|^{2\alpha}} \\
& + \left(\lambda_\alpha + \mu_\alpha\right) \left(\sum_{j:\, j \neq i} \frac{\partial^{\alpha,-}}{\partial |x_j|^{\alpha}} \frac{\partial^{\alpha,-} u_j}{\partial |x_i|^{\alpha}} + \frac{\partial^{2\alpha,+} u_i}{\partial |x_i|^{2\alpha}} \right) + \mu_\alpha \sum_{j=1}^{3} \frac{\partial^{2\alpha,+} u_i}{\partial |x_i|^{2\alpha}} \\
& - \left(\lambda_\alpha + \mu_\alpha\right) l_2^2(\alpha) \sum_{j:\, j \neq i} \left(\frac{\partial^{\alpha,-}}{\partial |x_j|^{\alpha}} \frac{\partial^{3\alpha,-} u_j}{\partial |x_i|^{3\alpha}} + \frac{\partial^{3\alpha,-}}{\partial |x_i|^{3\alpha}} \frac{\partial^{\alpha,-} u_j}{\partial |x_j|^{\alpha}} + \frac{\partial^{2\alpha,+}}{\partial |x_j|^{\alpha}} \frac{\partial^{2\alpha,+} u_i}{\partial |x_i|^{\alpha}} \right) \\
& - \left(\lambda_\alpha + \mu_\alpha\right) l_2^2(\alpha) \left(\sum_{\substack{k,j: \\ j \neq i; j \neq k; k \neq i}} \frac{\partial^{2\alpha,+}}{\partial |x_k|^{2\alpha}} \frac{\partial^{\alpha,-}}{\partial |x_j|^{\alpha}} \frac{\partial^{\alpha,-} u_i}{\partial |x_i|^{\alpha}} + \frac{\partial^{4\alpha,+} u_i}{\partial |x_i|^{4\alpha}} \right) \\
& - \mu_\alpha\, l_3^2(\alpha) \left(\sum_{\substack{k,l \\ k \neq l}} \frac{\partial^{2\alpha,+}}{\partial |x_k|^{2\alpha}} \frac{\partial^{2\alpha,+} u_i}{\partial |x_j|^{2\alpha}} + \sum_{j=1}^{3} \frac{\partial^{4\alpha,+} u_i}{\partial |x_i|^{4\alpha}} \right) + f_i,
\end{aligned} \tag{203}$$

where $u_i = u_i(\mathbf{r}, t)$ are components of the displacement field for the continuum and $f_i = f_i(\mathbf{r}, t)$ are the components of the body force.

For $\alpha = 1$, equations (203) give the differential equations of elasticity for the continuum

$$\begin{aligned} \rho\, \ddot{u}_i = \rho\, l_1^2 \sum_{j=1}^{3} \partial_j^2 \ddot{u}_i \\ &+ (\lambda + \mu) \left(\sum_{j:\, j \neq i} \partial_j\, \partial_i u_j + \partial_i^2 u_i \right) + \mu \sum_{j=1}^{3} \partial_j^2 u_i \\ &- (\lambda + \mu)\, l_2^2 \sum_{j:\, j \neq i} \left(\partial_j\, \partial_i^3 u_j + \partial_i^3\, \partial_j u_j + \partial_j^2\, \partial_i^2 u_i \right) \\ &- (\lambda + \mu)\, l_2^2 \left(\sum_{\substack{k,j: \\ j \neq i; j \neq k; k \neq i}} \partial_k^2 \partial_j\, \partial_i u_i + \partial_i^4 u_i \right) \\ &- \mu\, l_3^2 \left(\sum_{\substack{k,l \\ k \neq l}} \partial_k^2\, \partial_j^2 u_i + \sum_{j=1}^{3} \partial_j^4 u_i \right) + f_i\,. \end{aligned} \tag{204}$$

In equations (204) the derivatives of the integer orders with respect to the same spatial coordinates are clearly marked. Equations (204) can be rewritten in the form (197).

If the lattice equations (199) are written only through even lattice derivatives $\mathbb{D}^{+}\left[\begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$, then the correspondent continuum equations contain the Riesz derivatives $\partial^{\alpha+} / \partial |x_j|^{\alpha}$ of orders 1 and 3 that are non-local operators. In this case, we cannot get the usual Mindlin's model with derivatives of integer orders. Therefore, we suggest the equations of the lattice model that contain two type of lattice fractional derivatives $\mathbb{D}^{\pm}\left[\begin{smallmatrix} \alpha \\ j \end{smallmatrix} \right]$,

In the lattice model (199) all lattice derivatives are fractional orders. For a wide class of non-local elastic material the fractional derivatives are important only if short- and long-range particle interactions are present at the same time. This means that the lattice equations should include the lattice derivatives of integer and non-integer orders. To describe these types of material we can consider the lattice equation in the form

$$\begin{aligned} M\, \ddot{u}_i(\mathbf{n}, t) = A_0^L \sum_{j=1}^{3} \mathbb{D}^{+}\left[\begin{smallmatrix} 2 \\ j \end{smallmatrix} \right] \ddot{u}_i(\mathbf{m}, t) \\ &+ A_1^L \sum_{j} \mathbb{D}^{-,-}\left[\begin{smallmatrix} 1\, 1 \\ j\, i \end{smallmatrix} \right] u_j(\mathbf{m}, t) + A_2^L \sum_{j} \mathbb{D}^{+}\left[\begin{smallmatrix} 2 \\ j \end{smallmatrix} \right] u_i(\mathbf{m}, t) - \\ &+ B_1^L \sum_{j,m,i} \mathbb{D}^{-,+,-}\left[\begin{smallmatrix} 1\, \alpha\, 1 \\ j\, m\, i \end{smallmatrix} \right] u_j(\mathbf{m}, t) \\ &+ B_2^L \sum_{j,m,i} \mathbb{D}^{-,+,-}\left[\begin{smallmatrix} 1\, \alpha\, 1 \\ j\, m\, j \end{smallmatrix} \right] u_i(\mathbf{m}, t) + F_i(\mathbf{n}, t), \end{aligned} \tag{205}$$

where the displacement for the lattice is $u_i(\mathbf{m}, t) = u_i(m_1, m_2, m_3, t)$, and A_0^L, A_1^L, A_2^L, B_1^L and B_2^L are the coupling constants of the lattice long-range interactions. This three-dimensional lattice model in the continuum limit gives a fractional generalization of Mindlin's model of the first gradient elasticity. Using proposition 1, the operations $\mathcal{F}^{-1} \bigcirc \mathrm{Lim} \bigcirc \mathcal{F}_{\Delta}$ for lattice equations (205) give the continuum equations of the fractional gradient elasticity in the form

$$\rho\, \ddot{u}_i - A_0^C \sum_{j=1}^{3} \frac{\partial^2 \ddot{u}_i}{\partial x_j^2} = A_1^C \sum_{j=1}^{3} \frac{\partial^2 u_j}{\partial x_j \partial x_i} + A_2^C \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2}$$

$$+ B_1^C \sum_{j=1}^{3} \sum_{m=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial^{\alpha,+}}{\partial |x_m|^{\alpha}} \frac{\partial u_j}{\partial x_i} + B_2^C \sum_{j=1}^{3} \sum_{m=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial^{\alpha,+}}{\partial |x_m|^{\alpha}} \frac{\partial u_i}{\partial x_j} + f_i, \qquad (206)$$

where the constants fot continuum are defined by

$$A_i^C = \frac{a^2 \rho}{M} A_i^L \quad (i = 0, 1, 2), \quad B_j^C = \frac{a^{2+\alpha} \rho}{M} B_j^L \quad (j = 1; 2). \qquad (207)$$

Note that the definition of the lattice derivatives $\mathbb{D}^{\pm}\left[{\alpha \atop j} \right]$ includes $1/a_j^{\alpha}$. This means that we represent all real coupling constants of the lattice model in the form A_i^L/a^2 and $B_j^L/a^{2+\alpha}$. Therefore, the values of $|\mathbf{a}_j|$ do not exist in the relations (207). The Lame constants λ and μ are defined by the lattice coupling constants using the equation

$$\mu = \frac{\rho}{M} A_2^L, \quad \lambda = \frac{\rho}{M} \left(A_1^L - A_2^L \right). \qquad (208)$$

The three additional parameters l_1, $l_2(\alpha)$ and $l_3(\alpha)$ of Mindlin's model are

$$l_1^2 = \frac{A_0}{M}, \quad l_2^2(\alpha) = \frac{\left|B_1^L\right|}{\left|A_1^L\right|}, \quad l_3^2(\alpha) = \frac{\left|B_2^L\right|}{\left|A_2^L\right|}. \qquad (209)$$

Note that x_k, a, l_1^2, $l_2^2(\alpha)$ and $l_3^2(\alpha)$ are dimensionless values. Equations (206) can be considered as a generalization of the fractional Mindlin's equations.

For $\alpha = 2$, the three-dimensional lattice equations (205) give the well-known Mindlin's equation (197) for the displacement field $u_i = u_i(\mathbf{r}, t)$ of the continuum, where we take into account $\partial^{2,+}/\partial|x_m|^2 = -\partial^2/\partial x_m^2$.

For $\alpha = 1$, equations (206) give differential equations with a non-local operator since these equations contain the Riesz derivatives of odd orders that are non-local operators for odd integer α.

6.3. Three-dimensional lattice models for fractional generalization of Aifantis gradient elasticity

A simplified model of gradient elasticity has been suggested by Aifantis [68, 69], where the length-scales of Mindlin's models are taken equal to each other. The gradient terms are used to take into account so-called weak non-locality. In order to describe a weak non-locality of power-law type, we should use terms with fractional gradients and fractional Laplace operators [46, 48]. The one-dimensional lattice models for fractional elasticity and the correspondent continuum equations were suggested in [46–48, 50]. In this section we apply the suggested lattice vector calculus to generalize one-dimensional lattice models of fractional elasticity for three-dimensional lattices. To generalize these models for three-dimensional lattices, we consider for simplicity a primitive orthorhombic Bravais lattice with long-range interactions, where $\mathbf{a}_i = a_i\, \mathbf{e}_i$, and $\mathbf{e}_i$ is the basis of the Cartesian coordinate system.

As a microstructural basis of the three-dimensional fractional gradient elasticity for the anisotropic case, we can consider the following equations of three-dimensional lattice with long-range interactions

$$M\frac{\partial^2 u_i(\mathbf{n},t)}{\partial t^2} = \sum_{j,l} A^L_{ijkl}\ \mathbb{D}^{-,-}\begin{bmatrix} 1\ 1 \\ j\ l \end{bmatrix} u_k(\mathbf{m},t)$$

$$+\sum_{j,m,l} B^L_{ijkl}\ \mathbb{D}^{-,+,-}\begin{bmatrix} 1\ \alpha\ 1 \\ j\ m\ l \end{bmatrix} u_k(\mathbf{m},t) + F_i(\mathbf{n},t), \tag{210}$$

where $u_k(\mathbf{m},t) = u_k(m_1, m_2, m_3, t)$ is the displacement for the lattice and A^L_{ijkl} and B^L_{ijkl} are the lattice coupling constants. We assume that the fourth-order tensors A^L_{ijkl} and B^L_{ijkl} have the same type of symmetry as the fourth-order elastic stiffness tensor C_{ijkl}:

$$A^L_{ijkl} = A^L_{jikl} = A^L_{ijlk} = A^L_{klij}, \quad B^L_{ijkl} = B^L_{jikl} = B^L_{ijlk} = B^L_{klij}. \tag{211}$$

For a primitive orthorhombic Bravais lattice [7], we have nine coupling constants A^L_{ijkl} and nine gradient coupling constants B^L_{ijkl}.

Using the statement (102), (119) and (120), the operations $\mathcal{F}^{-1} \circ \mathrm{Lim} \circ \mathcal{F}_\Delta$ for lattice equations (210) give the continuum equations for the fractional gradient elasticity in the form

$$\rho\,\frac{\partial^2 u_i(\mathbf{r},t)}{\partial t^2} = \sum_{j,l} A^C_{ijkl}\,\frac{\partial^2 u_k(\mathbf{r},t)}{\partial x_j \partial x_l} + \sum_{j,m,l} B^C_{ijkl}\,\frac{\partial}{\partial x_j}\frac{\partial^{\alpha+}}{\partial |x_m|^{2\alpha}}\frac{\partial u_k(\mathbf{r},t)}{\partial x_l} + f_i(\mathbf{r},t), \tag{212}$$

where $u_i(\mathbf{r},t)$ are the components of the displacement vector field for the continuum, and A^C_{ijkl} and B^C_{ijkl} are the coupling constants for the non-local continuum. The coupling constants of the continuum are defined by the lattice coupling constants A^L_{ijkl} and B^L_{ijkl} by the relations

$$A^C_{ijkl} = \frac{\rho}{M}\,A^L_{ijkl}, \quad B^C_{ijkl} = \frac{\rho}{M}\,B^L_{ijkl}. \tag{213}$$

Note that the definition of the lattice derivatives $\mathbb{D}^+\begin{bmatrix} \alpha \\ j \end{bmatrix}$ includes $1/a_j^\alpha$. This means that we represent all real coupling constants of the lattice model in the form $A^L_{ijkl}/a_j a_l$ and $B^L_{ijkl}/a_j a_l a_m^\alpha$. Therefore, the values of $|\mathbf{a}_j|$ do not exist in the relations (213).

In the case $a_1 = a_2 = a_3 = a$, we obtain the fourth-order elastic stiffness tensor C_{ijkl} in the form

$$C_{ijkl} = A^C_{ijkl} = \frac{\rho}{M}\,A^L_{ijkl}. \tag{214}$$

If $B^L_{ijkl} = g_B\,A^L_{ijkl}$, then the scale parameter l_α^2 is $l_\alpha^2 = g_B$ and we have $B^C_{ijkl} = l_\alpha^2\ C_{ijkl}$. Note that x_k, a_k and l_α^2 are dimensionless values.

If $\alpha = 2$, then equation (212) gives the well-known continuum equation of gradient elasticity

$$\rho\,\ddot{u}_i(\mathbf{r},t) = \sum_{j,k,l} C_{ijkl}\,\partial_j\,\partial_l\,u_k(\mathbf{r},t) \pm l_\alpha^2 \sum_{j,k,l,m} C_{ijkl}\,\partial_j\,\partial^2_m\,\partial_l\,u_k(\mathbf{r},t) + f_i(\mathbf{r},t). \tag{215}$$

For isotropic materials, C_{ijkl} are expressed in terms of the Lame constants λ and μ by

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right). \tag{216}$$

Let us give the stress–strain constitutive relation for fractional gradient elasticity (212). Equation (212) can be represented in the form

$$\rho\,\ddot{u}_i(\mathbf{r},t) = \sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \tag{217}$$

where σ_{ij} is the stress tensor that is connected with the strain tensor

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \tag{218}$$

by the constitutive relation

$$\sigma_{ij} = \sum_{k,l} A^C_{ijkl}\ \varepsilon_{kl} + \sum_{k,l,m} B^C_{ijkl} \frac{\partial^{\alpha+}}{\partial|x_m|^{\alpha}} \varepsilon_{kl}. \tag{219}$$

If we use (214) and assume that

$$B^C_{ijkl} = \pm\ l_\alpha^2\ A^C_{ijkl}, \tag{220}$$

then relation (219) can be rewritten as

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\left(1 \pm l_\alpha^2\ \Delta_C^{\alpha,+}\right)\varepsilon_{kl}, \tag{221}$$

where $\Delta)_C^{\alpha,+}$ is the lattice Laplacian defined by (125) in the form

$$\Delta_C^{\alpha,+} = \sum_{m=1}^{3} \frac{\partial^{\alpha+}}{\partial|x_m|^{\alpha}} \tag{222}$$

which is the fractional Laplacian. Equation (221) gives the constitutive relation for fractional gradient elasticity. For $\alpha = 2$, relation (221) gives

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\left(1 \mp l_\alpha^2\ \Delta\right)\ \varepsilon_{kl}. \tag{223}$$

This is the well-known stress–strain constitutive relation for gradient elasticity [68, 69]. If we consider the case with

$$u_x(\mathbf{r}, t) = u(x, t), \quad u_y(\mathbf{r}, t) = u_z(\mathbf{r}, t) = 0, \tag{224}$$

$$f_x(\mathbf{r}, t) = f(x, t), \quad f_y(\mathbf{r}, t) = f_z(\mathbf{r}, t) = 0, \tag{225}$$

then we get the one-dimensional fractional elasticity models suggested in [46, 48, 50]. The lattice models (205) and (210) are three-dimensional generalizations of the one-dimensional lattice models proposed in [46, 48, 50]. In addition, the equation (210) of the lattice with long-range interactions allows us to derive the stress–strain constitutive relations for fractional non-local elasticity by using the usual law (217).

7. Conclusion

In this paper an extension of fractional vector calculus for three-dimensional unbounded lattices with long-range interactions is suggested. The main advantage of the suggested lattice fractional calculus is the possibility of using this calculus to formulate a lot of microstructural models of fractional non-local continua. The lattice analogs of fractional partial derivatives are represented by kernels of long-range interactions of lattice particles. The Fourier series transforms of these kernels have a power-law form with respect to the components of the wave vector. The proposed form of the long-range interactions allows us to use the lattice equations not only for the integer but also for the fractional order of lattice partial derivatives. The continuous limit for these lattice partial derivatives gives the fractional derivative of Riesz type with respect to space coordinates. The advantage of the suggested types of inter-particle interactions (α-interactions) in the lattice is a possibility to formulate different lattice models for a wide class of fractional non-local generalizations of local continuum models in different areas of physics and mechanics.

Let us note some possible generalizations and extensions of the suggested lattice fractional vector calculus which are partially discussed in section 5.

(a) A discrete fractional vector calculus can be developed using the central and generalized fractional differences suggested in [55–57]. In the continuous limit these differences give the central and generalized fractional derivatives.
(b) A lattice fractional vector calculus based on the Grünwald–Letnikov fractional differences (see section 20 in [8]) can be developed using the lattice models with long-range interaction of the Grünwald–Letnikov type and the Grünwald–Letnikov–-Riesz type proposed in [47].
(c) We assume that it is possible to formulate a lattice fractional vector calculus based on the classical $O(N)$ lattice models (the classical Heisenberg lattice model or the XY-model) with spin–spin long-range interactions. This calculus can be connected with the Weyl fractional derivatives (see section 9 in [8]), or the Grünwald–Letnikov fractional differences for the periodic case (see section 20.2 in [8]).
(d) We assume that the suggested lattice fractional vector calculus for unbounded physical lattices can be extended for bounded lattices and the correspondent continuum models. This extension can be developed using the Grünwald–Letnikov fractional differences on finite intervals (see section 20.4 in [8]). A consistent description of possible boundary conditions, the extensivity and additivity for bounded lattices with long-range interactions and their connections with the correspondent continuum models are open questions at this time.
(e) A vector difference calculus of integer order for physical lattice models is suggested in [41–43]. This calculus is considered for models defined on a general triangulating graph using discrete field quantities and differential operators analogous to differential forms and exterior differential calculus. We assume that the approach suggested in [41–43] can be generalized for fractional operators of non-integer orders. To this aim, it is possible to use a fractional generalization of exterior differential calculus of differential forms suggested in [27, 30, 45] and the fractional-order differences [8, 9].

In this paper, we give some examples of applications of the suggested lattice fractional vector calculus. Using the lattice calculus, we propose three-dimensional lattice models with long-range interactions for the fractional Maxwell equations of non-local continuous media and for the fractional generalization of the Mindlin and Aifantis continuum models of three-dimensional gradient elasticity. Lattice fractional vector calculus also allows us to consider lattice models and the correspondent fractional generalizations of continuum equations for a wide class of long-range interactions of particles. The suggested lattice vector calculus is based on long-range inter-particle interactions of the power-law type. Therefore it can be used to describe the non-local properties of materials at the microscale and nanoscale, where inter-atomic and inter-molecular interactions are prevalent in determining the properties of these materials.

Appendix. Interaction kernels for lattice derivatives of integer orders

The inverse relations to the definition of $\hat{K}_\alpha^{\pm}(k)$ by equation (17) for $\hat{K}_\alpha^{+}(0) - \hat{K}_\alpha^{+}(k) = |k|^\alpha$ and by equation (19) for $\hat{K}_\alpha^{-}(0) - \hat{K}_\alpha^{-}(k) = \mathrm{i}\,\mathrm{sgn}\,(k)\,|k|^\alpha$ has the form

$$K_\alpha^+(n) = \frac{1}{\pi}\int_0^\pi k^\alpha \ \cos(n\,k)\,\mathrm{d}k, \quad K_\alpha^-(n) = -\frac{1}{\pi}\int_0^\pi k^\alpha \ \sin(n\,k)\,\mathrm{d}k, \tag{A.1}$$

where $s \in \mathbb{N}$. Using the integral (see section 2.5.3.5 in [51]) of the form

$$\int_0^\pi x^m \ \cos(n\,x)\,\mathrm{d}x = \frac{(-1)^{n+2}}{n^{m+1}} \sum_{k=0}^{[(m-1)/2]} \frac{(-1)^k\, m!}{(m-2n-1)!}\,(\pi\,n)^{m-2k-1}$$
$$+\frac{(-1)^{[(m+1)/2]}\, m!}{n^{m+1}}\,(2[(m+1)/2]-m), \quad (m\in\mathbb{N}) \tag{A.2}$$

$$\int_0^\pi x^m \ \sin(n\,x)\,\mathrm{d}x = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{[m/2]} \frac{(-1)^k\, m!}{(m-2n)!}\,(\pi\,n)^{m-2k}$$
$$+\frac{(-1)^{[m/2]}\, m!}{n^{m+1}}\,(2[m/2]-m+1), \quad (m\in\mathbb{N}), \tag{A.3}$$

where $[x]$ is the integer part of the value x, we get $K_\alpha^{\pm}(n)$ for integer positive $\alpha = m$ by the equation

$$K_\alpha^+(n) = \sum_{k=0}^{[(\alpha-1)/2]} \frac{(-1)^{n+k}\,\alpha!\,\pi^{\alpha-2k-2}}{(\alpha-2n-1)!}\,\frac{1}{n^{2k+2}}$$
$$+\frac{(-1)^{[(\alpha+1)/2]}\,\alpha!\,(2[(\alpha+1)/2]-s)}{\pi\,n^{\alpha+1}}, \tag{A.4}$$

and

$$K_\alpha^-(n) = -\sum_{k=0}^{[\alpha/2]} \frac{(-1)^{n+k+1}\,\alpha!\,\pi^{\alpha-2k-1}}{(\alpha-2n)!}\,\frac{1}{n^{2k+2}}$$
$$-\frac{(-1)^{[\alpha/2]}\,\alpha!\,(2[\alpha/2]-\alpha+1)}{\pi\,n^{\alpha+1}}. \tag{A.5}$$

Here $2[(\alpha+1)/2]-\alpha = 1$ for odd $\alpha = m$ and $2[(\alpha+1)/2]-\alpha = 0$ for even $\alpha = m$.

Direct integration (A.1) for $\alpha = 1, 2, 3, 4$, or equation (A.5), gives the examples of the kernels $K_\alpha^+(n)$ in the form

$$K_1^+(n) = \frac{1}{\pi}\int_0^\pi k \ \cos(n\,k)\,\mathrm{d}k = -\frac{1-(-1)^n}{\pi\,n^2}, \tag{A.6}$$

$$K_2^+(n) = \frac{1}{\pi}\int_0^\pi k^2 \ \cos(n\,k)\,\mathrm{d}k = \frac{2\,(-1)^n}{n^2}, \tag{A.7}$$

$$K_3^+(n) = \frac{1}{\pi}\int_0^\pi k^3 \ \cos(n\,k)\,\mathrm{d}k = \frac{3\,\pi\,(-1)^n}{n^2} + \frac{6\,\left(1-(-1)^n\right)}{\pi\,n^4}, \tag{A.8}$$

$$K_4^+(n) = \frac{1}{\pi}\int_0^\pi k^3 \ \cos(n\,k)\,\mathrm{d}k = \frac{4\,\pi^2\,(-1)^n}{n^2} - \frac{24\,(-1)^n}{n^4}, \tag{A.9}$$

where $n \in \mathbb{N}$. Note that

$$K_m^+(0) = \frac{\pi^m}{m+1}, \quad m \in \mathbb{N}.$$

Note that $(1-(-1)^n) = 2$ for odd n and $((-1)^n - 1) = 0$ for even n.

The examples of the kernels $K_\alpha^-(n)$ have the form

$$K_1^-(n) = -\frac{1}{\pi}\int_0^\pi k \ \sin(n\,k)\,\mathrm{d}k = \frac{(-1)^n}{n}, \tag{A.10}$$

$$K_2^-(n) = -\frac{1}{\pi}\int_0^\pi k^2 \ \sin(n\,k)\,\mathrm{d}k = \frac{(-1)^n\,\pi}{n} + \frac{2\big(1-(-1)^n\big)}{\pi\,n^3}, \tag{A.11}$$

$$K_3^-(n) = -\frac{1}{\pi}\int_0^\pi k^3 \ \sin(n\,k)\,\mathrm{d}k = \frac{(-1)^n\,\pi^2}{n} - \frac{6\,(-1)^n}{n^3}, \tag{A.12}$$

$$K_4^-(n) = -\frac{1}{\pi}\int_0^\pi k^3 \ \sin(n\,k)\,\mathrm{d}k = \frac{(-1)^n\,\pi^3}{n} - \frac{12\,(-1)^n\pi}{n^3} - \frac{24\,\big(1-(-1)^n\big)}{\pi\,n^5}, \tag{A.13}$$

where $n \in \mathbb{N}$. Note that $K_m^-(0) = 0$ for all $m \in \mathbb{N}$.

We can see that

$$K_3^+(n) = \frac{3\pi}{2}K_2^+(n) - \frac{6}{n^2}K_1^+(n), \tag{A.14}$$

$$K_4^+(n) = 2\,\pi^2\,K_2^+(n) - \frac{12}{n^2}K_2^+(n), \tag{A.15}$$

and

$$K_3^-(n) = \pi^2\,K_1^-(n) - \frac{6}{n^2}K_1^-(n), \tag{A.16}$$

$$K_4^-(n) = \pi^3\,K_1^-(n) - \frac{12}{n^2}K_2^-(n). \tag{A.17}$$

For $\alpha = 2$, we can also use the long-range interactions in the following power-law form

$$K_2^+(n-m) = \frac{1}{\zeta(\alpha-1)|n-m|^{\alpha+1}}, \quad (\alpha > 2, \quad \alpha \neq 3, 4, 5, \ldots), \tag{A.18}$$

where $\zeta(z)$ is the Riemann zeta-function. For details see section 8.11–8.12 in [27].

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