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Fractional Liouville and BBGKI Equations

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Abstract. We consider the fractional generalizations of Liouville equation. The normalization condition, phase volume, and average values are generalized for fractional case. The interpretation of fractional analog of phase space as a space with fractal dimension and as a space with fractional measure are discussed. The fractional analogs of the Hamiltonian systems are considered as a special class of non-Hamiltonian systems. The fractional generalization of the reduced distribution functions are suggested. The fractional analogs of the BBGKI equations are derived from the fractional Liouville equation.

1. Introduction

Fractional integrals and derivatives have many applications in statistical mechanics and kinetics [1]. G.M. Zaslavsky [2, 1, 3] proved that the chaotic dynamics can be described by using the Fokker-Planck equation with the coordinate fractional derivatives. The fractional Zaslavsky's equation can be used to describe dynamics on fractal [4, 5]. It is known that Fokker-Planck equation can be derived from the Liouville and BBGKI equations [6, 7, 8]. The Liouville equation is derived from the normalization condition and the Hamilton equations [9]. In the Hamilton equations we have only the time derivatives. Therefore the usual normalization condition leads to the usual Liouville equation. The BBGKI equations can be derived from the Liouville equation and the definition of average value [17, 18, 19, 20]. Therefore Zaslavsky's equation [2, 1, 3] can be derived from a fractional generalization of the Liouville and BBGKI equations. To derive the fractional generalization of the Liouville and BBGKI equations, we use a fractional generalization of normalization condition [10] and fractional generalization of the definition of average values. In these generalizations, we use the integrals of fractional order. To use these integrals we must have the physical interpretation of the fractional order of integrals. We can consider the fractional integrals as integrals for the function on a fractional space. In order to use this interpretation of the fractional integrals, we must define a fractional space. The interpretation of the fractional space is connected with fractal dimension space. We can replace the distribution on fractal with fractal mass dimension by some continuous distribution that is described by fractional integrals. This procedure is a fractional generalization of Christensen approach [11] that is averaging procedure over fractals. Suggested procedure leads to the fractional integration and differentiation to describe the distributions on fractal. The fractional integrals allow us to take into account the fractality of the distribution. The functions on fractals can be averaged and the distribution on fractal can be replaced by some "fractional" continuous distribution. In order to describe the averaged distribution on fractal with non-integer mass dimension, we must use the fractional integrals. Smoothing of the microscopic characteristics over the physically infinitesimal volume transforms the initial distribution on fractal into distribution on space with measure that uses the fractional integrals. The order of fractional integral is equal to the fractal mass dimension. The consistent approach to describe the distribution on fractal is connected with the mathematical definition the integrals on fractals. In Ref. [12], was proved that integrals on net of fractals can be approximated by fractional integrals. In Ref. [10], we prove that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor. We use the well-known formulas of dimensional regularizations [16]. Note that almost all systems with fractional phase space are non-Hamiltonian dissipative systems in the usual phase space (q, p). Therefore we have the other interpretation of the fractional phase space. This interpretation follows from the fractional measure of phase space [10] that is used in the fractional integrals. The fractional phase space can be considered as phase space that is described by the fractional powers of coordinates and momenta. Using this phase space we can consider wide class of non-Hamiltonian systems as generalized Hamiltonian systems. In this case, the fractional normalization condition and the fractional average values are considered as condition and values for the generalized Hamiltonian systems that are non-Hamiltonian systems in the usual phase space.

In Sec. 2, we consider the fractional generalization of normalization condition. In Sec. 3, we derive the fractional generalization of continuity equation and Liouville equation. In Sec. 4, we consider the physical interpretations of fractional phase space as a space with fractal dimension. In Sec. 5, the fractional phase space is considered as space with fractional measure. The fractional generalization of the Hamiltonian systems are suggested. In Sec. 6, we consider the fractional generalization of average values. In Sec. 7., the fractional generalization of BBGKI equations are derived from the fractional average value and the fractional Liouville equation. Finally, a short conclusion is given in Sec. 8.

2. Fractional Generalization of Normalization Condition

Let us consider a distribution function $\rho(x,t)$ for x in 1-dimensional Euclidean space E^1. Let $\rho(x,t) \in L_1(E^1)$, where t is a parameter. Normalization condition has the form

$$\int_{-\infty}^{+\infty} \rho(x,t)dx = 1.$$

This condition can be rewritten in the form

$$\int_{-\infty}^{y} \rho(x,t)dx + \int_{y}^{+\infty} \rho(x,t)dx = 1, \tag{1}$$

where $y \in (-\infty, +\infty)$.

Let $\rho(x,t) \in L_p(E^1)$, where $1 < p < 1/\alpha$. Fractional integrations on $(-\infty, y)$ and $(y, +\infty)$ are defined [14] by the equations

$$(I^{\alpha}_{+}\rho)(y,t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{y} \frac{\rho(x,t)dx}{(y-x)^{1-\alpha}}, \quad (I^{\alpha}_{-}\rho)(y,t) = \frac{1}{\Gamma(\alpha)} \int_{y}^{+\infty} \frac{\rho(x,t)dx}{(x-y)^{1-\alpha}}. \tag{2}$$

Using these notations, we rewrite Eq. (1) in an equivalent form

$$(I^{1}_{+}\rho)(y,t) + (I^{1}_{-}\rho)(y,t) = 1.$$

The fractional analog of normalization condition (1) can be represented by the following equation

$$(I^{\alpha}_{+}\rho)(y,t) + (I^{\alpha}_{-}\rho)(y,t) = 1.$$

Equations (2) can be rewritten in the form

$$(I^{\alpha}_{\pm}\rho)(y,t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} \rho(y \mp x, t) dx. \tag{3}$$

This leads to the fractional generalization of the normalization condition

$$\int_{-\infty}^{+\infty} \tilde{\rho}(x,t) d\mu_{\alpha}(x) = 1, \tag{4}$$

where we use the following notations

$$\tilde{\rho}(x,t) = T_x \rho = \frac{1}{2}\Big(\rho(y-x,t) + \rho(y+x,t)\Big), \quad d\mu_{\alpha}(x) = \frac{|x|^{\alpha-1}}{\Gamma(\alpha)} dx. \tag{5}$$

The normalization in the phase space is derived by analogy with a normalization in the configuration space. The fractional normalization condition in the phase space

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\rho}(q,p,t) d\mu_{\alpha}(q,p) = 1, \tag{6}$$

where $d\mu_{\alpha}(q,p)$ has the form

$$d\mu_{\alpha}(q,p) = d\mu_{\alpha}(q) \wedge d\mu_{\alpha}(q) = \frac{(qp)^{\alpha-1}}{\Gamma^2(\alpha)} dq \wedge dp. \tag{7}$$

The distribution function $\tilde{\rho}(q,p,t)$ in the phase space is defined by

$$\tilde{\rho}(q,p,t) = T_q T_p \rho(q,p,t), \tag{8}$$

where the operators T_q and T_p are defined by the equation

$$T_{x_k} f(..., x_k, ...) = \frac{1}{2}\Big(f(..., x'_k - x_k, ...) + f(..., x'_k + x_k, ...)\Big). \tag{9}$$

This operator allows us to rewrite the distribution functions

$$\tilde{\rho}(q,p,t) = \frac{1}{4}\Big(\rho(q'-q, p'-p, t) + \rho(q'+q, p'-p, t) + \rho(q'-q, p'+p, t) + \rho(q'+q, p'+p, t)\Big)$$

in the simple form $\tilde{\rho}(q,p,t) = T_q T_p \rho(q,p,t)$.

3. Fractional Generalization of Liouville Equation

3.1. Fractional Generalization Continuity Equation

Let us consider a region W_0 for the time $t = 0$. In the Hamilton picture we have

$$\int_{W_t} \tilde{\rho}(x_t, t) d\mu_{\alpha}(x_t) = \int_{W_0} \tilde{\rho}(x_0, 0) d\mu_{\alpha}(x_0).$$

Using the replacement of variables $x_t = x_t(x_0)$, where x_0 is a Lagrangian variable, we get

$$\int_{W_0} \tilde{\rho}(x_t, t) |x_t|^{\alpha-1} \frac{\partial x_t}{\partial x_0} dx_0 = \int_{W_0} \tilde{\rho}(x_0, 0) |x_0|^{\alpha-1} dx_0.$$

Since W_0 is an arbitrary region, we have

$$\tilde{\rho}(x_t,t)d\mu_\alpha(x_t) = \tilde{\rho}(x_0,0)d\mu_\alpha(x_0). \tag{10}$$

This condition leads us to the equation

$$\tilde{\rho}(x_t,t)|x_t|^{\alpha-1}\frac{\partial x_t}{\partial x_0} = \tilde{\rho}(x_0,0)|x_0|^{\alpha-1}.$$

Differentiating this equation in time t, we obtain

$$\frac{d\tilde{\rho}(x_t,t)}{dt}|x_t|^{\alpha-1}\frac{\partial x_t}{\partial x_0} + \tilde{\rho}(x_t,t)\frac{d}{dt}\Big(|x_t|^{\alpha-1}\frac{\partial x_t}{\partial x_0}\Big) = 0.$$

As the result we have the fractional generalization of the continuity equation

$$\frac{d\tilde{\rho}(x_t,t)}{dt} + \Omega_\alpha(x_t,t)\tilde{\rho}(x_t,t) = 0, \tag{11}$$

where the omega function

$$\Omega_\alpha(x_t,t) = \frac{d}{dt}ln\Big(|x_t|^{\alpha-1}\frac{\partial x_t}{\partial x_0}\Big)$$

describes the velocity of phase volume change. Eq. (11) is a fractional continuity equation in the Hamilton picture. If the equation of motion has the form

$$\frac{dx_t}{dt} = F_t(x),$$

then the function Ω_α is defined by

$$\Omega_\alpha(x_t,t) = \frac{d}{dt}\Big(ln\ |x|_t^{\alpha-1} + ln\ \frac{\partial x_t}{\partial x_0}\Big) = \frac{\alpha-1}{x_t}\frac{dx_t}{dt} + \frac{\partial}{\partial x_t}\frac{dx_t}{dt} = \frac{(\alpha-1)F_t}{x_t} + \frac{\partial F_t}{\partial x_t}. \tag{12}$$

3.2. Fractional Continuity Equation for Phase Space
Using phase space analog of Eq. (10) in the form

$$\tilde{\rho}_t d\mu_\alpha(q_t,p_t) = \tilde{\rho}_0 d\mu_\alpha(q_0,p_0),$$

we get the relation

$$\tilde{\rho}_t\frac{|q_tp_t|^{\alpha-1}}{\Gamma^2(\alpha)}dq_t\wedge dp_t = \tilde{\rho}_0\frac{|q_0p_0|^{\alpha-1}}{\Gamma^2(\alpha)}dq_0\wedge dp_0. \tag{13}$$

Let us use the well known transformation

$$dq_t\wedge dp_t = \{q_t,p_t\}_0 dq_0\wedge dp_0, \tag{14}$$

where $\{q_t,p_t\}_0$ is Jacobian which is defined by the determinant

$$\{q_t,p_t\}_0 = det\frac{\partial(q_t,p_t)}{\partial(q_0,p_0)} = det\begin{pmatrix} \partial q_{kt}/\partial q_{l0} & \partial q_{kt}/\partial p_{l0} \\ \partial p_{kt}/\partial q_{l0} & \partial p_{kt}/\partial p_{l0}\end{pmatrix}.$$

As the result we have condition (13) in the form

$$\tilde{\rho}_t|q_tp_t|^{\alpha-1}\{q_t,p_t\}_0 = |q_0p_0|^{\alpha-1}\tilde{\rho}_0. \tag{15}$$

This equation can be rewritten in more simple form

$$\tilde{\rho}_t \{q_t^\alpha, p_t^\alpha\}_0 = \alpha^2 |q_0 p_0|^{\alpha-1} \tilde{\rho}_0. \tag{16}$$

We use the following notation for fractional power of coordinates and momenta

$$x^\alpha = \beta(x)(x)^\alpha = sgn(x)|x|^\alpha, \tag{17}$$

where $\beta(x) = (sgn(x))^{\alpha-1}$. The function $sgn(x)$ is equal to $+1$ for $x \ge 0$, and -1 for $x < 0$. The total time derivatives of Eq. (16) lead us to the fractional generalization of Liouville equation in the form

$$\frac{d\tilde{\rho}}{dt} + \Omega_\alpha \tilde{\rho} = 0, \tag{18}$$

where the omega function Ω_α is defined by

$$\Omega_\alpha = \{q_t^\alpha, p_t^\alpha\}_0^{-1} \frac{d}{dt} \{q_t^\alpha, p_t^\alpha\}_0 = \frac{d}{dt} ln \{q_t^\alpha, p_t^\alpha\}_0. \tag{19}$$

In the usual notations we have

$$\Omega_\alpha = \frac{d}{dt} ln \ det \frac{\partial(q_t^\alpha, p_t^\alpha)}{\partial(q_0, p_0)}. \tag{20}$$

Using well-known relation $ln \ det \ A = Sp \ ln \ A$, we can write the omega function Ω_α in the form

$$\Omega_\alpha = \{\frac{dq_t^\alpha}{dt}, p_t^\alpha\}_\alpha + \{q_t^\alpha, \frac{dp_t^\alpha}{dt}\}_\alpha,$$

where $\{ \ , \ \}_\alpha$ is the fractional generalization of the Poisson brackets in the form

$$\{A, B\}_\alpha = \frac{\partial A}{\partial q^\alpha} \frac{\partial B}{\partial p^\alpha} - \frac{\partial A}{\partial p^\alpha} \frac{\partial B}{\partial q^\alpha}.$$

In the general case ($\alpha \ne 1$) the function Ω_α is not equal to zero ($\Omega_\alpha \ne 0$) for the systems that are Hamiltonian systems in the usual phase space. If $\alpha = 1$, we have $\Omega_\alpha \ne 0$ only for non-Hamiltonian systems. If the Hamilton equations have the form

$$\frac{dq_t}{dt} = K(q_t, p_t), \quad \frac{dp_t}{dt} = F(q_t, p_t), \tag{21}$$

then the omega function Ω_α is defined by

$$\Omega_\alpha(q, p) = (\alpha - 1)\Big(q^{-1} K(q, p) + p^{-1} F(q, p)\Big) + \{K, p\}_1 + \{q, F\}_1. \tag{22}$$

This relation allows to derive Ω_α for all dynamical systems (21). It is easy to see that the usual nondissipative system

$$\frac{dq}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = f(q), \tag{23}$$

has the omega function

$$\Omega_\alpha(q, p) = (\alpha - 1)(mqp)^{-1}(p^2 + mqf(q))$$

and can be considered as a dissipative system in the fractional phase space (q^α, p^α).

3.3. Fractional Liouville Equation for N-particle System

Let us consider N-particle system. Suppose k-particle is described by the generalized coordinates $\mathbf{q}_k = (q_{k1}, ..., q_{kn})$ and generalized momenta $\mathbf{p}_k = (p_{k1}, ..., p_{kn})$ that satisfy the Hamilton equations in the form

$$\frac{d\mathbf{q}^{\alpha}_k}{dt} = \mathbf{K}_k(\mathbf{q}^{\alpha}, \mathbf{p}^{\alpha}), \quad \frac{d\mathbf{p}^{\alpha}_k}{dt} = \mathbf{F}_k(\mathbf{q}^{\alpha}, \mathbf{p}^{\alpha}, t). \tag{24}$$

Here we use the notation (17) for fractional power of coordinates and momenta. The evolution of N-particle distribution function ρ_N is described by the Liouville equation. Using the fractional normalization condition

$$\hat{I}^{\alpha}[1, ..., N]\tilde{\rho}_N(\mathbf{q}, \mathbf{p}, t) = 1, \tag{25}$$

we can derive the fractional Liouville equation [10] for N-particle distribution function in the form

$$\frac{d\tilde{\rho}_N}{dt} + \Omega_{\alpha}\tilde{\rho}_N = 0, \tag{26}$$

where d/dt is a total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^{N} \frac{d\mathbf{q}_k}{dt}\frac{\partial}{\partial \mathbf{q}_k} + \sum_{k=1}^{N} \frac{d\mathbf{p}_k}{dt}\frac{\partial}{\partial \mathbf{p}_k}.$$

Here we use the following notations

$$\mathbf{A}_k\mathbf{B}_k = \sum_{a=1}^{n} A_{ka}B_{ka}.$$

Using Eq. (24), the total time derivative can be written for the fractional powers in the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^{N} \mathbf{K}_k\frac{\partial}{\partial \mathbf{q}^{\alpha}_k} + \sum_{k=1}^{N} \mathbf{F}_k\frac{\partial}{\partial \mathbf{p}^{\alpha}_k}. \tag{27}$$

The omega function Ω_{α} is defined by the equation

$$\Omega_{\alpha} = \sum_{k=1}^{N}\sum_{a=1}^{n}\Big(\{K^k_a, p^{\alpha}_{ka}\}_{\alpha} + \{q^{\alpha}_{ka}, F^k_a\}_{\alpha}\Big). \tag{28}$$

Here we use the following notations for the brackets

$$\{A, B\}_{\alpha} = \sum_{k=1}^{N}\sum_{a=1}^{n}\Big(\frac{\partial A}{\partial q^{\alpha}_{ka}}\frac{\partial B}{\partial p^{\alpha}_{ka}} - \frac{\partial A}{\partial p^{\alpha}_{ka}}\frac{\partial B}{\partial q^{\alpha}_{ka}}\Big). \tag{29}$$

Using Eqs. (26), (28) and (27), we can rewrite the Liouville equation in the equivalent form

$$\frac{\partial\tilde{\rho}_N}{\partial t} = \mathcal{L}_N\tilde{\rho}_N, \tag{30}$$

where $\mathcal{L}_N$ is Liouville operator that is defined by the equation

$$\mathcal{L}_N\tilde{\rho}_N = -\sum_{k=1}^{N}\Big(\frac{\partial(\mathbf{K}_k\tilde{\rho}_N)}{\partial \mathbf{q}^{\alpha}_k} + \frac{\partial(\mathbf{F}_k\tilde{\rho}_N)}{\partial \mathbf{p}^{\alpha}_k}\Big). \tag{31}$$

4. Fractional Space as Space with Fractal Dimension

The interpretation of the fractional space can be connected with fractal mass dimension. Fractal dimension can be best calculated by box counting method which means drawing a box of size R and counting the mass inside. The mass fractal dimension [4, 5] can be easy measured. The properties of the fractal like mass obeys a power law relation

$$M(R) = kR^{\alpha}, \tag{32}$$

where M is the mass of fractal, R is a box size (or a sphere radius), and α is a mass fractal dimension. Amount of mass inside a box of size R has a power law relation (32).

Let us consider the region W in 3-dimensional Euclidean space E^3. The volume of the region W is denoted by $V(W)$. The mass of the region W of the distribution on fractal is denoted by $M(W)$. The fractality means that the mass of this distribution in any region W of Euclidean space E^3 increase more slowly that the volume of this region E^3. For the ball region of the distribution on fractal, this property can be described by the power law (32), where R is the radius of the ball W.

The distribution on fractal is called homogeneous if the power law (32) does not depends on the translation and rotation of the region W. The homogeneity property of the distribution can be formulated in the form: For all regions W_1 and W_2 such that the volumes are equal $V(W_1) = V(W_2)$, we have that the mass of these regions are equal $M(W_1) = M(W_2)$. In order to describe the homogeneous distribution on fractals, we must use the continuous distribution such that fractality and homogeneity properties can be realized in the form:

(1) Fractality: The mass of the ball region W for the distribution on fractal obeys a power law relation (32). In the general case, we have the scaling law relation

$$dM_{\alpha}(\lambda W) = \lambda^{\alpha} dM_{\alpha}(W),$$

where $\lambda W = \{\lambda x, \quad x \in W\}$.

(2) Homogeneity: The local density of homogeneous fractal distribution is translation and rotation invariant value that have the form $\rho(x) = \rho_0 = const$.

We can realize these requirements by the fractional generalization of the equation

$$M_3(W) = \int_W \rho(x) d^3x. \tag{33}$$

Let us define the fractional integral in Euclidean space E^3 in the Riesz form [14] by the equation

$$(I^{\alpha}\rho)(y) = \int_W \rho(x) d\mu_{\alpha}(x), \tag{34}$$

where $d\mu_{\alpha}(x) = c_3(\alpha, x, y) d^3x$, and

$$c_3(\alpha, x, y) = \frac{2^{3-\alpha}\Gamma(3/2)}{\Gamma(\alpha/2)} |x-y|^{\alpha-3}, \quad |x-y| = \sqrt{\sum_{k=1}^{3} (x_k - y_k)^2}.$$

The point $y \in W$ is the initial point of the fractional integral. We will use the initial points in the integrals are set to zero ($y = 0$). The numerical factor in Eq. (34) has this form in order to derive usual integral in the limit $\alpha \to (3-0)$. Note that the usual numerical factor $\gamma_3^{-1}(\alpha) = \Gamma(1/2)/2^{\alpha}\pi^{3/2}\Gamma(\alpha/2)$, which is used in Ref. [14], leads us to $\gamma_3^{-1}(3-0) = \Gamma(1/2)/2^3\pi^{3/2}\Gamma(3/2)$ in the limit $\alpha \to (3-0)$.

Using notations (34), we can rewrite Eq. (33) in the form $M_3(W) = (I^3\rho)(y)$. Therefore the fractional generalization of this equation can be defined in the form

$$M_\alpha(W) = (I^\alpha\rho)(y) = \frac{2^{3-\alpha}\Gamma(3/2)}{\Gamma(\alpha/2)} \int_W \rho(x)|x-y|^{\alpha-3}d^3x. \qquad (35)$$

If we consider the homogeneous fractal distribution ($\rho(x) = \rho_0 = const$) and the ball region $W = \{x: \ |x| \le R\}$, then we have

$$M_\alpha(W) = \rho_0 \frac{2^{3-\alpha}\Gamma(3/2)}{\Gamma(\alpha/2)} \int_W |x|^{\alpha-3}d^3x.$$

Using the spherical coordinates, we get

$$M_\alpha(W) = \frac{\pi 2^{5-\alpha}\Gamma(3/2)}{\Gamma(\alpha/2)}\rho_0 \int_W |x|^{\alpha-1}d|x| = \frac{2^{5-\alpha}\pi\Gamma(3/2)}{\alpha\Gamma(\alpha/2)}\rho_0 R^\alpha.$$

As the result, we have $M(W) \sim R^\alpha$, i.e., we derive Eq. (32) up to the numerical factor. Therefore the distribution on fractal with non-integer mass dimension α can be described by fractional integral of order α.

Note that the interpretation of the fractional integration is connected with fractional dimension [10]. This interpretation follows from the well-known formulas for dimensional regularizations [16]:

$$\int f(x)d^\alpha x = \frac{2\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^{+\infty} f(x)x^{\alpha-1}dx. \qquad (36)$$

Using Eq. (36), we get that the fractional integral

$$\int_W f(x)d\mu_\alpha(x),$$

can be considered as integral in the fractional dimension space

$$\frac{\Gamma(\alpha/2)}{2\pi^{\alpha/2}\Gamma(\alpha)} \int f(x)d^\alpha x \qquad (37)$$

up to the numerical factor $\Gamma(\alpha/2)/(2\pi^{\alpha/2}\Gamma(\alpha))$.

5. Fractional Space as Space with Fractional Measure

The interpretation of the fractional space is connected with the fractional measure that is used in the fractional integrals. The parameter α defines the space with the fractional measure (volume) of the region W. It is easy to prove that the velocity of the fractional measure (volume) change is defined by the omega function (12).

5.1. Fractional Phase Volume for Configuration Space

The usual phase volume of the region $W = \{x: \ x \in [a;b]\}$ in Euclidean space E^1 is defined by

$$\mu_1(W) = \int_a^b dx = \int_a^y dx + \int_y^b dx, \qquad (38)$$

where $y \in [a;b]$. Using the fractional integrals [14] in the form

$$(I^{\alpha}_{a+}1)(y) = \frac{1}{\Gamma(\alpha)} \int_a^y \frac{dx}{(y-x)^{1-\alpha}}, \quad (I^{\alpha}_{b-}1)(y) = \frac{1}{\Gamma(\alpha)} \int_y^b \frac{dx}{(x-y)^{1-\alpha}},$$

we get the phase volume (38) in the equivalent form

$$\mu_1(W) = (I^1_{a+}1)(y) + (I^1_{b-}1)(y). \tag{39}$$

The fractional generalization of the phase volume can be defined by

$$\mu_\alpha(W) = (I^{\alpha}_{a+}1)(y) + (I^{\alpha}_{b-}1)(y). \tag{40}$$

The fractional phase volume integral can be represented by the following equation

$$\mu_\alpha(W) = \int_{-(b-y)}^{+(b-y)} g(\alpha) d\mu_\alpha(x). \tag{41}$$

Here we use the notations

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1}dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)}, \qquad g(\alpha) = \frac{1}{2}\Big(1 + \Big(\frac{y-a}{b-y}\Big)^\alpha\Big). \tag{42}$$

We use the notation (17) for fractional power of coordinates.

5.2. Fractional Phase Volume for Phase Space

The fractional measure for the region W of 2n-dimensional phase space can be defined by the equation

$$\mu_\alpha(W) = \int_W g(\alpha) d\mu_\alpha(q,p), \tag{43}$$

where $d\mu_\alpha(q,p)$ is a phase volume element

$$d\mu_\alpha(q,p) = \prod_{k=1}^n \frac{dq^\alpha_k \wedge dp^\alpha_k}{(\alpha\Gamma(\alpha))^2}, \tag{44}$$

and $g(\alpha)$ is a numerical multiplier. If the region W of the phase space is defined by $q_k \in E^1$ and $p_k \in E^1$, then $g(\alpha) = 1/4^n$. If this region is defined by $q_k \in [q_{ak}; q_{bk}]$ and $p_k \in [p_{ak}; p_{bk}]$, then

$$g(\alpha) = \frac{1}{4^n} \prod_{k=1}^n g_k(\alpha), \quad g_k(\alpha) = \Big(1 + \Big(\frac{q_{bk}-y_k}{y_k - q_{ak}}\Big)^\alpha\Big)\Big(1 + \Big(\frac{p_{bk}-y'_k}{y'_k - p_{ak}}\Big)^\alpha\Big) \tag{45}$$

It is easy to see that the fractional measure depends on the fractional powers of coordinates and momenta (q^α, p^α).

For example, the phase volume for the two-dimensional phase space has the form

$$\mu_\alpha(W) = \int_{-(q_b-y')}^{(q_b-y)} \int_{-(p_b-y')}^{(p_b-y)} g_1(\alpha) \frac{dq^\alpha \wedge dp^\alpha}{(\alpha\Gamma(\alpha))^2} = \int_{-(q_b-y')}^{(q_b-y)} \int_{-(p_b-y')}^{(p_b-y)} g_1(\alpha)|qp|^{\alpha-1} \frac{dq \wedge dp}{(\Gamma(\alpha))^2}. \tag{46}$$

Note that the volume element of fractional phase space can be realized by fractional exterior derivatives [13] that are defined by

$$d^\alpha = \sum_{k=1}^n dq^\alpha_k \frac{\partial^\alpha}{(\partial(q_k - y_k))^\alpha} + \sum_{k=1}^n dp^\alpha_k \frac{\partial^\alpha}{(\partial(p_k - y'_k))^\alpha}.$$

For example, the two-dimensional phase space is defined by

$$d\mu_\alpha(q,p) = \frac{1}{(\alpha\Gamma(\alpha))^2} dq^\alpha \wedge dp^\alpha = \frac{1}{(\alpha\Gamma(\alpha))^2}\Big(\frac{4}{\Gamma^2(2-\alpha)} + \frac{1}{\Gamma^2(1-\alpha)}\Big)^{-1} (qp)^{\alpha-1} d^\alpha q \wedge d^\alpha p.$$

5.3. Fractional Phase Volume Change

The interpretation of the fractional phase space is connected with the fractional measure of phase space. The parameter α defines the space with the fractional phase measure (43) and (44). It is easy to prove that the velocity of the fractional phase volume change is defined by the equation

$$\frac{d\mu_\alpha(W)}{dt} = \int_W \Omega_\alpha(q,p,t)g(\alpha)d\mu_\alpha(q,p).$$

where the omega function Ω_α has the form

$$\Omega_\alpha = \{\frac{dq^\alpha_t}{dt}, p^\alpha_t\}_\alpha + \{q^\alpha_t, \frac{dp^\alpha_t}{dt}\}_\alpha. \tag{47}$$

Here we use the fractional brackets:

$$\{A,B\}_\alpha = \sum_{k=1}^{n}\Big(\frac{\partial A}{\partial q^\alpha_k}\frac{\partial B}{\partial p^\alpha_k} - \frac{\partial A}{\partial p^\alpha_k}\frac{\partial B}{\partial q^\alpha_k}\Big), \tag{48}$$

where we use notations (17).

The form of the omega function allows us to consider new class of the systems that are described by the fractional powers of coordinates and momenta. The system can be called a fractional dissipative system if a fractional phase volume changes, i.e., $\Omega_\alpha \neq 0$. The system which is a nondissipative system in the usual phase space, can be a dissipative system in the fractional phase space. The usual nondissipative systems (23) are dissipative in the fractional phase space.

5.4. Fractional Generalization of Hamiltonian Systems

Fractional phase space can be considered as a phase space of the systems that are described by the fractional powers of coordinates and momenta. Let us consider the class of non-Hamiltonian systems that are described by the fractional powers of coordinates and momenta. A system is called a fractional system if the phase space of the system is described by the fractional powers of coordinates and momenta

$$q^\alpha_k = \beta(q)(q_k)^\alpha = sgn(q_k)|q_k|^\alpha, \quad p^\alpha_k = \beta(p)(p_k)^\alpha = sgn(p_k)|p_k|^\alpha. \tag{49}$$

Here $k = 1, ..., n$, and $\beta(x)$ is defined by Eq. (17). We can consider the fractional systems in the usual phase space (q,p) and in the fractional phase space (q^α, p^α). In the second case, the equations of motion for the fractional systems have more simple form. Therefore we use the fractional phase space that is a space with the fractional measure that is used in the fractional integrals. We consider the fractional power of the coordinates as a convenient way to describe systems in the space with measure that is defined by fractional integrals.

A classical system (in the usual phase space) is called Hamiltonian if the right-hand sides of the equations

$$\frac{dq_k}{dt} = K_k(q,p), \quad \frac{dp_k}{dt} = F_k(q,p) \tag{50}$$

satisfy the following Helmholtz conditions [15]:

$$\frac{\partial K_k}{\partial p_l} - \frac{\partial K_l}{\partial p_k} = 0, \quad \frac{\partial K_k}{\partial q_l} - \frac{\partial F_l}{\partial p_k} = 0, \quad \frac{\partial F_k}{\partial q_l} - \frac{\partial F_l}{\partial q_k} = 0. \tag{51}$$

In this case, we can rewrite Eqs. (50) in the form

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k},$$

Using the Poisson brackets $\{\ ,\ \}_1$, we can rewrite these equations in an equivalent form

$$\frac{dq_k}{dt} = \{q_k, H\}_1, \quad \frac{dp_k}{dt} = \{p_k, H\}_1.$$

The fractional generalization of the Hamiltonian system is described by the equation

$$\frac{dq_k^\alpha}{dt} = \frac{\partial H}{\partial p_k^\alpha}, \quad \frac{dp_k^\alpha}{dt} = -\frac{\partial H}{\partial q_k^\alpha}, \tag{52}$$

where H is a function that can be considered as a fractional analog of the Hamiltonian. Note that the function H such that $\partial H/\partial t = 0$ is the invariant of the motion. Using the brackets (48), we can rewrite Eq. (52) in the equivalent form

$$\frac{dq_k^\alpha}{dt} = \{q_k^\alpha, H\}_\alpha, \quad \frac{dp_k^\alpha}{dt} = \{p_k^\alpha, H\}_\alpha. \tag{53}$$

These equations describe the system in the fractional phase space (q^α, p^α). For the usual phase space (q, p), the fractional Hamiltonian systems (52) are described by the equations

$$\frac{dq_k}{dt} = \frac{(q_k p_k)^{1-\alpha}}{\alpha^2} \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{(q_k p_k)^{1-\alpha}}{\alpha^2} \frac{\partial H}{\partial q_k}. \tag{54}$$

The fractional Hamiltonian systems (54) are non-Hamiltonian systems in the usual phase space (q, p). It is easy to prove that the Helmholtz conditions (51) are not satisfied. Therefore fractional Hamiltonian system (54) is a non-Hamiltonian system in the usual phase space (q, p). The fractional phase space allows us to write the equations of motion for the non-Hamiltonian systems (54) in the generalized Hamiltonian form (52).

The omega function for the system (50) in the usual phase space (q, p) is defined by the equation

$$\Omega = \sum_{k=1}^{n} \Big(\frac{\partial K_k}{\partial q_k} + \frac{\partial F_k}{\partial p_k}\Big). \tag{55}$$

If the omega function is negative $\Omega < 0$, then the system is called a dissipative system. If $\Omega \neq 0$, then the system is a generalized dissipative system. For the fractional Hamiltonian systems (54), the omega function (55) is not equal to zero. Therefore the fractional Hamiltonian systems are the general dissipative systems in the usual phase space.

The function $H = H(q^\alpha, p^\alpha)$ can be considered as a fractional analog of the Hamiltonian function. For example, we can use

$$H(q^\alpha, p^\alpha) = \sum_{k=1}^{n} \frac{p_k^{2\alpha}}{2m} + U(q^\alpha). \tag{56}$$

It is easy to see that fractional systems with Hamiltonian (56) lead us to the non-Gaussian statistics. The interest in and relevance of fractional kinetic equations is a natural consequence of the realization of the importance of non-Gaussian statistics of many dynamical systems. There is already a substantial literature studying such equations in one or more space dimensions.

Note that the classical dissipative non-Hamiltonian systems can have canonical Gibbs distribution as a solution of stationary Liouville equations for this dissipative system [23]. Using the methods [23], it is easy to prove that some of fractional dissipative systems can have fractional generalization of the canonical Gibbs distribution in the form

$$\rho(q, p) = Z(T) exp - \frac{H(q^\alpha, p^\alpha)}{kT},$$

as a solution of the fractional Liouville equations

$$\frac{\partial \rho}{\partial t} + \frac{p_k^\alpha}{m} \frac{\partial \rho}{\partial q_k^\alpha} + \frac{\partial}{\partial p_k^\alpha} \Big(F_k(q^\alpha, p^\alpha) \rho \Big) = 0. \tag{57}$$

Here the function $H(q^\alpha, p^\alpha)$ is defined by (56).

6. Fractional Generalization of Average Values

The usual average value for the configuration space

$$< f >_1 = \int_{-\infty}^{+\infty} f(x,t)\rho(x,t)dx \tag{58}$$

can be written in the form

$$< f >_1 = \int_{-\infty}^{y} f(x,t)\rho(x,t)dx + \int_{y}^{+\infty} f(x,t)\rho(x,t)dx, \tag{59}$$

where $y \in (-\infty, +\infty)$. Using the notations (2), we can rewrite the average value (59) in the form

$$< f >_1 (y,t) = (I_+^1 f\rho)(y,t) + (I_-^1 f\rho)(y,t).$$

The fractional generalization of this equation is defined by

$$< f >_\alpha (y,t) = (I_+^\alpha f\rho)(y,t) + (I_-^\alpha f\rho)(y,t). \tag{60}$$

We can rewrite Eq. (60) in the form

$$< f >_\alpha (y,t) = \int_{-\infty}^{+\infty} T_x(f\rho)(x,t)d\mu_\alpha(x), \tag{61}$$

where we use

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1}dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)}, \quad T_x f = \frac{1}{2}\Big(f(y-x) + f(y+x)\Big), \tag{62}$$

and x^α is defined by Eq. (17).

We can define the integral operator $\hat{I}_x^\alpha$ by the equation

$$\hat{I}_x^\alpha f(x) = \int_{-\infty}^{+\infty} f(x) d\mu_\alpha(x). \tag{63}$$

In this case, the fractional generalization of average value (61) can be written in the form

$$< f >_\alpha (y,t) = \hat{I}_x^\alpha T_x f(x)\rho(x,t).$$

Let us consider k-particle that is described by generalized coordinates $\mathbf{q}_k = (q_{k1}, ..., q_{kn})$ and generalized momenta $\mathbf{p}_k = (p_{k1}, ..., p_{kn})$. We can define the phase space integral operator for k-particle by the equation

$$\hat{I}^\alpha[k] = \hat{I}^\alpha_{q_{k1}} \hat{I}^\alpha_{p_{k1}} ... \hat{I}^\alpha_{q_{kn}} \hat{I}^\alpha_{p_{kn}}.$$

For the N-particle system, we use the operators

$$\hat{I}^\alpha[1, ..., N] = \hat{I}^\alpha[1] ... \hat{I}^\alpha[N], \quad T[1, ..., N] = T[1] ... T[N],$$

where the operator $T[k]$ is defined by the relation

$$T[k] = T_{q_{k1}} T_{p_{k1}} ... T_{q_{kn}} T_{p_{kn}}.$$

Here the operator T_{xk} is defined by Eq. (62).

Using these notations, we can define fractional analog of the average values for the phase space of N-particle system by the relation

$$< f >_\alpha (y,t) = \hat{I}^\alpha [1,...,N] T[1,...,N] f \rho_N. \tag{64}$$

In the simple case ($N = m = 1$), the fractional average value (64) is defined by the equation

$$< f >_\alpha (y,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mu_\alpha(q,p) \ T_q T_p f(q,p) \rho(q,p,t). \tag{65}$$

Note that the fractional normalization condition is a special case of this definition of the average value $< 1 >_\alpha (y,t) = 1$.

7. Fractional Generalization of BBGKI Equations

The state of N-particle system is described by N-particle distribution function

$$\rho_N(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}_1, \mathbf{p}_1, ..., \mathbf{q}_N, \mathbf{p}_N, t).$$

We use the tilde distribution functions

$$\tilde{\rho}_N(\mathbf{q}, \mathbf{p}, t) = T[1,...,N] \rho_N(\mathbf{q}, \mathbf{p}, t). \tag{66}$$

The fractional generalization of 1-particle reduced distribution function $\tilde{\rho}_1$ can be defined by the equation

$$\tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = \tilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}^\alpha [2,...,N] \tilde{\rho}_N(\mathbf{q}, \mathbf{p}, t). \tag{67}$$

Obviously, that 1-particle distribution function satisfies the fractional normalization condition

$$\hat{I}^\alpha [1] \tilde{\rho}_1(\mathbf{q}, \mathbf{p}, t) = 1. \tag{68}$$

The BBGKI equations [17, 18, 19, 20] are equations for the reduced distribution functions. These equations can be derived from the Liouville equation. Let us derive the fractional generalization of the first BBGKI equation from the fractional Liouville equation (30).

In order to derive the equation for 1-particle distribution function $\tilde{\rho}_1$ we differentiate Eq. (67) with respect to time:

$$\frac{\partial \tilde{\rho}_1}{\partial t} = \frac{\partial}{\partial t} \hat{I}^\alpha [2,...,N] \tilde{\rho}_N = \hat{I}^\alpha [2,...,N] \frac{\partial \tilde{\rho}_N}{\partial t}.$$

Using the Liouville equation (30) for N-particle distribution function $\tilde{\rho}_N$, we have

$$\frac{\partial \tilde{\rho}_1}{\partial t} = \hat{I}^\alpha [2,...,N] \mathcal{L}_N \tilde{\rho}_N(\mathbf{q}, \mathbf{p}, t). \tag{69}$$

Substituting (31) in Eq. (69), we get

$$\frac{\partial \tilde{\rho}_1}{\partial t} = -\hat{I}^\alpha [2,...,N] \sum_{k=1}^{N} \Big(\frac{\partial (\mathbf{K}_k \tilde{\rho}_N)}{\partial \mathbf{q}_k^\alpha} + \frac{\partial (\mathbf{F}_k \tilde{\rho}_N)}{\partial \mathbf{p}_k^\alpha} \Big). \tag{70}$$

Let us consider in Eq. (70) the integration over $\mathbf{q}_k$ and $\mathbf{p}_k$ for k-particle term. Since the coordinates and momenta are independent variables, we can derive

$$\hat{I}^{\alpha}[\mathbf{q}_k]\frac{\partial}{\partial \mathbf{q}_k^{\alpha}}(\mathbf{K}_k\tilde{\rho}_N) = \frac{1}{\alpha\Gamma(\alpha)}\Big(\mathbf{K}_k\tilde{\rho}_N\Big)_{-\infty}^{+\infty} = 0. \tag{71}$$

Here we use that the distribution $\tilde{\rho}_N$ in the limit $\mathbf{q}_k \to \pm\infty$ is equal to zero. It follows from the normalization condition. If the limit is not equal to zero, then the integration over phase space is equal to infinity. Similarly, we have

$$\hat{I}^{\alpha}[\mathbf{p}_k]\Big(\frac{\partial}{\partial \mathbf{p}_k^{\alpha}}(\mathbf{F}_k\tilde{\rho}_N)\Big) = \frac{1}{\alpha\Gamma(\alpha)}\Big(\mathbf{F}_k\tilde{\rho}_N\Big)_{-\infty}^{+\infty} = 0.$$

Then all terms in Eq. (70) with $k = 2, ..., n$ are equal to zero. We have only term for $k = 1$. Therefore Eq. (70) has the form

$$\frac{\partial \tilde{\rho}_1}{\partial t} = -\hat{I}^{\alpha}[2, ..., N]\Big(\frac{\partial(\mathbf{K}_1\tilde{\rho}_N)}{\partial \mathbf{q}_1^{\alpha}} + \frac{\partial(\mathbf{F}_1\tilde{\rho}_N)}{\partial \mathbf{p}_1^{\alpha}}\Big). \tag{72}$$

Since the variable $\mathbf{q}_1$ is an independent of $\mathbf{q}_2, ..., \mathbf{q}_N$ and $\mathbf{p}_2, ..., \mathbf{p}_N$, the first term in Eq. (72) can be written in the form

$$\hat{I}^{\alpha}[2, ..., N]\frac{\partial(\mathbf{K}_1\tilde{\rho}_N)}{\partial \mathbf{q}_1^{\alpha}} = \frac{\partial}{\partial \mathbf{q}_1^{\alpha}}\mathbf{K}_1\hat{I}^{\alpha}[2, ..., N]\tilde{\rho}_N = \frac{\partial(\mathbf{K}_1\tilde{\rho}_1)}{\partial \mathbf{q}_1^{\alpha}}.$$

The force $\mathbf{F}_1$ acts on the first particle. This force is a sum of the internal forces $\mathbf{F}_{1k} = \mathbf{F}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_k, \mathbf{p}_k, t)$, and the external force $\mathbf{F}_1^e = \mathbf{F}^e(\mathbf{q}_1, \mathbf{p}_1, t)$. In the case of binary interactions, we have

$$\mathbf{F}_1 = \mathbf{F}_1^e + \sum_{k=2}^{N} \mathbf{F}_{1k}. \tag{73}$$

Using Eq. (73), the second term in Eq. (72) can be rewritten in the form

$$\hat{I}^{\alpha}[2, ..., N]\Big(\frac{\partial(\mathbf{F}_1\tilde{\rho}_N)}{\partial \mathbf{p}_1^{\alpha}}\Big) = \hat{I}^{\alpha}[2, ..., N]\Big(\frac{\partial(\mathbf{F}_1^e\tilde{\rho}_N)}{\partial \mathbf{p}_1^{\alpha}} + \sum_{k=2}^{N}\frac{\partial(\mathbf{F}_{1k}\tilde{\rho}_N)}{\partial \mathbf{p}_1^{\alpha}}\Big) =$$

$$= \frac{\partial(\mathbf{F}_1^e\tilde{\rho}_1)}{\partial \mathbf{p}_1^{\alpha}} + \sum_{k=2}^{N}\frac{\partial}{\partial \mathbf{p}_1^{\alpha}}\hat{I}^{\alpha}[2, ..., N]\Big(\mathbf{F}_{1k}\tilde{\rho}_N\Big). \tag{74}$$

We assume that distribution function is invariant under the permutations of identical particles [21]:

$$\rho_N(..., \mathbf{q}_k, \mathbf{p}_k, ..., \mathbf{q}_l, \mathbf{p}_l, ..., t) = \rho_N(..., \mathbf{q}_l, \mathbf{p}_l, ..., \mathbf{q}_k, \mathbf{p}_k, ..., t).$$

In this case, the N-particle distribution function $\tilde{\rho}_N$ is a symmetric function for the identical particles and we have that all $(N-1)$ terms of sum (74) are identical. Therefore the sum can be replaced by one term with the multiplier $(N-1)$:

$$\sum_{k=2}^{N} \hat{I}^{\alpha}[2, ..., N]\ \frac{\partial}{\partial \mathbf{p}_{1s}^{\alpha}}\Big(\mathbf{F}_{1k}\tilde{\rho}_N\Big) = (N-1)\hat{I}^{\alpha}[2, ..., N]\ \frac{\partial}{\partial \mathbf{p}_1^{\alpha}}\Big(\mathbf{F}_{12}\tilde{\rho}_N\Big). \tag{75}$$

Using $\hat{I}^{\alpha}[2, ..., N] = \hat{I}^{\alpha}[2]\hat{I}^{\alpha}[3, ..., N]$, we can rewrite the right hand side of Eq. (75) in the form

$$\hat{I}^{\alpha}[2]\ \frac{\partial}{\partial \mathbf{p}_1^{\alpha}}\Big(\mathbf{F}_{12}\hat{I}^{\alpha}[3, ..., N]\tilde{\rho}_N\Big) = \frac{\partial}{\partial \mathbf{p}_1^{\alpha}}\hat{I}^{\alpha}[2]\ \Big(\mathbf{F}_{12}\tilde{\rho}_2\Big). \tag{76}$$

Here we use definition of 2-particle distribution function $\tilde{\rho}_2$. This distribution is defined by the fractional integration of the N-particle distribution function over all $\mathbf{q}_k$ and $\mathbf{p}_k$, except $k = 1, 2$:

$$\tilde{\rho}_2 = \tilde{\rho}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \hat{I}^{\alpha}[3, ..., N]\tilde{\rho}_N(\mathbf{q}, \mathbf{p}, t). \quad (77)$$

Since $\mathbf{p}_1$ is independent of $\mathbf{q}_2$, $\mathbf{p}_2$, we can change (76) the order of the integrations and the differentiations:

$$\hat{I}^{\alpha}[2] \, \frac{\partial}{\partial \mathbf{p}_1^{\alpha}} \Big(\mathbf{F}_{12}\tilde{\rho}_2 \Big) = \frac{\partial}{\partial \mathbf{p}_1^{\alpha}} \hat{I}^{\alpha}[2]\mathbf{F}_{12}\tilde{\rho}_2.$$

Finally, we obtain the equation for one-particle reduced distribution function

$$\frac{\partial \tilde{\rho}_1}{\partial t} + \frac{\partial (\mathbf{K}_1 \tilde{\rho}_1)}{\partial \mathbf{q}_1^{\alpha}} + \frac{\partial (\mathbf{F}_1^e \tilde{\rho}_1)}{\partial \mathbf{p}_1^{\alpha}} = I(\tilde{\rho}_2). \quad (78)$$

Here $I(\tilde{\rho}_2)$ is a term with 2-particle reduced distribution function

$$I(\tilde{\rho}_2) = -(N-1) \frac{\partial}{\partial \mathbf{p}_1^{\alpha}} \hat{I}^{\alpha}[2]\mathbf{F}_{12}\tilde{\rho}_2. \quad (79)$$

Therefore the fractional generalization of the first BBGKI equation has the form

$$\frac{\partial \tilde{\rho}_1}{\partial t} = \mathcal{L}_1 \tilde{\rho}_1 + I(\tilde{\rho}_2),$$

where $\mathcal{L}_1$ is 1-particle Liouville operator

$$\mathcal{L}_1 \tilde{\rho}_2 = -\frac{\partial (\mathbf{K}_1 \tilde{\rho}_2)}{\partial \mathbf{q}_1^{\alpha}} - \frac{\partial (\mathbf{F}_1^e \tilde{\rho}_2)}{\partial \mathbf{p}_1^{\alpha}}, \quad (80)$$

The physical meaning of the term $I(\tilde{\rho}_2)$ is following. The term $I(\tilde{\rho}_2) d\mu_{\alpha}(\mathbf{q}, \mathbf{p})$ is a velocity of particle number change in $4m$-dimensional elementary phase volume $d\mu_{\alpha}(\mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2, \mathbf{p}_2)$. This change is caused by the interactions between particles. If $\alpha = 1$, then we have the first BBGKI equation for non-Hamiltonian systems.

Let us consider the particles as statistical independent systems. In this case, we have

$$\tilde{\rho}_2(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \tilde{\rho}_1(\mathbf{q}_1, \mathbf{p}_1, t)\tilde{\rho}_1(\mathbf{q}_2, \mathbf{p}_2, t). \quad (81)$$

Substituting (81) in (79), we get

$$I(\tilde{\rho}_2) = -\frac{\partial}{\partial \mathbf{p}_1^{\alpha}} \tilde{\rho}_1 \hat{I}^{\alpha}[2]\mathbf{F}_{12}\tilde{\rho}_1(\mathbf{q}_2, \mathbf{p}_2, t),$$

where $\rho_1 = \rho_1(\mathbf{q}_1, \mathbf{p}_1, t)$. As the result we have the effective force

$$\mathbf{F}^{eff}(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}^{\alpha}[2]\mathbf{F}_{12}\tilde{\rho}_1(\mathbf{q}_2, \mathbf{p}_2, t).$$

In this case, we can rewrite the term (79) in the form

$$I(\tilde{\rho}_2) = -\frac{\partial}{\partial \mathbf{p}_1^{\alpha}} (\tilde{\rho}_1 \mathbf{F}^{eff}). \quad (82)$$

Substituting (82) in Eq. (78), we get

$$\frac{\partial \tilde{\rho}_1}{\partial t} + \frac{\partial (\mathbf{K}_1 \tilde{\rho}_1)}{\partial \mathbf{q}_1^{\alpha}} + \frac{\partial}{\partial \mathbf{p}_1^{\alpha}} \Big((\mathbf{F}_1^e + (N-1)\mathbf{F}^{eff})\tilde{\rho}_1 \Big) = 0. \quad (83)$$

This equation is a closed equation for 1-particle distribution function with the external force $\mathbf{F}^e_1$ and the effective force $\mathbf{F}^{eff}$. Equation (83) is a fractional generalization of the Vlasov equation.

Let us differentiate this equation (77) that defines the two-particle reduced distribution function $\tilde{\rho}_2$. The fractional Liouville equation allows us to derive equation for 2-particle reduced distribution function $\tilde{\rho}_2$ in the form

$$\frac{\partial \tilde{\rho}_2}{\partial t} = \mathcal{L}_1 \tilde{\rho}_2 + \mathcal{L}_2 \tilde{\rho}_2 + \mathcal{L}_{12} \tilde{\rho}_2 + I(\tilde{\rho}_3), \tag{84}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are 1-particle Liouville operators (80) and $\mathcal{L}_{12}$ is 2-particle Liouville operator that is defined by equation

$$\mathcal{L}_{12} \tilde{\rho}_2 = \frac{\partial}{\partial \mathbf{p}^\alpha_1} \Big(\mathbf{F}_{12} \tilde{\rho}_2 \Big) + \frac{\partial}{\partial \mathbf{p}^\alpha_2} \Big(\mathbf{F}_{21} \tilde{\rho}_2 \Big),$$

and $I(\tilde{\rho}_3)$ is a term with the 3-particle reduced distribution

$$I(\tilde{\rho}_3) = \frac{(N-1)(N-2)}{2} \hat{I}^\alpha[3] \ \Big(\frac{\partial (\mathbf{F}_{13} \tilde{\rho}_3)}{\partial \mathbf{p}^\alpha_1} + \frac{\partial (\mathbf{F}_{23} \tilde{\rho}_3)}{\partial \mathbf{p}^\alpha_2} \Big). \tag{85}$$

The derivation of Eq. (84) is the analogous to the derivation of Eq. (78). It is easy to see that the system of Eqs. (78) and (84) are not closed. The system of these equations for the reduced distribution functions can be called the fractional generalization of the BBGKI equations.

8. Conclusion

In this paper, we consider the fractional generalizations of Liouville and BBGKI equations. The normalization condition, phase volume, and average values are generalized for fractional case. These generalizations lead us to the fractional analog of phase space. The space can be considered as a fractal dimensional space. The physical interpretation of the fractional phase space is discussed. The fractional generalization of average values is derived.

In this paper the fractional analogs of the BBGKI equations are derived. In order to derive these analogs we use the fractional Liouville equation [10], we define the fractional average values and the fractional reduced distribution functions. The fractional analog of Vlasov equation is considered.

The fractional Liouville, BBGKI and Vlasov equations are better approximation than its classical analogs for the systems with the fractional phase spaces. For example, the systems that live on some fractals can be described by the suggested fractional equations. Note that the fractional system is non-Hamiltonian systems. The fractional harmonic oscillator is an oscillator in the fractional phase space that can be considered as a fractal medium. Therefore the fractional oscillator can be interpreted as an elementary excitation of some fractal medium with non-integer mass dimension.

It is not hard to prove that the fractional systems are connected with the non-Gaussian statistics. That the dissipative and non-Hamiltonian systems can have stationary states of the Hamiltonian systems [22]. Classical dissipative and non-Hamiltonian systems can have the canonical Gibbs distribution as a solution of the stationary Liouville equations for this dissipative system [23]. Using the methods [23], it is easy to prove that some fractional dissipative systems can have fractional analog of the Gibbs distribution (non-Gaussian statistic) as a solution of the fractional Liouville equations. Using the methods [23], it is easy to find the stationary solutions of the fractional BBGKI equations for the fractional systems. Note that the interest in and relevance of fractional kinetic equations is a natural consequence of the realization of the importance of non-Gaussian statistics of many dynamical systems. There is already a substantial literature studying such equations in one or more space dimensions.

Note that the quantization of the fractional systems is a quantization of non-Hamiltonian dissipative systems. Using the method, which is suggested in Refs. [24, 25], we can realize the Weyl quantization for the fractional systems. The suggested fractional Hamilton and Liouville equations allow us to derive the fractional generalization for the quantum systems by methods suggested in Refs. [24, 25].

The fractional BBGKI equations can be used to derive the Enskog transport equations. The fractional analog of the hydrodynamic equations can be derived from the first fractional BBGKI equation. It is known that the Fokker-Planck equation can be derived from the BBGKI equations [6]. The fractional Zaslavsky's equation [1, 2] can be derived from the fractional generalization of the BBGKI equation.

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