

# Large lattice fractional Fokker–Planck equation

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**Abstract.** An equation of long-range particle drift and diffusion on a 3D physical lattice is suggested. This equation can be considered as a lattice analog of the space-fractional Fokker–Planck equation for continuum. The lattice approach gives a possible microstructural basis for anomalous diffusion in media that are characterized by the non-locality of power law type. In continuum limit the suggested 3D lattice Fokker–Planck equations give fractional Fokker–Planck equations for continuous media with power law non-locality that is described by derivatives of non-integer orders. The consistent derivation of the fractional Fokker–Planck equation is proposed as a new basis to describe space-fractional diffusion processes.

**Keywords:** diffusion

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**1. Introduction**

Fokker–Planck equations are usually used to describe the Brownian motion of particles [1]. These equations describe the change of probability of a random function in space and time in diffusion processes. The Fokker–Planck equation is usually the second-order partial differential equation of parabolic type. In many studies of diffusion processes in complex media, the usual second-order Fokker–Planck equation may not be adequate. In particular, the probability density may have a thicker tail than the Gaussian probability density and correspondent correlation functions may decay to zero much slower than the functions for usual diffusion processes resulting in long-range dependence. This phenomenon is known as anomalous diffusion [2–4]. Anomalous diffusion processes can be characterized by a power law mean squared displacement of the form [2–4]

$$\langle x^2(t) \rangle = \frac{2K(\alpha)t^\alpha}{\Gamma(\alpha + 1)}, \quad (1)$$

where  $\Gamma(z)$  is the gamma function,  $\alpha$  is the anomalous diffusion exponent and  $K(\alpha)$  is the anomalous diffusion constant. In equation (1), we use the second moment that is defined in terms of the ensemble average. Depending on the value of  $\alpha$ , we usually distinguish sub-diffusion for  $0 < \alpha < 1$  or super-diffusion for  $\alpha > 1$ . There are two limit cases such as the normal diffusion ( $\alpha = 1$ ) and the ballistic motion ( $\alpha = 2$ ). One possible approach to describe the anomalous diffusion is based on the continuous time random walk models [5] in which the particles are considered as random walkers with step lengths  $\mathbf{r}$  and waiting times  $t$ . An important role is played by the anomalous diffusion processes with the Poissonian waiting time and the Lévy distribution for the jump length. The Lévy flights [2] are random walks in which the step lengths (long jumps) have a probability distribution that is heavy-tailed. The Lévy motion can be described by a generalized diffusion equation

with space derivatives of non-integer orders  $\mu$ , [4]. The fractional moment of order  $\delta$  for Lévy flights has the form  $\langle |x(t)|^\delta \rangle \sim t^{\delta/\mu}$ , where  $0 < \delta < \mu \leq 2$ .

Derivatives of non-integer orders [6–13] play an important role in describing particle transport in anomalous diffusion [4, 14–19] and have a wide application in various areas of physics (see, for example, [20–28]). Various approaches lead to different types of space–time fractional Fokker–Planck equations. Usually the space-fractional Fokker–Planck equations are obtained from the second-order differential equations by replacing the first-order and second-order space derivatives by fractional-order derivatives [7]. Fractional Fokker–Planck equations with coordinate derivatives of non-integer order have been suggested in [29]. The solutions and properties of these equations are described in [15, 30]. The Fokker–Planck equation with fractional coordinate derivatives was also considered in [14, 31–35]. It should be noted that the fractional Fokker–Planck equations can be derived from the probabilistic continuous time random walk [36–38]. In this paper we propose a consistent derivation of the space-fractional Fokker–Planck equation based on a lattice model with long-range drift and diffusion that is considered as a microstructural basis to describe fractional diffusion processes in continua.

A discrete lattice version of the Fokker–Planck equation in analogy with the lattice Boltzmann models has been suggested in [40–42]. These models are used to solve the equations of hydrodynamics and cavity flow simulations [43]. The lattice Fokker–Planck equation is applied to the study of electro-rheological transport of 1D charged fluid [44] and it is used in phase-space descriptions of inertial polymer dynamics [45]. All these lattice Fokker–Planck equations are based on the lattice Boltzmann discretization approach.

In this paper, we propose a lattice equation for probability density of particle in unbounded homogeneous 3D lattice with long-range drift and diffusion to  $\mathbf{n}$ -site from all other  $\mathbf{m}$ -sites ( $\mathbf{m} \neq \mathbf{n}$ ). We prove that continuous limit for the suggested lattice Fokker–Planck equation gives the space-fractional Fokker–Planck equation for non-local continuum. The fractional differential equation for continuum contains generalized conjugate Riesz derivatives of non-integer orders.

Continuum mechanics [46] can be considered as a continuous limit of lattice dynamics [47–50], where the length-scales of a continuum element are much larger than the distances between the lattice particles. The first self-consistent derivation of the Fokker–Planck equation based on the microscopic dynamics for classical and quantum systems was obtained by Bogolyubov and Krylov [51, 52]. Long-range interactions are important for different problems in statistical mechanics [53–55], kinetic theory and nonequilibrium statistical mechanics [56, 57], theory of non-equilibrium phase transitions [58, 59]. As it was shown in [60, 61] (see also [62–64] and [65–70]), the continuum equations with fractional derivatives can be directly connected to lattice models with long-range properties. A connection between the dynamics of a lattice system of particles with long-range properties and the fractional continuum equations are proved by using the transform operation [60, 61]. The papers [60, 61] deal with the 1D lattice models and the correspondent 1D continuum equations. In this paper, we suggest 3D lattice models for space-fractional diffusion processes. We propose a general form of 3D lattice Fokker–Planck equation, which leads to a continuum fractional Fokker–Planck equation with space derivatives of non-integer orders by continuous limit. The suggested approach to derive the space fractional Fokker–Planck equations can serve as a microstructural basis to describe the spatial-fractional diffusion processes.

## 2. Lattice with long-range drift and diffusion

The lattice is characterized by space periodicity. In an unbounded lattice we can define three non-coplanar vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , that are the shortest vectors by which a lattice can be displaced and be brought back into itself. All space lattice sites can be defined by the vector  $\mathbf{n} = (n_1, n_2, n_3)$ , where  $n_i$  are integer. For simplification, we consider a lattice with mutually perpendicular primitive lattice vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ . We choose directions of the axes of the Cartesian coordinate system to coincide with the vector  $\mathbf{a}_i$ . Then  $\mathbf{a}_i = a_i \mathbf{e}_i$ , where  $a_i = |\mathbf{a}_i|$  and  $\mathbf{e}_i$  are the basis vectors of the Cartesian coordinate system. This simplification means that the lattice is a primitive orthorhombic Bravais lattice with long-range drift and diffusion of particles.

If we choose the coordinate origin at one of the sites, then the position vector of an arbitrary lattice site with  $\mathbf{n} = (n_1, n_2, n_3)$  is written  $\mathbf{r}(\mathbf{n}) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ . The lattice sites are numbered by  $\mathbf{n}$ , so that the vector  $\mathbf{n}$  can be considered as a number vector of the corresponding particle. We assume that the positions of particles coincide with the lattice sites  $\mathbf{r}(\mathbf{n})$ . The probability density for the lattice site will be denoted by  $f(\mathbf{n}, t) = f(n_1, n_2, n_3, t)$ , where the site is defined by the vector  $\mathbf{n} = (n_1, n_2, n_3)$ . The function  $f(\mathbf{n}, t)$  satisfies the conditions

$$\sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \sum_{n_3=-\infty}^{+\infty} f(n_1, n_2, n_3, t) = 1, \quad f(n_1, n_2, n_3, t) \geq 0 \quad (2)$$

for all  $t \in \mathbb{R}$ .

The equation for probability density of particle in unbounded homogeneous lattice is

$$\frac{\partial f(\mathbf{n}, t)}{\partial t} = - \sum_{i=1}^3 \sum_{m_i \neq n_i} g_i K_{\alpha_i}^i(\mathbf{n} - \mathbf{m}) f(\mathbf{m}, t) + \sum_{i,j=1}^3 \sum_{m_i \neq n_i} \sum_{m_j \neq n_j} g_{ij} K_{\alpha_i, \beta_j}^{ij}(\mathbf{n} - \mathbf{m}) f(\mathbf{m}, t), \quad (3)$$

where  $f(\mathbf{n}, t)$  is the probability density function to find the test particle at site  $\mathbf{n}$  at time  $t$ . The italics  $i, j \in \{1; 2; 3\}$  are the coordinate indices,  $g_i$  and  $g_{ij}$  are lattice coupling constants. The coefficients  $K_{\alpha_i}^i(\mathbf{n} - \mathbf{m})$  and  $K_{\alpha_i, \beta_j}^{ij}(\mathbf{n} - \mathbf{m})$  describe the particle drift and diffusion on the lattice and it can be called the drift and diffusion kernels for lattice step length  $\mathbf{n} - \mathbf{m}$ . These kernels describe the long-range drift and diffusion to  $\mathbf{n}$ -site from all other  $\mathbf{m}$ -sites. The parameters  $\alpha_i$  and  $\beta_j$  in the kernels are positive real numbers that characterize how quickly the intensity of the drift and diffusion processes in the lattice decrease with increasing the value  $\mathbf{n} - \mathbf{m}$ . These parameters also can be considered as degrees of the power law of lattice spatial dispersion [65, 68] that is described by non-integer power of the wave vector components.

Equation (3) describes fractional diffusion processes on the physical lattices, where long-range jumps can be realized. The Lévy motion (flights) for these lattices can be described by the lattice Fokker–Planck equation (3), which is considered as a lattice analog of the fractional diffusion processes with the Poissonian waiting time and the Lévy distribution for the jump length [4].

For simplification, we consider the kernels in the form

$$K_{\alpha_i}^i(\mathbf{n} - \mathbf{m}) = K_{\alpha_i}(n_i - m_i), \quad K_{\alpha_i, \beta_j}^{ij}(\mathbf{n} - \mathbf{m}) = K_{\alpha_i}(n_i - m_i) K_{\beta_j}(n_j - m_j), \quad (4)$$

where  $i, j = 1, 2, 3$ . The kernels  $K_{\alpha_i}(n_i - m_i)$ , where  $i = 1, 2, 3$ , describe long-range jumps in the direction  $\mathbf{a}_i$  with lattice step length  $n_i - m_i$  in the lattice. The correspondent terms with kernels  $K_{\alpha_i}(n_i - m_i)$  can be considered as lattice analogs of fractional derivatives of order  $\alpha_i$  with respect to coordinate  $x_i = (\mathbf{r}, \mathbf{a}_i)$ . We will consider even and odd types of the kernels  $K_{\alpha_i}(n_i - m_i)$ ,  $i = 1, 2, 3$ , that will be denoted by  $K_{\alpha_i}^+(n_i - m_i)$  and  $K_{\alpha_i}^-(n_i - m_i)$  respectively.

We assume that the kernels  $K_{\alpha}^{\pm}(n)$  satisfy the following conditions:

- (a) The kernels  $K_{\alpha}^{\pm}(n)$  are real-valued functions of integer variable  $n \in \mathbb{Z}$ . The kernels  $K_{\alpha}^+(n)$  and  $K_{\alpha}^-(n)$  are even and odd functions such that

$$K_{\alpha}^+(-n) = +K_{\alpha}^+(n), \quad K_{\alpha}^-(-n) = -K_{\alpha}^-(n) \tag{5}$$

hold for all  $n \in \mathbb{Z}$ .

- (b) The kernels  $K_{\alpha}^{\pm}(n)$  belong to the Hilbert space of square-summable sequences,

$$\sum_{n=1}^{\infty} |K_{\alpha}^{\pm}(n)|^2 < \infty. \tag{6}$$

- (c) The Fourier series transforms  $\hat{K}_{\alpha}^{\pm}(k)$  of the kernels  $K_{\alpha}^{\pm}(n)$  in the form

$$\hat{K}_{\alpha}^+(k) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e^{-ikn} K_{\alpha}^+(n) = 2 \sum_{n=1}^{\infty} K_{\alpha}^+(n) \cos(kn), \tag{7}$$

$$\hat{K}_{\alpha}^-(k) = \sum_{n=-\infty}^{+\infty} e^{-ikn} K_{\alpha}^-(n) = -2i \sum_{n=1}^{\infty} K_{\alpha}^-(n) \sin(kn) \tag{8}$$

satisfying the conditions

$$\hat{K}_{\alpha}^+(k) = |k|^{\alpha} + o(|k|^{\alpha}), \quad (k \rightarrow 0), \tag{9}$$

and

$$\hat{K}_{\alpha}^-(k) = i \operatorname{sgn}(k) |k|^{\alpha} + o(|k|^{\alpha}), \quad (k \rightarrow 0) \tag{10}$$

respectively. Here the little-o notation  $o(|k|^{\alpha})$  means the terms that include higher powers of  $|k|$  than  $|k|^{\alpha}$ . The suggested forms (9) and (10) of the Fourier series transforms of the kernels  $K_{\alpha}^{\pm}(n)$  mean that we consider lattices with weak spatial dispersion [65]. The conditions (9) and (10) allow us to consider a wide class of kernels to describe the long-range lattice drift and diffusion.

In general, the type of dependence of the function  $\hat{K}_{\alpha}^{\pm}(k)$  on the wave vector  $k$  is defined by the type of spatial dispersion in the lattice [65, 68]. For a wide class of processes in the lattice, the wavelength  $\lambda$  holds the relation  $a_0/\lambda \sim ka_0 \ll 1$ , where  $a_0$  is the characteristic size of the lattice distance such that  $a_0 = \max\{|\mathbf{a}_1|, |\mathbf{a}_2|, |\mathbf{a}_3|\}$ . In the case  $ka_0 \ll 1$ , where

$a_0$ , the spatial dispersion of the lattice is weak. To describe lattices with such property it is enough to know the dependence of the function  $\hat{K}_\alpha^\pm(k)$  only for small values  $k$  and we can replace this function by Taylor’s polynomial series. The weak spatial dispersion of the lattices with a power law type of spatial dispersion cannot be described by the usual Taylor approximation. In this case, we should use a fractional Taylor series [65, 68]. The fractional Taylor series is more adequate for approximation of non-integer power law functions. For example, the usual Taylor series for the power law function  $\hat{K}_\alpha^+(k) = a_\alpha k^\alpha$  has the infinite by many terms for non-integer  $\alpha$ . The fractional Taylor series of order  $\alpha$  has a finite number of terms for this function and the fractional Taylor’s approximation is exact. We can use the fractional Taylor’s series in the Riemann–Liouville form (see chapter 1 section 2.6 [6]) that can be represented as

$$\hat{K}_\alpha^+(k_j) = b_j(\alpha) |k_j|^\alpha + o(|k_j|^\alpha), \tag{11}$$

where

$$b_j(\alpha) = \frac{({}_0^{\text{RL}}D_{k_j}^\alpha \hat{K}_\alpha^+)(0)}{\Gamma(\alpha + 1)}, \tag{12}$$

and  ${}_0^{\text{C}}D_k^\alpha$  is the Riemann–Liouville fractional derivative [7] of order  $0 < \alpha < 1$  with respect to  $k$ . This derivative is defined by

$$({}_0^{\text{RL}}D_k^\alpha \hat{K}_\alpha^+)(k) = \left(\frac{d}{dk}\right)^n \left({}_0I_k^{n-\alpha} \hat{K}_\alpha^+\right)(k), \tag{13}$$

where  ${}_0I_k^\alpha$  is the left-sided Riemann–Liouville fractional integral of order  $\alpha > 0$  with respect to  $k$  of the form

$$({}_0I_k^\alpha \hat{K}_\alpha^+)(k) = \frac{1}{\Gamma(\alpha)} \int_0^k \frac{\hat{K}_\alpha^+(k') dk'}{(k - k')^{1-\alpha}}, \quad (k > 0). \tag{14}$$

Using the approximation (11), we neglect a frequency dispersion for simplification, i.e. the parameters  $b_j(\alpha)$  do not depend on the frequency  $\omega$ . In suggested lattice models, we define the kernels such that the constants  $g_i$  and  $g_{ij}$  include the factor  $b_j(\alpha)$  and as a result the conditions (9) and (10) hold.

For simplification, we can consider the lattice kernels that are defined by the explicit expressions in the form

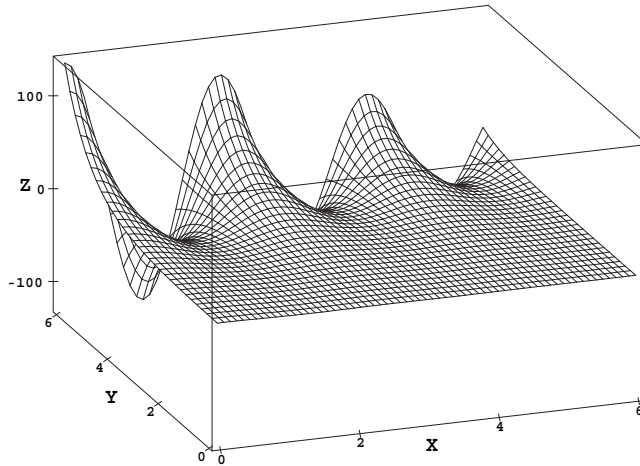
$$\hat{K}_\alpha^+(k) = |k|^\alpha, \quad \hat{K}_\alpha^-(k) = i \operatorname{sgn}(k) |k|^\alpha. \tag{15}$$

In this case, the inverse relations to the definitions of  $\hat{K}_\alpha^\pm(k)$  by equations (7) and (8) have the forms

$$K_\alpha^+(n) = \frac{1}{\pi} \int_0^\pi k^\alpha \cos(nk) dk, \quad K_\alpha^-(n) = -\frac{1}{\pi} \int_0^\pi k^\alpha \sin(nk) dk. \tag{16}$$

For non-integer real values of the parameter  $\alpha$ , the expressions for the kernels  $K_\alpha^\pm(n - m)$  are

$$K_\alpha^+(n - m) = \frac{\pi^\alpha}{\alpha + 1} {}_1F_2\left(\frac{\alpha + 1}{2}; \frac{1}{2}, \frac{\alpha + 3}{2}; -\frac{\pi^2(n - m)^2}{4}\right), \quad \alpha > -1, \tag{17}$$



**Figure 1.** Plot of the function  $F_+(x, y)$  (19) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 6]$ .

$$K_\alpha^-(n - m) = -\frac{\pi^{\alpha+1} (n - m)}{\alpha + 2} {}_1F_2 \left( \frac{\alpha + 2}{2}; \frac{3}{2}, \frac{\alpha + 4}{2}; -\frac{\pi^2 (n - m)^2}{4} \right), \quad \alpha > -2, \quad (18)$$

where  ${}_1F_2$  is the Gauss hypergeometric function (see Chapter 2 in [72]). Note that expressions can be used not only for  $\alpha > 0$ , but also for some negative values of  $\alpha$ .

To visualize the properties of the kernels (17) and (18), we give the plots of the functions

$$F_+(x, y) = \frac{\pi^y}{y + 1} {}_1F_2 \left( \frac{y + 1}{2}; \frac{1}{2}, \frac{y + 3}{2}; -\frac{\pi^2 x^2}{4} \right), \quad (19)$$

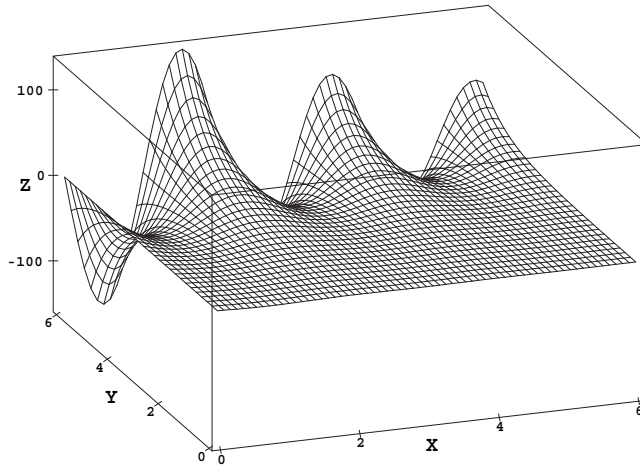
$$F_-(x, y) = -\frac{\pi^{y+1} x}{y + 2} {}_1F_2 \left( \frac{y + 2}{2}; \frac{3}{2}, \frac{y + 4}{2}; -\frac{\pi^2 x^2}{4} \right), \quad (20)$$

where

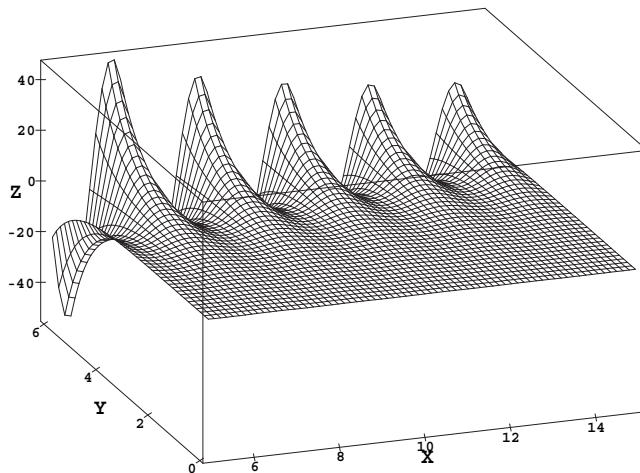
$$K_\alpha^\pm(n - m) = F_\pm(n - m, \alpha). \quad (21)$$

We present the plots of the function (19) in figures 1, 3, 5 and the plots of (20) in figures 2, 4, 6 for the same ranges of  $x$  and  $y > 0$ .

Let us note some qualitative properties that can be seen from figures 1–6. We should note that the functions (19) and (20) represent the kernels with (17) and (18) that describe the long-range drift and diffusion to  $n$ -site from all other  $m$ -sites, where  $m \in \mathbb{N}$ . Oscillations tell us that the inflow and outflow of probability periodically change each other, when the distance  $x = n - m$  between sites increases. The negative values of  $F_\pm(x, \alpha)$  can be interpreted as the probability flux from the site and the positive values of  $F_\pm(x, \alpha)$  can be interpreted as the flux to the site. Maximums and minimums of  $F_\pm(x, \alpha)$  characterize an amplitude of oscillation of the probability flux from the site and into the site. The amplitudes as functions of the parameter  $\alpha$  are increasing functions for a fixed value  $x = n$ . Plots of the functions (19) and (20) with  $\alpha = 1.5$  and  $\alpha = 1$  for the range  $x \in [0, 7]$  are presented in figures 7 and 8, where the graphics of functions with  $\alpha = 1.5$  have larger amplitudes than the graphics of the functions with  $\alpha = 1$ . The amplitudes as



**Figure 2.** Plot of the function  $F_-(x, y)$  (20) for the range  $x \in [0, 6]$  and  $y = \alpha \in [0, 6]$ .



**Figure 3.** Plot of the function  $F_+(x, y)$  (19) for the range  $x \in [5, 15]$  and  $y = \alpha \in [0, 6]$ .

functions of the values  $n$  are decreasing functions for a fixed value  $\alpha$  and this decreasing has a power law form. During the transition from a non-local to local case for the functions  $F_{\pm}(x, \alpha)$ , a sharp jump does not occur. We can only state that the fractional power law decreasing is transformed into the decreasing of the integer power form.

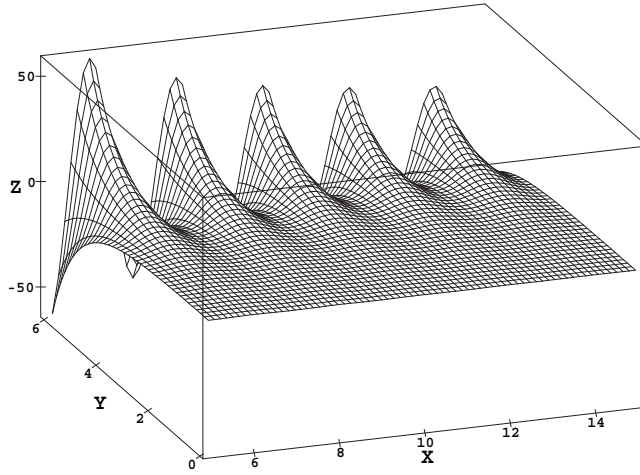
It should be noted that the kernels  $K_{\alpha}^{+}(n)$  give the local operators for continuum limit for even  $\alpha$  only and  $K_{\alpha}^{-}(n)$  give the local operators for odd  $\alpha$  only. The kernels  $K_{\alpha}^{\pm}(n)$  for integer values of  $\alpha$  (see also section 2.5.3.5 in [71]) can be represented by the equations

$$K_{\alpha}^{+}(n) = \sum_{k=0}^{[(\alpha-1)/2]} \frac{(-1)^{n+k} \alpha! \pi^{\alpha-2k-2}}{(\alpha-2n-1)!} \frac{1}{n^{2k+2}} + \frac{(-1)^{[(\alpha+1)/2]} \alpha! (2[(\alpha+1)/2] - \alpha)}{\pi n^{\alpha+1}}, \quad (22)$$

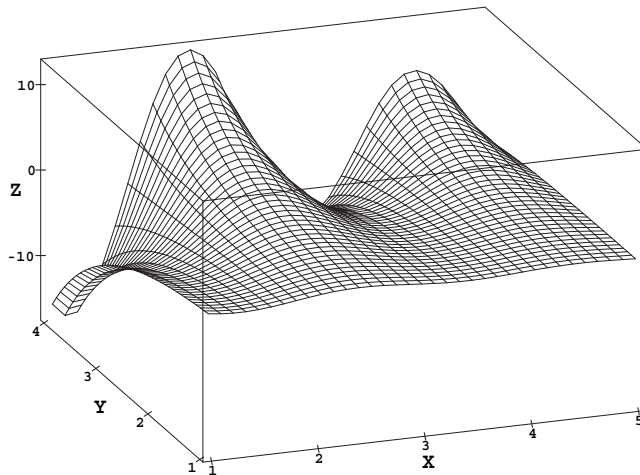
and

$$K_{\alpha}^{-}(n) = - \sum_{k=0}^{[\alpha/2]} \frac{(-1)^{n+k+1} \alpha! \pi^{\alpha-2k-1}}{(\alpha-2n)!} \frac{1}{n^{2k+2}} - \frac{(-1)^{[\alpha/2]} \alpha! (2[\alpha/2] - \alpha + 1)}{\pi n^{\alpha+1}}, \quad (23)$$





**Figure 4.** Plot of the function  $F_-(x, y)$  (20) for the range  $x \in [5, 15]$  and  $y = \alpha \in [0, 6]$ .



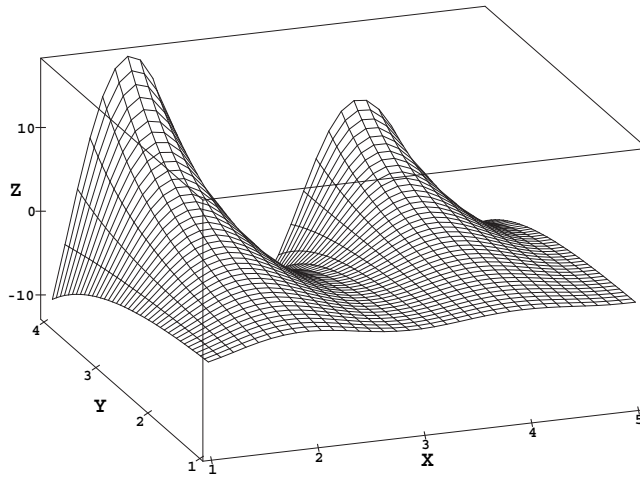
**Figure 5.** Plot of the function  $F_+(x, y)$  (19) for the range  $x \in [1, 5]$  and  $y = \alpha \in [1, 4]$ .

where  $[z]$  is the integer part of the value  $z$  and  $2[(\alpha + 1)/2] - \alpha = 1$  for odd  $n$ , and  $2[(\alpha + 1)/2] - \alpha = 0$  for even  $n$ . We can give examples of kernel with some integer  $\alpha$ . Using equation (22) or direct integration (16) for  $\alpha \in \{1; 2; 3\}$ , we give  $K_\alpha^+(n)$  in the form

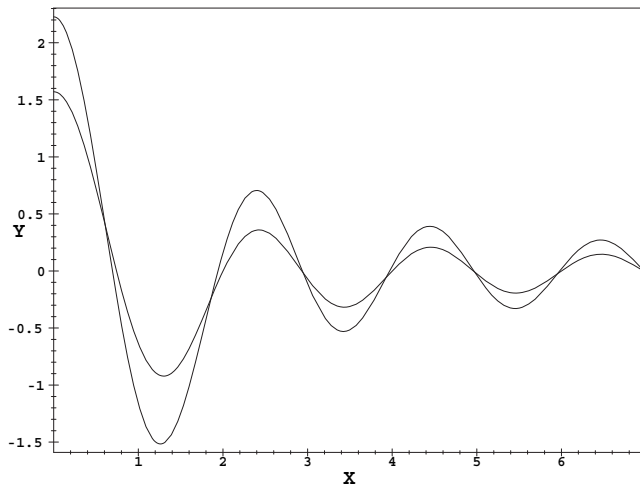
$$K_1^+(n) = -\frac{1 - (-1)^n}{\pi n^2}, \quad K_2^+(n) = \frac{2(-1)^n}{n^2}, \quad K_3^+(n) = \frac{3\pi(-1)^n}{n^2} + \frac{6(1 - (-1)^n)}{\pi n^4}, \quad (24)$$

$$K_1^-(n) = \frac{(-1)^n}{n}, \quad K_2^-(n) = \frac{(-1)^n \pi}{n} + \frac{2(1 - (-1)^n)}{\pi n^3}, \quad K_3^-(n) = \frac{(-1)^n \pi^2}{n} - \frac{6(-1)^n}{n^3}, \quad (25)$$

where  $(1 - (-1)^n) = 2$  for odd  $n$  and  $((-1)^n - 1) = 0$  for even  $n$ .



**Figure 6.** Plot of the function  $F_-(x, y)$  (20) for the range  $x \in [1, 5]$  and  $y = \alpha \in [1, 4]$ .

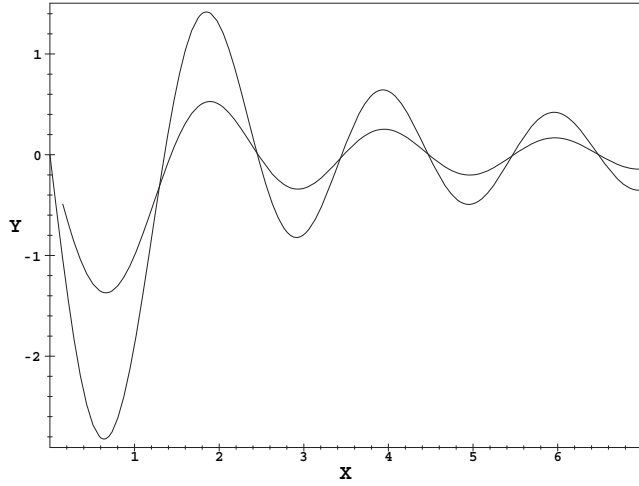


**Figure 7.** Plot of the function  $F_+(x)$  (19) with  $\alpha = 1.5$  and  $\alpha = 1$  for the range  $x \in [0, 7]$ .

For simplification of the form of lattice equation, we use the lattice operators  $\mathbb{K}^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right]$  such that the action of these operators on the lattice probability density  $f(\mathbf{m}, t)$  is

$$\mathbb{K}^\pm \left[ \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \right] f(\mathbf{m}, t) = \sum_{\substack{m_i = -\infty \\ m_i \neq n_i}}^{+\infty} K_\alpha^\pm(n_i - m_i) f(\mathbf{m}, t), \quad (i = 1, 2, 3). \quad (26)$$

The values  $i = 1, 2, 3$  specify one of the three variables  $n_1, n_2, n_3$  of the lattice site that are similar to  $x_i$  of the space  $\mathbb{R}^3$ . If  $\alpha_i = 1$ , then  $\mathbb{K}^+$  is a non-local operator and if  $\alpha_i = 2$ , then  $\mathbb{K}^-$  are non-local operators also. Note that the operators  $\mathbb{K}^+ \left[ \begin{smallmatrix} \alpha_i \\ i \end{smallmatrix} \right]$  for odd integer values of  $\alpha_i$  and  $\mathbb{K}^- \left[ \begin{smallmatrix} \alpha_i \\ i \end{smallmatrix} \right]$  for even integer values of  $\alpha_i$  are non-local. For example, the operators  $\mathbb{K}^+ \left[ \begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]$  and  $\mathbb{K}^- \left[ \begin{smallmatrix} 2 \\ i \end{smallmatrix} \right]$  cannot be considered as local operators of integer orders.



**Figure 8.** Plot of the function  $F_-(x)$  (20) with  $\alpha = 1.5$  and  $\alpha = 1$  for the range  $x \in [0, 7]$ .

We also can consider combinations of the lattice operators

$$\mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix} = \mathbb{K}^{\pm} \begin{bmatrix} \alpha_i \\ i \end{bmatrix} \mathbb{K}^{\pm} \begin{bmatrix} \beta_j \\ j \end{bmatrix}, \quad (27)$$

where  $i, j$  take values from the set  $\{1; 2; 3\}$ . The action of the operator (27) on the lattice probability density  $f(\mathbf{m}, t)$  is

$$\mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix} f(\mathbf{m}, t) = \sum_{\substack{m_i=-\infty \\ m_i \neq n_i}}^{+\infty} \sum_{\substack{m_j=-\infty \\ m_j \neq n_j}}^{+\infty} K_{\alpha_i}^{\pm}(n_i - m_i) K_{\beta_j}^{\pm}(n_j - m_j) f(\mathbf{m}, t). \quad (28)$$

This is the mixed lattice operators.

Using the lattice operators (26) and (28), the equation for probability density (3) takes the form

$$\frac{\partial f(\mathbf{n}, t)}{\partial t} = - \sum_{i=1}^3 g_i \mathbb{K}^{\pm} \begin{bmatrix} \alpha_i \\ i \end{bmatrix} f(\mathbf{m}, t), + \sum_{i,j=1}^3 g_{ij} \mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix} f(\mathbf{m}, t). \quad (29)$$

This is the 3D lattice Fokker–Planck equation in the operator form to describe fractional diffusion and drift with the lattice jump length  $(\mathbf{n} - \mathbf{m})$ .

To describe the long-range drift and diffusion for the lattice with memory, we can use the equation

$$\frac{\partial f(\mathbf{n}, t)}{\partial t} = - \sum_{i=1}^3 g_i \mathbb{K}^{\pm} \begin{bmatrix} \alpha_i \\ i \end{bmatrix} f(\mathbf{m}, t), + \sum_{i,j=1}^3 g_{ij} \mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix} {}_0^{\text{RL}}D_t^{1-\gamma} f(\mathbf{m}, t), \quad (30)$$

where  ${}_0^{\text{RL}}D_t^{1-\gamma}$  is the Riemann–Liouville fractional derivative of order  $(1 - \gamma)$  with respect to time [7]. Note that the time-fractional derivative  ${}_0^{\text{RL}}D_t^{1-\gamma}$  is present only in the diffusion term. This fractional derivative describes the long-term memory of power law type. Equation (30) describes anomalous diffusion processes with the waiting time  $t$  and the lattice jump length  $(\mathbf{n} - \mathbf{m})$ .

We can consider the time-fractional derivatives  ${}_0^{\text{RL}}D_t^{1-\gamma}$  in the first and second terms of the right side of the lattice Fokker–Planck equation (29). In this case the time-fractional lattice Fokker–Planck equation has the form

$$\frac{\partial f(\mathbf{n}, t)}{\partial t} = {}_0^{\text{RL}}D_t^{1-\gamma} \mathcal{L}_{\text{LFP}}^{\alpha, \beta} f(\mathbf{n}, t), \quad (31)$$

where  $\mathcal{L}_{\text{LFP}}^{\alpha, \beta}$  is the lattice Fokker–Planck operator

$$\mathcal{L}_{\text{LFP}}^{\alpha, \beta} = - \sum_{i=1}^3 g_i \mathbb{K}^{\pm} \begin{bmatrix} \alpha_i \\ i \end{bmatrix} + \sum_{i,j=1}^3 g_{ij} \mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix}. \quad (32)$$

Equation (31) describes long-range diffusion and drift with power law memory on orthorhombic Bravais lattices.

### 3. Continuum limit for lattice equations

#### 3.1. Continuum limit for lattice probability density

In order to transform a lattice probability density  $f(\mathbf{n}, t)$  into a probability density  $f(\mathbf{r}, t)$  of continuum, we use the approach suggested in [60, 61]. We propose to consider  $f(\mathbf{n}, t)$  as Fourier series coefficients of some function  $\hat{f}(\mathbf{k}, t)$  for  $k_j \in [-k_{j0}/2, k_{j0}/2]$ , where  $k_{j0} = 2\pi/a_j$ . Then we use the continuous limit  $\mathbf{k}_0 \rightarrow \infty$  to obtain  $f(\mathbf{k}, t)$  and finally we apply the inverse Fourier integral transformation to obtain the probability density  $f(\mathbf{r}, t)$ . For clarity, we have presented the set of transformations of the probability density in figure 9.

The transformation of a lattice probability density into a continuum probability density is realized by a sequence of the following three steps:

The first step is the Fourier series transform  $\mathcal{F}_{\Delta} : f(\mathbf{n}, t) \rightarrow \mathcal{F}_{\Delta}\{f(\mathbf{n}, t)\} = \hat{f}(\mathbf{k}, t)$  that is defined by

$$\hat{f}(\mathbf{k}, t) = \sum_{n_1, n_2, n_3 = -\infty}^{+\infty} f(\mathbf{n}, t) e^{-i(\mathbf{k}, \mathbf{r}(\mathbf{n}))} = \mathcal{F}_{\Delta}\{f(\mathbf{n}, t)\}, \quad (33)$$

where the inverse transformation is

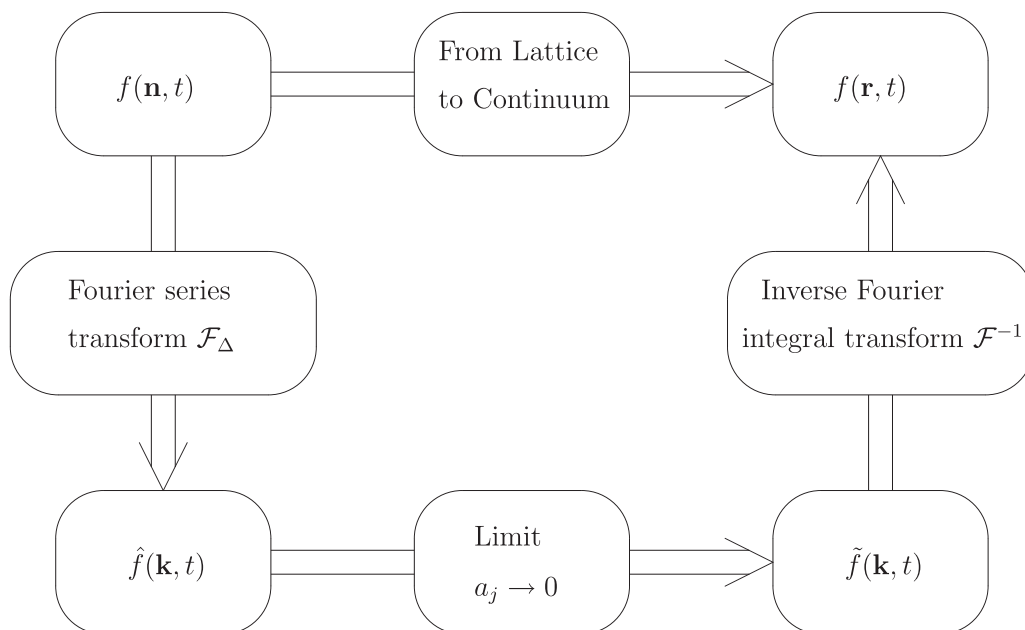
$$f(\mathbf{n}, t) = \left( \prod_{j=1}^3 \frac{1}{k_{j0}} \right) \int_{-k_{10}/2}^{+k_{10}/2} dk_1 \dots \int_{-k_{30}/2}^{+k_{30}/2} dk_3 \hat{f}(\mathbf{k}, t) e^{i(\mathbf{k}, \mathbf{r}(\mathbf{n}))} = \mathcal{F}_{\Delta}^{-1}\{\hat{f}(\mathbf{k}, t)\}, \quad (34)$$

and  $\mathbf{r}(\mathbf{n}) = \sum_{j=1}^3 n_j \mathbf{a}_j$  and  $k_{j0} = 2\pi/a_j$ . We assume that all lattice particles have the same inter-particle distance  $a_j$  in the direction  $\mathbf{a}_j$  for simplification.

The second step is the passage to the limit  $a_j \rightarrow 0$  ( $k_{j0} \rightarrow \infty$ ) denoted by  $\text{Lim} : \hat{f}(\mathbf{k}, t) \rightarrow \text{Lim}\{\hat{f}(\mathbf{k}, t)\} = \tilde{f}(\mathbf{k}, t)$ . The function  $\tilde{f}(\mathbf{k}, t)$  can be derived from  $\hat{f}(\mathbf{k}, t)$  in the limit  $a_i \rightarrow 0$ . Note that  $\tilde{f}(\mathbf{k}, t)$  is a Fourier integral transform of the probability density  $f(\mathbf{r}, t)$  and  $\hat{f}(\mathbf{k}, t)$  is a Fourier series transform of  $f(\mathbf{n}, t)$ , where we use

$$f(\mathbf{n}, t) = \prod_{j=1}^3 \frac{2\pi}{k_{j0}} f(\mathbf{r}(\mathbf{n}), t)$$

considering  $\mathbf{r}(\mathbf{n}) = \sum_{j=1}^3 n_j \mathbf{a}_j = 2\pi \sum_{j=1}^3 n_j/k_{j0} \mathbf{e}_j \rightarrow \mathbf{r}$ .



**Figure 9.** Transformation of lattice probability density to continuum probability density.

The third step is the inverse Fourier integral transform  $\mathcal{F}^{-1} : \tilde{f}(\mathbf{k}, t) \rightarrow \mathcal{F}^{-1}\{\tilde{f}(\mathbf{k}, t)\} = f(\mathbf{r}, t)$  is defined by

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3\mathbf{k} e^{i\sum_{j=1}^3 k_j x_j} \tilde{f}(\mathbf{k}, t) = \mathcal{F}^{-1}\{\tilde{f}(\mathbf{k}, t)\} \quad (35)$$

that corresponds to the transformation

$$\tilde{f}(\mathbf{k}, t) = \int_{-\infty}^{+\infty} d^3\mathbf{r} e^{-i\sum_{j=1}^3 k_j x_j} f(\mathbf{r}, t) = \mathcal{F}\{f(\mathbf{r}, t)\}. \quad (36)$$

Note that the Fourier series transform equations (33) and (34) in the limit  $a_j \rightarrow 0$  ( $k_{j0} \rightarrow \infty$ ) give the Fourier integral transform equations (36) and (35), where the sum is replaced by the integral.

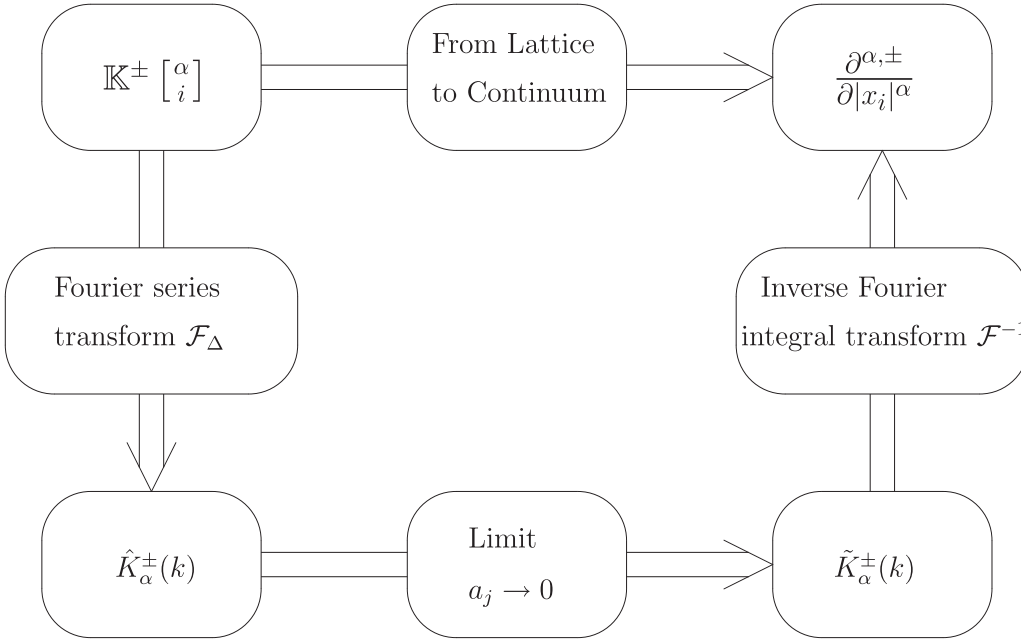
The lattice probability density  $f(\mathbf{n}, t)$  is transformed by the combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  into a probability density  $f(\mathbf{r}, t)$  of continuum,

$$\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta (f(\mathbf{n}, t)) = f(\mathbf{r}, t). \quad (37)$$

The combination of the operations  $\mathcal{F}^{-1}$ , Lim and  $\mathcal{F}_\Delta$  allows us to map the lattice functions and operators into functions and operators for continuum.

### 3.2. Continuum limit of lattice operators

Let us consider transformations of lattice operators (26) and (27) into continuum operators. The transformations  $\mathcal{F}^{-1} \circ \text{Limit} \circ \mathcal{F}_\Delta$  map the lattice operators into the fractional derivatives with respect to coordinates. We represent these transformations in figure 10.



**Figure 10.** Transformation of lattice operators to fractional derivatives.

Using the methods suggested in [60, 61], we can prove the connection between the lattice operators and fractional derivatives of non-integer orders with respect to coordinates.

The lattice operators (26), where  $K_\alpha^\pm(n - m)$  are defined by (17) and (18), are transformed by the combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  into the fractional derivatives of order  $\alpha$  with respect to coordinate  $x_i$  as

$$\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta \left( \mathbb{K}^\pm \begin{bmatrix} \alpha \\ i \end{bmatrix} \right) = a_i^\alpha \frac{\partial^{\alpha, \pm}}{\partial |x_i|^\alpha}, \quad (38)$$

where  $a_i = |\mathbf{a}_i|$  are the primitive lattice vectors,  $\partial^{\alpha, +} / \partial |x_i|^\alpha$  is the Riesz fractional derivative of order  $\alpha > 0$  with respect to  $x_i$  and  $\partial^{\alpha, -} / \partial |x_i|^\alpha$  is the generalized conjugate Riesz derivative of order  $\alpha > 0$ . The order of the partial derivative  $\partial^{\alpha, \pm} / \partial |x_i|^\alpha$  is defined by the order of lattice operator  $\mathbb{K}^\pm \begin{bmatrix} \alpha \\ i \end{bmatrix}$  and it can be integer and non-integer.

Using the independence of the site vectors of lattice site  $\mathbf{n}_1 = (n_1, 0, 0)$ ,  $\mathbf{n}_2 = (0, n_2, 0)$ ,  $\mathbf{n}_3 = (0, 0, n_3)$  and the statement (38), we can prove that the continuum limits for the mixed lattice operators (27) have the form

$$\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta \left( \mathbb{K}^{\pm, \pm} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix} \right) = a_i^{\alpha_i} a_j^{\alpha_j} \frac{\partial^{\alpha_i, \pm}}{\partial |x_i|^{\alpha_i}} \frac{\partial^{\beta_j, \pm}}{\partial |x_j|^{\beta_j}}, \quad (39)$$

As a result, we obtain continuum limits for the lattice fractional derivatives in the form of the fractional derivatives of the Riesz type with respect to coordinates.

The Riesz fractional derivative of the order  $\alpha$  is defined [6, 7] by the equation

$$\frac{\partial^{\alpha, +} f(\mathbf{r})}{\partial |x_i|^\alpha} = \frac{1}{d_1(m, \alpha)} \int_{\mathbb{R}} \frac{1}{|z_i|^{\alpha+1}} (\Delta_i^m f)(\mathbf{z}_i) dz_i, \quad (0 < \alpha < m), \quad (40)$$

where  $(\Delta_i^m f)(\mathbf{z}_i)$  is a finite difference of order  $m$  of a function  $f(\mathbf{r})$  with the vector step

$\mathbf{z}_i = x_i \mathbf{e}_i \in \mathbb{R}^3$  for the point  $\mathbf{r} \in \mathbb{R}^3$ . The non-centered difference is

$$(\Delta_i^m f)(\mathbf{z}_i) = \sum_{k=0}^m (-1)^k \frac{m!}{k!(m-k)!} f(\mathbf{r} - k \mathbf{z}_i), \quad (41)$$

and the centered difference is

$$(\Delta_i^m f)(\mathbf{z}_i) = \sum_{k=0}^m (-1)^k \frac{m!}{k!(m-k)!} f(\mathbf{r} - (m/2 - k) \mathbf{z}_i). \quad (42)$$

The constant  $d_1(m, \alpha)$  is defined by

$$d_1(m, \alpha) = \frac{\pi^{3/2} A_m(\alpha)}{2^\alpha \Gamma(1 + \alpha/2) \Gamma((1 + \alpha)/2) \sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = 2 \sum_{j=0}^m (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha$$

in the case of the non-centered difference (41) and

$$A_m(\alpha) = 2 \sum_{j=0}^{[m/2]} (-1)^{j-1} \frac{m!}{j!(m-j)!} (m/2 - j)^\alpha$$

in the case of the centered difference (42). The constants  $d_1(m, \alpha)$  is different from zero for all  $\alpha > 0$  in the case of an even  $m$  and centered difference  $(\Delta_i^m f)$  (see Theorem 26.1 in [6]). In the case of a non-centered difference the constant  $d_1(m, \alpha)$  vanishes if and only if  $\alpha = 1, 3, 5, \dots, 2[m/2] - 1$ . Note that the integral (40) does not depend on the choice of  $m > \alpha$ . The Fourier transform  $\mathcal{F}$  of the Riesz fractional derivative is given by

$$\mathcal{F} \left( \frac{\partial^{\alpha,+} f(\mathbf{r})}{\partial |x_i|^\alpha} \right) (\mathbf{k}) = |k_i|^\alpha (\mathcal{F} f)(\mathbf{k}). \quad (43)$$

Equation (43) can be considered as a definition of the Riesz fractional derivative of order  $\alpha$ .

Using  $(-i)^{2j} = (-1)^j$ , the Riesz derivatives for even  $\alpha = 2j$  are

$$\frac{\partial^{2j,+} f(\mathbf{r})}{\partial |x_i|^{2j}} = (-1)^j \frac{\partial^{2j} f(\mathbf{r})}{\partial x_i^{2j}}. \quad (44)$$

For  $\alpha = 2$  the Riesz derivative looks like the Laplace operator. The fractional derivatives  $\partial^{\alpha,+} / \partial |x_i|^\alpha$  for even orders  $\alpha$  are local operators. Note that the Riesz derivative  $\partial^{1,+} / \partial |x_i|$  cannot be considered as a derivative of first-order with respect to  $|x_i|$ . For  $\alpha = 1$  it looks like ‘the square root of the Laplacian’. The Riesz derivatives for odd orders  $\alpha = 2j + 1$  are non-local operators that cannot be considered as usual derivatives  $\partial^{2j+1} / \partial x_i^{2j+1}$ .

We also define the new fractional derivatives  $\partial^{\alpha,-} / \partial |x_i|^\alpha$  by the equation

$$\frac{\partial^{\alpha,-}}{\partial |x_i|^\alpha} = \begin{cases} \frac{\partial}{\partial x_i} \mathbf{I}_i^{1-\alpha} & 0 < \alpha < 1, \\ \frac{\partial}{\partial x_i} & \alpha = 1, \\ \frac{\partial}{\partial x_i} \frac{\partial^{\alpha-1,+}}{\partial |x_i|^{\alpha-1}} & \alpha > 1, \end{cases} \quad (45)$$

where  $\partial/\partial x_i$  is the usual derivative of first-order with respect to coordinate  $x_i$  and  $\mathbf{I}_i^{1-\alpha}$  is the Riesz potential of order  $(1 - \alpha)$  (see appendix A) with respect to  $x_i$ ,

$$\mathbf{I}_i^{1-\alpha} f(\mathbf{r}) = \int_{\mathbb{R}^1} R_{1-\alpha}(x_i - z_i) f(\mathbf{r} + (z_i - x_i)\mathbf{e}_i) dz_i, \quad (0 < \alpha < 1), \quad (46)$$

where  $\mathbf{e}_i$  is the basis of the Cartesian coordinate system. For  $0 < \alpha < 1$  the operator  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  is called the conjugate Riesz derivative [9]. Therefore, the operator  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  for all  $\alpha > 0$  can be called the generalized conjugate Riesz derivative.

The Fourier transform  $\mathcal{F}$  of the fractional derivative (45) is given by

$$\mathcal{F}\left(\frac{\partial^{\alpha,-} f(\mathbf{r})}{\partial|x_i|^\alpha}\right)(\mathbf{k}) = i k_i |k_i|^{\alpha-1} (\mathcal{F}f)(\mathbf{k}) = i \operatorname{sgn}(k_i) |k_i|^\alpha (\mathcal{F}f)(\mathbf{k}). \quad (47)$$

Using (44) and (45), we get

$$\frac{\partial^{2j+1,-} f(\mathbf{r})}{\partial|x_i|^{2j+1}} = (-1)^j \frac{\partial^{2j+1} f(\mathbf{r})}{\partial x_i^{2j+1}}. \quad (48)$$

The fractional derivatives  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  for odd orders  $\alpha$  are local operators. Note that the generalized conjugate Riesz derivative  $\partial^{2,-}/\partial|x_i|^2$  cannot be considered as a derivative of second-order with respect to  $|x_i|$ . The derivatives  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  for even orders  $\alpha = 2j$  are non-local operators that cannot be considered as usual derivatives  $\partial^{2j}/\partial x_i^{2j}$ . For  $\alpha = 2$  the generalized conjugate Riesz derivative is not the Laplacian.

Equations (44) and (48) allow us to state that the usual local partial derivatives of integer orders are obtained from the operators  $\partial^{\alpha,\pm}/\partial|x_i|^\alpha$  in the following two cases: (1) for odd values  $\alpha = 2j + 1 > 0$  by  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  only; (2) for even values  $\alpha = 2j > 0$  by  $\partial^{\alpha,+}/\partial|x_i|^\alpha$  only. The operators  $\partial^{\alpha,+}/\partial|x_i|^\alpha$  with integer odd  $\alpha = 2j + 1$  and  $\partial^{\alpha,-}/\partial|x_i|^\alpha$  with integer even  $\alpha = 2j$ , where  $n \in \mathbb{N}$ , are non-local operators. Therefore we consider the lattice equations with the lattice operators  $\mathbb{K}^- \begin{bmatrix} \alpha_i \\ i \end{bmatrix}$  and  $\mathbb{K}^{-,-} \begin{bmatrix} \alpha_i & \beta_j \\ i & j \end{bmatrix}$  as main lattice models to have the usual equations with local spatial derivatives in the case  $\alpha_i = \beta_i = 1$  for all  $i = 1, 2, 3$ .

#### 4. Fractional Fokker–Planck equation for continuum

Using the statements (37), (38) and (39), where  $K_\alpha^-(n - m)$  are defined by (18), the lattice Fokker–Planck equation (29) are transformed by the combination  $\mathcal{F}^{-1} \circ \operatorname{Lim} \circ \mathcal{F}_\Delta$  into the fractional Fokker–Planck equation with derivatives of non-integer orders with respect to space coordinates. This space-fractional Fokker–Planck equation for the probability density  $f(\mathbf{r}, t)$  has the form

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = - \sum_{i=1}^3 D_i(\alpha) \frac{\partial^{\alpha_i,-}}{\partial|x_i|^{\alpha_i}} f(\mathbf{r}, t) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 D_{ij}(\alpha, \beta) \frac{\partial^{\alpha_i,-}}{\partial|x_i|^{\alpha_i}} \frac{\partial^{\beta_j,-}}{\partial|x_j|^{\beta_j}} f(\mathbf{r}, t), \quad (49)$$

where  $D_i(\alpha)$  is the drift vector and  $D_{ij}(\alpha, \beta)$  is the diffusion tensor for the continuum that are defined by the lattice coupling constants  $g_i$  and  $g_{ij}$  by the relations

$$D_i(\alpha) = a_i^{\alpha_i} g_i, \quad D_{ij}(\alpha, \beta) = 2 a_i^{\alpha_i} a_j^{\beta_j} g_{ij}. \quad (50)$$



Using the definition (45), the fractional Fokker–Planck equation (49) can be represented as the well-known continuity equation

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = - \sum_{i=1}^3 \frac{\partial J_i(\mathbf{r}, t)}{\partial x_i}, \quad (51)$$

where  $J_i$  is the probability flow

$$J_i(\mathbf{r}, t) = \begin{cases} D_i(\alpha) \mathbf{I}_i^{1-\alpha_i} f(\mathbf{r}, t) - \frac{1}{2} \sum_{j=1}^3 D_{ij}(\alpha, \beta) \mathbf{I}_i^{1-\alpha_i} \frac{\partial^{\beta_j, -}}{\partial |x_j|^{\beta_j}} f(\mathbf{r}, t) & 0 < \alpha_i < 1, \\ D_i(\alpha) f(\mathbf{r}, t) - \frac{1}{2} \sum_{j=1}^3 D_{ij}(\alpha, \beta) \frac{\partial^{\beta_j, -}}{\partial |x_j|^{\beta_j}} f(\mathbf{r}, t) & \alpha_i = 1, \\ D_i(\alpha) \frac{\partial^{\alpha_i-1, +} f(\mathbf{r}, t)}{\partial |x_j|^{\alpha_i-1}} - \frac{1}{2} \sum_{j=1}^3 D_{ij}(\alpha, \beta) \frac{\partial^{\alpha_i-1, +}}{\partial |x_j|^{\alpha_i-1}} \frac{\partial^{\beta_j, -}}{\partial |x_j|^{\beta_j}} f(\mathbf{r}, t) & \alpha_i > 1, \end{cases} \quad (52)$$

Note that coincidence of orders of fractional derivatives in the first and second terms allows us to represent the fractional Fokker–Planck equation (49) in the form of the space-fractional continuity equation. The fractional Fokker–Planck equation (49) can be represented as the fractional continuity equation

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = - \sum_{i=1}^3 \frac{\partial^{\alpha_i, -} J_i^{(\text{frac})}(\mathbf{r}, t)}{\partial |x_i|^{\alpha_i}}, \quad (53)$$

where  $J_i^{(\text{frac})}$  is the probability flow

$$J_i^{(\text{frac})}(\mathbf{r}, t) = D_i(\alpha) f(\mathbf{r}, t) - \frac{1}{2} \sum_{j=1}^3 D_{ij}(\alpha, \beta) \frac{\partial^{\beta_j, -}}{\partial |x_j|^{\beta_j}} f(\mathbf{r}, t). \quad (54)$$

If  $\alpha_i = 1$ , the continuity equation (53) has the standard form.

For the 1D case with  $D_i(\alpha) = 0$  and  $f(\mathbf{r}, t) = f(x, t)$ , equation (49) can be represented in the form

$$\frac{\partial f(x, t)}{\partial t} = K(\mu) \nabla^\mu f(x, t), \quad (55)$$

where  $K(\mu)$  is the generalized diffusion constant,

$$K(\mu) = \frac{1}{2} D_{11}(\alpha, \beta), \quad (56)$$

and  $\nabla^\mu$  is the fractional derivative of order  $\mu$ ,

$$\nabla^\mu = \frac{\partial^{\alpha_1, -}}{\partial |x|^{\alpha_1}} \frac{\partial^{\beta_1, -}}{\partial |x|^{\beta_1}}, \quad \mu = \alpha_1 + \beta_1. \quad (57)$$

Note that for sufficiently good functions, the operator (57) can be represented in the form  $\nabla^\mu = \partial^{\mu, +} / \partial |x|^\mu$ , but it cannot be done in the general case. Equation (55) describes the fractional diffusion processes with the Poissonian waiting time and the Lévy

distribution for the jump length (see section 3.5 of [4]). In [4] the space-fractional diffusion equation (55) contains the Weyl fractional derivative  $\nabla^\mu$  of order  $\mu$ , which is equivalent to the Riesz operator  $\partial^{\mu,+}/\partial|x|^\mu$  in 1D. The solution of equation (55) can be obtained analytically by using the Fox function  $H_{2,2}^{1,1}$  (for details see section 3.5 in [4] and [73]). The exact calculation of fractional moments [4] gives

$$\langle |x(t)|^\delta \rangle = \frac{2(K(\mu))^{\delta/\mu} \Gamma(-\delta/\mu) \Gamma(1+\delta)}{\mu \Gamma(-\delta/2) \Gamma(1+\delta/2)} t^{\delta/\mu}, \quad (58)$$

where  $0 < \delta < \mu \leq 2$ .

The time-fractional lattice Fokker–Planck equation (30) are transformed by the combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  into the space-time fractional Fokker–Planck equation

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = -\sum_{i=1}^3 D_i(\alpha) \frac{\partial^{\alpha_i,-}}{\partial|x_i|^{\alpha_i}} f(\mathbf{r}, t) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 D_{ij}(\alpha, \beta, \gamma) \frac{\partial^{\alpha_i,-}}{\partial|x_i|^{\alpha_i}} \frac{\partial^{\beta_j,-}}{\partial|x_j|^{\beta_j}} {}_0^{\text{RL}} D_t^{1-\gamma} f(\mathbf{r}, t), \quad (59)$$

where  ${}_0^{\text{RL}} D_t^{1-\gamma}$  is the Riemann–Liouville fractional derivative with respect to time that describes the power law memory. For the 1D case with  $D_i(\alpha) = 0$  and  $f(\mathbf{r}, t) = f(x, t)$ , equation (59) can be represented in the form

$$\frac{\partial f(x, t)}{\partial t} = {}_0^{\text{RL}} D_t^{1-\gamma} K(\mu, \gamma) \nabla^\mu f(x, t), \quad (60)$$

where

$$K(\mu, \gamma) = \frac{1}{2} D_{11}(\alpha, \beta, \gamma), \quad (61)$$

and the fractional derivative  $\nabla^\mu$  is defined by (57). Equation (60) describes a random walk characterized by waiting time and jump length (see section 3.6 in [4]). The competition between long rests (waiting events) and long jumps (motion events) in the Lévy walks processes is given [74] as

$$\langle x^2(t) \rangle \sim \begin{cases} t^{2+\gamma-\mu} & 0 < \gamma < 1, \\ t^{3-\mu} & \gamma > 1, \end{cases} \quad (62)$$

where  $1 < \mu < 2$ . It should be noted that the continuum form of the Lévy flights is described by the drift term with the first-order derivative ( $\alpha_i = 1$ ) as proposed in [39] and derived from the continuous time random walk in [36]. The solutions of Cauchy problems for the space–time fractional diffusion equation with the Riesz–Feller fractional derivatives are described in [14].

The time-fractional lattice Fokker–Planck equation (31) is transformed by the combination  $\mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta$  into the space–time fractional continuum Fokker–Planck equation

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = {}_0^{\text{RL}} D_t^{1-\gamma} \mathcal{L}_{\text{CFP}}^{\alpha, \beta} f(\mathbf{r}, t), \quad (63)$$

where

$$\mathcal{L}_{\text{CFP}}^{\alpha, \beta} = \mathcal{F}^{-1} \circ \text{Lim} \circ \mathcal{F}_\Delta \left( \mathcal{L}_{\text{LFP}}^{\alpha, \beta} \right) \quad (64)$$

is the continuum Fokker–Planck operator of the form

$$\mathcal{L}_{\text{CFP}}^{\alpha,\beta} = - \sum_{i=1}^3 D_i(\alpha) \frac{\partial^{\alpha_i,-}}{\partial |x_i|^{\alpha_i}} + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 D_{ij}(\alpha, \beta, \gamma) \frac{\partial^{\alpha_i,-}}{\partial |x_i|^{\alpha_i}} \frac{\partial^{\beta_j,-}}{\partial |x_j|^{\beta_j}}. \quad (65)$$

For  $\alpha_i = \beta_i = 1$ , equation (63) takes the form of the time-fractional Fokker–Planck equation that is suggested in [36–38].

If  $\alpha_i = \beta_i = \gamma = 1$  for all  $i = 1, 2, 3$ , then equations (49), (59) and (63) for the probability density  $f(\mathbf{r}, t)$  give the well-known Fokker–Planck equation in the form

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = - \sum_{i=1}^3 D_i \frac{\partial}{\partial x_i} f(\mathbf{r}, t) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{r}, t), \quad (66)$$

where  $D_i = D_i(1)$  is the usual drift vector and  $D_{ij} = D_{ij}(1, 1)$  is the usual diffusion tensor.

## 5. Conclusion

In this paper 3D lattice models with long-range drift and diffusion of particles are suggested. These proposed lattice models can be considered as a possible microscopic basis to describe the anomalous diffusion in continuum. The suggested type of lattice long-range drift and diffusion can be considered for non-integer (fractional) values of the parameters  $\alpha_i, \beta_i, \gamma$ . This allows us to have lattice equations for the fractional non-local diffusion and transport processes. The proposed forms of the drift and diffusion of particles in lattice allow us to obtain the continuum equations with the generalized conjugate Riesz derivatives of fractional orders by using the approaches and methods proposed in [60, 61]. The suggested 3D models with long-range lattice drift and diffusion of the types (17) and (18) can be considered as lattice analogs of the fractional diffusion and drift in non-local continuum. Different fractional generalizations of the Fokker–Planck equation for continuum can be obtained by using the suggested lattice approach. We expect that the proposed 3D lattice Fokker–Planck equations can play an important role in the description of non-local processes in microscale and nanoscale because at these scales the interatomic interactions can be prevalent in determining the properties of media.

Let us note some possible generalizations of the proposed lattice models. We assume that the suggested lattice Fokker–Planck equations can be generalized in the form of a lattice Kramers–Moyal equation for the case of the high-order terms by using the different fractional-order derivatives. The suggested lattice models can be generalized for the lattices with dislocation and disclinations that are connected with non-commutativity of the lattice operators (26). In this paper, we consider the primitive orthorhombic Bravais lattice for simplification. It is interesting to generalize the suggested consideration for other types of Bravais lattices such as triclinic, monoclinic, rhombohedral and hexagonal. The suggested models of unbounded lattices can be generalized for the bounded physical lattices. We also assume that the proposed lattice approach to the fractional diffusion can be generalized for lattices, which are characterized by fractal spatial dispersion [75–77] and correspondent models for fractal media [78, 79] (see also [80–82]).

We also can note some remaining challenges and questions in the suggested approach to fractional diffusion. The function  $f(\mathbf{n}, t)$  has the meaning of probability density on the lattice and it should be positively defined. It is known that this condition for the continuum case leads to restriction for the parameters  $\alpha, \beta, \gamma, \mu$ . For example, we have the condition  $0 < \mu \leq 2$  for Lévy processes on continuum. A rigorous consideration of positiveness of  $f(\mathbf{n}, t)$  for a set of these parameter does not exist at the present time. Exact mathematical conditions of existence of solutions for the lattice Fokker–Planck equation can be important to describe anomalous long-range particle drift and diffusion on 3D physical lattices.

## Appendix A. Riesz fractional integral

The Riesz fractional integration is defined by

$$\mathbf{I}_x^\alpha f(x) = \mathcal{F}^{-1}\left(|k|^{-\alpha}(\mathcal{F}f)(k)\right). \quad (\text{A.1})$$

The fractional integration (A.1) can be realized in the form of the Riesz potential defined as the Fourier’s convolution of the form

$$\mathbf{I}_x^\alpha f(x) = \int_{\mathbb{R}^n} R_\alpha(x-z)f(z)dz, \quad (\alpha > 0), \quad (\text{A.2})$$

where the function  $R_\alpha(x)$  is the Riesz kernel. If  $\alpha > 0$ , the function  $R_\alpha(x)$  is defined by

$$R_\alpha(x) = \begin{cases} \gamma_n^{-1}(\alpha)|x|^{\alpha-n} & \alpha \neq n + 2k, \\ -\gamma_n^{-1}(\alpha)|x|^{\alpha-n} \ln|x| & \alpha = n + 2k, \end{cases} \quad (\text{A.3})$$

where  $n \in \mathbb{N}$  and the constant  $\gamma_n(\alpha)$  has the form

$$\gamma_n(\alpha) = \begin{cases} 2^\alpha \pi^{n/2} \Gamma(\alpha/2) / \Gamma\left(\frac{n-\alpha}{2}\right) & \alpha \neq n + 2k, \\ (-1)^{(n-\alpha)/2} 2^{\alpha-1} \pi^{n/2} \Gamma(\alpha/2) \Gamma(1 + [\alpha-n]/2) & \alpha = n + 2k. \end{cases} \quad (\text{A.4})$$

The Fourier transform of the Riesz fractional integration is given by

$$\mathcal{F}\left(\mathbf{I}_x^\alpha f(x)\right) = |k|^{-\alpha}(\mathcal{F}f)(k). \quad (\text{A.5})$$

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