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Three-dimensional lattice models with long-range interactions of Grünwald–Letnikov type for fractional generalization of gradient elasticity

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Abstract Models of three-dimensional lattices with long-range interactions of Grünwald–Letnikov type for fractional gradient elasticity of non-local continuum are suggested. The lattice long-range interactions are described by fractional-order difference operators. Continuous limit of suggested three-dimensional lattice equations gives continuum differential equations with the Grünwald–Letnikov derivatives of non-integer orders. The proposed lattice models give a new microstructural basis for elasticity of materials with power-law type of non-locality. Moreover these lattice models allow us to have a unified microscopic description for fractional and usual (non-fractional) gradient elasticity continuum.

Keywords Gradient elasticity · Nonlocal continuum · Fractional derivatives · Lattice model · Long-range interactions · Fractional-order difference

1 Introduction

Elastic deformations in materials are described by a microscopic approach based on the lattice mechanics [1–4], and a macroscopic approach based on the continuum mechanics [5–7]. Continuum mechanics can be considered as a continuous limit of lattice dynamics, where the length-scales of an continuum element are much larger than the distances between the lattice particles. Continuum models for elasticity of materials with microstructure have been suggested by Mindlin [8]. In these models, two types of physical quantities are used to characterize properties of materials at the microscale and macroscale. The kinetic energy and the deformation energy densities for microstructured materials are considered for both scales. The continuum models of these materials differ by the relations between the microscopic and macroscopic quantities. The most popular continuum models of gradient elasticity suggested by Mindlin [8–10] are the first- and the second-gradient models. In the first model, the microscopic deformation gradient is assumed to be the first gradient of the macroscopic strain. In the second model, the microscopic deformation gradient is defined as the second gradient of the macroscopic displacement. Despite these differences between these gradient models, the equations for displacements are identical [8]. For simplification of gradient elasticity models, the length scales of the Mindlin models can be taken equal to each other [11–13].

In general, the models of gradient elasticity cannot be considered as a real nonlocal models since the equations of these models include a finite number of integer-order derivatives with respect to coordinates. An application of general infinite series with integer-

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order derivatives to describe a weak nonlocality is a difficult problem, This problem can be solved for power-law type of weak nonlocality by using fractional-order derivatives. It is important to note that the use of the derivatives of non-integer orders is actually equivalent to using an infinite number of derivatives of integer orders, which can be arbitrarily large values (for example, see Lemma 15.3 in [14, 15]).

The theory of derivatives and integrals of non-integer orders [14–19] has a wide application in different areas of mechanics and physics (for example see [20–28]). The fractional-order derivatives allow us to formulate fractional generalizations of elasticity models of continuum with weak nonlocality of power-law type. First time the fractional derivatives with respect to space coordinates have been applied in the elasticity theory of nonlocal continuum by Gubenko [29, 30] in 1957. Recently, the fractional-order derivatives is used to describe continuum with power-law type of non-locality (for example, see [26, 31–43]). Fractional models of integral non-local elasticity are considered in papers [44–50], where the microscopic models of fractional integral elasticity are also described. The fractional-order integration is used to describe continua with strong nonlocality. In this paper, we will consider lattice models of continua with weak nonlocality only.

Fractional calculus is a powerful tool to describe processes in continuously distributed media with nonlocal properties of power-law type. As it was shown in [26, 51, 52], the continuum equations with fractional derivatives are directly connected to lattice models with long-range interactions. As it was shown in [51, 52] the differential equations with fractional derivatives of non-integer orders can be derived from equation for lattice particles with long-range interactions in the continuous limit, where the distance between the lattice particles tends to zero. A direct connection between the lattice with long-range interaction and nonlocal continuum has been proved by using the special transform operation [51, 52] (see also [53–56]). The one-dimensional lattice models for fractional gradient elasticity and the correspondent continuum equations have been suggested in [40–43, 57, 58]. All proposed lattice models of fractional gradient elasticity describe one-dimensional lattices only. In this paper, we suggest three-dimensional models of lattices with long-range interactions and continua with power-law nonlocality.

Fractional-order differences and the correspondent derivatives have been first proposed by Grünwald [59] and by Letnikov [60]. At the present time these generalized differences and derivatives are called the Grünwald–Letnikov fractional differences and derivatives [14–16]. One-dimensional lattice models with long-range interactions of the Grünwald–Letnikov type and the correspondent fractional differential and integral continuum equations have been suggested in [42]. The suggested form of long-range interaction is based on the form of the left-sided and right-sided Grünwald–Letnikov fractional differences. In this paper, we generalize this approach for three-dimensional case. We suggest three-dimensional lattice models for fractional gradient elasticity of the Grünwald–Letnikov type and correspondent model of fractional nonlocal continuum. To give this three-dimensional generalizations, we use a lattice fractional vectors calculus based on the fractional-order differences of Grünwald–Letnikov type that has been suggested in Section 5.2 of [55]. In this paper, we apply this new mathematical tool to describe physical lattices with long-range interactions and correspondent fractional elasticity equations for nonlocal continuum with power-law nonlocality. A general form of lattice model with long-range interaction that gives a continuum equation with fractional-order derivatives in continuum limit is proposed. We consider a lattice model with long-range interactions for the Mindlin continuum model of first gradient elasticity for isotropic materials and its generalizations for fractional order nonlocalities.

2 Long-range interactions of lattice particles

Let us consider a three-dimensional unbounded and bounded lattices. Physical lattices are characterized by space periodicity. For unbounded lattices we can use three non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_2$, that are the shortest vectors by which a lattice can be displaced and be brought back into itself. Sites of this lattice can be characterized by the number vector $\mathbf{n} = (n_1, n_2, n_3)$, where n_i $(j = 1, 2, 3)$ are integer. For simplification, we consider a lattice with mutually perpendicular primitive lattice vectors $\mathbf{a}_j$ $(j = 1, 2, 3)$. This means that we use a primitive orthorhombic Bravais lattice. We choose directions of the axes of the Cartesian coordinate system coincide with the vector $\mathbf{a}_j$. In this case $\mathbf{a}_j = a_j \mathbf{e}_j$, where $a_i = |\mathbf{a}_j| > 0$ and $\mathbf{e}_j$ are the basis vectors of the Cartesian coordinate system. Then the vector $\mathbf{n}$ can be represented as $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$.

Choosing a coordinate origin at one of the lattice sites, then the positions of all other site with $\mathbf{n} = (n_1, n_2, n_3)$ is described by the vector $\mathbf{r}(\mathbf{n}) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$. The lattice sites are numbered by $\mathbf{n}$, so that the vector $\mathbf{n}$ can be considered as a number vector of the corresponding particle. We assume that the equilibrium positions of particles coincide with the lattice sites $\mathbf{r}(\mathbf{n})$. Coordinates $\mathbf{r}(\mathbf{n})$ of lattice sites differs from the coordinates of the corresponding particles, when particles are displaced relative to their equilibrium positions. To define the coordinates of a particle, we define displacement of a $\mathbf{n}$-particle with from its equilibrium position by the vector field $\mathbf{u}(\mathbf{n}, t) = \sum_{i=1}^{3} u_i(\mathbf{n}, t)\, \mathbf{e}_i$, where $\mathbf{e}_i$ is the basis of the Cartesian coordinate system, and $u_i(\mathbf{n}, t) = u_i(n_1, n_2, n_3, t)$ are components of the displacement vector for lattice particle that is defined by the vector $\mathbf{n} = (n_1, n_2, n_3)$.

In this paper, a new class of possible lattice models for nonlocal theory of elasticity is suggested. This class of lattice models can serve as a basis to describe elastic materials with power-law nonlocality. We consider a possibility of new type of materials with spatial dispersion by taking into account non-Debye screening of electromagnetic inter-atomic interactions that have long-range power-law type [61].

To simplify our consideration of these models, we use the some following assumptions for microscopic (lattice) structure. It is obvious that to describe an actual behavior of real materials we need to take into account that actual behavior depends on a real microscopic (lattice) structure of material.

In the paper, we consider primitive orthorhombic Bravais lattice for simplification. The suggested models can be generalized such that to consider other types of Bravais lattice. In this case, the primitive lattice vectors $\mathbf{a}_j$ $(j = 1, 2, 3)$ are not mutually perpendicular in general.

In the suggested lattice model, we consider pair-forces between lattice particles. We assume that lattice n-particle interacts by pair manner with all lattice m-particles that reflects the long-range nature of the interaction in the suggested non-local model of material. In the three-dimensional lattice models, these interactions can be described by using the a lattice fractional vectors calculus based that has been suggested in [55].

In the paper, we consider long-range interactions that have the same order for all space directions. This means that we use the difference operators which are suggested in the paper [55], with orders $\alpha_j = \alpha$ for all $j = 1, 2, 3$ for simplification. In general, it is possible to consider generalized model, where the long-range interactions are characterized by different orders α_j in different directions $\mathbf{e}_j = \mathbf{a}_j / |\mathbf{a}_j|$. In these models we should use the difference operators of orders α_j, where $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_3$.

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2.1 Long-range interaction of power-law type

Let us give a definition of the Long-range interaction of power-law type (for details see [51, 52] and Section 8 of [26]). An interaction of lattice particles is called the interaction of power-law type if the kernels $K(n-m)$ of this interaction satisfy the conditions

$$\lim_{k \to 0} \frac{\hat{K}_\alpha(k) - \hat{K}_\alpha(0)}{|k|^\alpha} = A_\alpha, \quad \alpha > 0, \quad 0 < |A_\alpha| < \infty, \tag{1}$$

where

$$\hat{K}_\alpha(k\Delta x) = \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} K(n) = 2 \sum_{n=1}^{\infty} K(n) \cos(kn\Delta x). \tag{2}$$

As a simple example of the power-law type of particle interaction, we can consider

$$K(n-m) = \frac{1}{|n-m|^\alpha}. \tag{3}$$

The other examples of power-law type interaction are considered in [26, 43, 51, 52, 55].

The power-law type of interactions is suggested in the papers [51, 52] (see also Section 8 of [26]). This type of interactions is characterized by the power-law asymptotic behavior of spatial dispersions in lattice.

The power-law type interaction can appear in crystal lattices and polymer materials. We assume that the power-law spatial dispersion in the lattice can be caused by non-Debye screening of electromagnetic inter-atomic interactions. Some aspects of the theory of this screening are described in the paper [61], where fractional-order power-law spatial dispersion in electrodynamics of continuum is discussed. Some elasticity models of materials with power-law spatial dispersion are discussed in [40, 41].

2.2 Long-range interaction of Grünwald–Letnikov type

The long-range interaction of Grünwald–Letnikov type is a special form of power-law type of interactions. The interaction of Grünwald–Letnikov type is a long-range interaction that can be represented as a finite or infinite countable sum of the fractional-order differences. The difference of a fractional order $\alpha > 0$ is defined by the infinite series (see Section 20 in [14, 15])

$$\nabla^{\alpha}_{h} f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} f(x-kh). \tag{4}$$

The space dispersion law of the suggested lattice model (lattice with Grünwald–Letnikov interactions) is defined by the following Fourier transform of the fractional difference that is given by

$$\mathcal{F}\{\nabla^{\alpha}_{h} f(x)\}(k) = (1-\exp\{ikh\})^{\alpha} \mathcal{F}\{f(x)\}(k)$$

for any function $f(x) \in L_1(\mathbb{R})$ (see Property in [16, p. 121]).

An important property of the Grünwald–Letnikov type of the power-law particle interactions is a the semigroup property that based on the corresponding property of fractional-order differences. For the fractional difference, the semigroup property

$$\nabla^{\alpha}_{h} \nabla^{\beta}_{h} f(x) = \nabla^{\alpha+\beta}_{h} f(x), \quad (\alpha > 0, \quad \beta > 0) \tag{5}$$

is valid for any bounded function $f(x)$ (see Property 2.29 in [16]). The semigroup property of Grünwald–Letnikov type of particle interaction allows us to use the superposition principle for this type of non-local interactions

$$ {}^{GL}\mathbb{K}^{\pm}_{L} \begin{bmatrix} \alpha_j \\ j \end{bmatrix} {}^{GL}\mathbb{K}^{\pm}_{L} \begin{bmatrix} \beta_j \\ j \end{bmatrix} = {}^{GL}\mathbb{K}^{\pm}_{L} \begin{bmatrix} \alpha_j+\beta_j \\ j \end{bmatrix}, \quad (\alpha_j > 0, \quad \beta_j > 0). \tag{6}$$

It should be noted that the superposition principle cannot be used for non-local interactions in general.

One-dimensional lattice models with long-range interactions of the Grünwald–Letnikov type and the correspondent fractional differential and integral continuum equations have been suggested in [42]. In this paper, we generalize this approach for three-dimensional case. We suggest three-dimensional lattice models for fractional gradient elasticity of the Grünwald–Letnikov type and correspondent model of fractional nonlocal continuum. To give this three-dimensional generalizations, we use a lattice fractional vectors calculus based on the fractional-order differences of Grünwald–Letnikov type that has been suggested in Section 5.2 of [55]. In this paper, we apply this new mathematical tool to describe physical lattices with long-range interactions and correspondent fractional elasticity equations for nonlocal continuum with power-law nonlocality.

2.3 Fractional-order difference operators of the Grünwald–Letnikov type

To describe dynamics in the lattice models with long-range interactions, we define fractional-order difference operators of the Grünwald–Letnikov type in the direction $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$ of the lattice. Fractional-order difference operators of the Grünwald–Letnikov type for unbounded lattice are the operators ${}^{GL}\mathbb{K}^{\pm}_{L} \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ that act on the function $u_i(\mathbf{m}, t)$ as

$$ {}^{GL}\mathbb{K}^{\pm}_{L} \begin{bmatrix} \alpha_j \\ j \end{bmatrix} u_i(\mathbf{m}, t) = \sum_{m_j=-\infty}^{+\infty} {}^{GL}K^{\pm}_{\alpha_j}(n_j - m_j) u_i(\mathbf{m}, t) \quad (\alpha_j > 0, \quad i,j = 1,2,3), \tag{7}$$

where the kernels ${}^{GL}K^{\pm}_{\alpha}(n)$ are defined by the equations

$$ {}^{GL}K^{\pm}_{\alpha_j}(n) = \frac{(-1)^n \Gamma(1+\alpha_j)(H[n] \pm H[-n])}{2\Gamma(|n|+1)\Gamma(1+\alpha_j-|n|)}, \quad (\alpha_j > 0), \tag{8}$$

and $\Gamma(z)$ is the gamma function, $H[n]$ is the discrete variable Heaviside step function that is defined as $H[n] = 1$ for $n \geq 0$, and $H[n] = 0$ for $n < 0$, where $n \in \mathbb{Z}$.

The parameter α_j is called the order of the operator. It should be notes that the definition of $H[0]=1$ for discrete variable Heaviside function is significant for us, since it allows us to write the kernels ${}^{GL}K^{+}_{\alpha}(n)$ in the simple form without allocating repeated zero terms. It is easy to see that the kernels ${}^{GL}K^{+}_{\alpha}(n)$ and ${}^{GL}K^{-}_{\alpha}(n)$ are even and odd functions such that ${}^{GL}K^{\pm}_{\alpha}(-n)=\pm {}^{GL}K^{\pm}_{\alpha}(n)$. The form of the lattice operators (7) can be represented in the form

$$ {}^{GL}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix} f(\mathbf{m},t)=\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\Gamma(1+\alpha_j)}{2\Gamma(m_j+1)\Gamma(1+\alpha_j-m_j)} \times \left(f(\mathbf{n}-m_j\mathbf{e}_j,t)\pm f(\mathbf{n}+m_j\mathbf{e}_j,t)\right). \quad (9)$$

We should note that in Eq. (9) the summation is realized over non-negative values m_j, in contrast to the sum over all integer values in Eq. (7).

It should be noted that one-dimensional lattice models with the long-range interaction of the form ${}^{GL}K^{+}_{\alpha}(n)$ and correspondent fractional nonlocal continuum models have been suggested in [42]. The lattice operators (7) recently have been proposed in [55].

For bounded physical lattice models the fractional-order difference operators of the Grünwald–Letnikov type for bounded lattice with $m^1_j \le m_j \le m^2_j$ are the operators ${}^{GL}_{B}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ that act on the function $u_i(\mathbf{m},t)$ can be defined in the form

$$ {}^{GL}_{B}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix} u_i(\mathbf{m},t)=\sum_{m_j=m^1_j}^{m^2_j} {}^{GL}K^{\pm}_{\alpha_j}(n_j-m_j)u_i(\mathbf{m},t) \quad (i,j=1,2,3), \quad (10)$$

where the kernels ${}^{GL}K^{\pm}_{\alpha_j}(n)$ are defined by Eq. (8). The suggested forms of fractional difference operators for bounded physical lattice models are based on the Grünwald–Letnikov fractional differences on finite intervals (see Section 20.4 in [14, 15]). For the finite interval $[x^1_j, x^2_j]$, the integer values m^1_j, m^2_j and m_j are defined by the equations

$$ m^1_j=\left[\frac{x^1_j}{a_j}\right], \quad m^2_j=\left[\frac{x^2_j}{a_j}\right], \quad m_j=\left[\frac{x_j}{a_j}\right], \quad (11)$$

where the brackets [ ] mean the floor function that maps a real number to the largest previous integer number.

To describe isotropic physical lattices we should use the difference operators ${}^{GL}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ and ${}^{GL}_{B}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ with orders $\alpha_j=\alpha$ for all $j=1,2,3$.

Using the semigroup property for fractional differences of non-negative orders (see Property 2.29 in [16]), it is easy to prove that the semi-group property holds for the fractional difference operators (7) in the form

$$ {}^{GL}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}, {}^{GL}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\beta_j\\ j\end{bmatrix} = {}^{GL}\mathbb{K}^{\pm}_{L}\begin{bmatrix}\alpha_j+\beta_j\\ j\end{bmatrix}, \quad (\alpha_j>0, \quad \beta_j>0). \quad (12)$$

Using this Eq. (12), it is easily to prove the commutativity and the associativity of the fractional-order difference operators (7) of the Grünwald–Letnikov type. The commutativity and associativity of the fractional operators (7) for different directions are obvious.

For simplification, we use the combination of the repeated fractional-order difference operators

$$ {}^{GL}\mathbb{K}^{\pm,\pm}\begin{bmatrix}\alpha_i & \beta_j\\ i & j\end{bmatrix} = {}^{GL}\mathbb{K}^{\pm}\begin{bmatrix}\alpha_i\\ i\end{bmatrix} {}^{GL}\mathbb{K}^{\pm}\begin{bmatrix}\beta_j\\ j\end{bmatrix}, \quad (13)$$

where i,j take values from the set $\{1;2;3\}$. The action of the operator (13) on the lattice fields $u_k(\mathbf{m},t)$ is

$$ {}^{GL}\mathbb{K}^{\pm,\pm}\begin{bmatrix}\alpha_i & \beta_j\\ i & j\end{bmatrix} u_k(\mathbf{m},t)=\sum_{m_i=-\infty}^{+\infty}\sum_{m_j=-\infty}^{+\infty} K^{\pm}_{\alpha_i}(n_i-m_i)K^{\pm}_{\beta_j}(n_j-m_j)u_k(\mathbf{m},t) \quad (14)$$

where $i,j,k\in\{1,2,3\}$. Analogously, we can define the repeated fractional-order difference operators ${}^{GL}\mathbb{K}^{\pm,\pm,\pm}\begin{bmatrix}\alpha_i & \beta_j & \gamma_l\\ i & j & l\end{bmatrix}$, ${}^{GL}\mathbb{K}^{\pm,\pm,\mp}\begin{bmatrix}\alpha_i & \beta_j & \gamma_l\\ i & j & l\end{bmatrix}$, and other.

3 Three-dimensional lattice models for fractional gradient elasticity

3.1 Three-dimensional lattice models for fractional generalization of Aifantis gradient elasticity

A simplest model of gradient elasticity has been suggested in [9, 62], where all length scales are taken equal to each other. The gradient terms are used to take into account so-called weak nonlocality. In order to describe a weak nonlocality of power-law type, we should use terms with the fractional gradients and fractional Laplace operators [55]. The one-dimensional lattice models for fractional elasticity and the correspondent continuum equations have been suggested in [40–43, 57, 58]. To generalize the one-dimensional lattice models of fractional elasticity for three-dimensional lattices we can apply the fractional-order difference of the Grünwald–Letnikov type. For simplification we will consider a primitive orthorhombic Bravais lattice with long-range interactions, where $\mathbf{a}_i = a_i\,\mathbf{e}_i$ and $\mathbf{e}_i$ is the basis of the Cartesian coordinate system.

For microstructural models of the three-dimensional fractional gradient elasticity of anisotropic continua, we use the lattice equations

$$M\frac{\partial^2 u_i(\mathbf{n},t)}{\partial t^2} = \sum_{j,l=1}^{3} A^L_{ijkl}\, {}^{GL}\mathbb{K}^{-,-}_L \begin{bmatrix} 1 & 1 \\ j & l \end{bmatrix} u_k(\mathbf{m},t) + \sum_{j,m,l=1}^{3} B^L_{ijkl}\, {}^{GL}\mathbb{K}^{-,+,-}_L \begin{bmatrix} 1 & \alpha & 1 \\ j & m & l \end{bmatrix} u_k(\mathbf{m},t) + F_i(\mathbf{n},t), \tag{15}$$

where $u_k(\mathbf{m},t) = u_k(m_1,m_2,m_3,t)$ is the displacement for the lattice, and A^L_{ijkl} and B^L_{ijkl} are the lattice coupling constants. We assume that the fourth-order tensors A^L_{ijkl} and B^L_{ijkl} have the same type of symmetry as the fourth-order elastic stiffness tensor C_{ijkl}:

$$A^L_{ijkl} = A^L_{jikl} = A^L_{ijlk} = A^L_{klij}, \quad B^L_{ijkl} = B^L_{jikl} = B^L_{ijlk} = B^L_{klij}. \tag{16}$$

For primitive orthorhombic Bravais lattice [7], we have nine coupling constants A^L_{ijkl} and nine gradient coupling constants B^L_{ijkl}.

For bounded lattice we can use the fractional difference operators (10), and the equations

$$M\frac{\partial^2 u_i(\mathbf{n},t)}{\partial t^2} = \sum_{j,l=1}^{3} A^L_{ijkl}\, {}^{GL}_{B}\mathbb{K}^{-,-}_L \begin{bmatrix} 1 & 1 \\ j & l \end{bmatrix} u_k(\mathbf{m},t) + \sum_{j,m,l=1}^{3} B^L_{ijkl}\, {}^{GL}_{B}\mathbb{K}^{-,+,-}_L \begin{bmatrix} 1 & \alpha & 1 \\ j & m & l \end{bmatrix} u_k(\mathbf{m},t) + F_i(\mathbf{n},t). \tag{17}$$

To describe anisotropic long-range interaction in lattices, we should use the difference operators ${}^{GL}\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ and ${}^{GL}_{B}\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ with with unequal orders α_j at least for one $j = 1, 2, 3$.

3.2 Three-dimensional lattice models for fractional generalization of Mindlin gradient elasticity

Mindlin [8] has been suggested a theory of elasticity with microstructure, where two different type of quantities are used for for the micro and macro scales. In the Mindlin theory of elasticity [8], the kinetic and the deformation energy densities are written in terms of the micro and macro scale quantities. Gradient elasticity models are special types of the elasticity theories with microstructure, in which the deformation energy density is represented in terms of the macroscopic displacements only. The Mindlin gradient elasticity models differ in the assumed relation between the microscopic deformation and the macroscopic displacement. It is important to note that despite the theoretical differences between these models, the equations for displacements of these models are identical [8]. In order to derive a fractional generalization of the Mindlin gradient models [8–10], and a correspondent there-dimensional lattice model, we assume that lattice is characterized by the mutually perpendicular vectors $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3$ with equal length $a_1 = a_2 = a_3 = a$. As lattice equations for the Mindlin gradient elasticity we consider the equation

$$
\begin{aligned}
M\ddot{u}_i(\mathbf{n},t) = {} & A_0^L(\alpha) \sum_{j=1}^{3} {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 2\alpha \\ j \end{bmatrix} \ddot{u}_i(\mathbf{m},t) - A_1^L(\alpha) \sum_{j:j\neq i}^{3} {}^{GL}\mathbb{K}_L^{-,-}\begin{bmatrix} \alpha & \alpha \\ j & i \end{bmatrix} u_i(\mathbf{m},t) \\
& - A_2^L(\alpha)\, {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 2\alpha \\ i \end{bmatrix} u_i(\mathbf{m},t) - A_3^L(\alpha) \sum_{j\neq i}^{3} {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 2\alpha \\ j \end{bmatrix} u_i(\mathbf{m},t) \\
& - B_1^L(\alpha) \sum_{j:j\neq i}^{3} \left({}^{GL}\mathbb{K}_L^{-,-}\begin{bmatrix} 3\alpha & \alpha \\ j & i \end{bmatrix} u_j(\mathbf{m},t) + {}^{GL}\mathbb{K}_L^{-,-}\begin{bmatrix} \alpha & 3\alpha \\ j & i \end{bmatrix} \right) u_j(\mathbf{m},t) \\
& - B_2^L(\alpha) \sum_{j:j\neq i}^{3} {}^{GL}\mathbb{K}_L^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ j & i \end{bmatrix} u_i(\mathbf{m},t) - B_3^L(\alpha)\, {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 4\alpha \\ i \end{bmatrix} u_i(\mathbf{m},t) \\
& - B_4^L(\alpha) \sum_{\substack{k,j \\ k\neq j; k\neq i; j\neq i}} {}^{3\ GL}\mathbb{K}_L^{-,-,+}\begin{bmatrix} \alpha & \alpha & 2\alpha \\ j & i & k \end{bmatrix} u_j(\mathbf{m},t) \\
& - B_5^L(\alpha) \sum_{\substack{k,j \\ k}} \neq j^{3\ GL}\mathbb{K}_L^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ j & k \end{bmatrix} u_i(\mathbf{m},t) - B_6^L(\alpha) \sum_{j=1}^{3} {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 4\alpha \\ j \end{bmatrix} u_i(\mathbf{m},t) + F_i(\mathbf{n},t),
\end{aligned}
\tag{18}
$$

where $A_1^L(\alpha)$, $A_2^L(\alpha)$, $A_3^L(\alpha)$, and $B_1^L(\alpha), \ldots, B_6^L(\alpha)$ are corresponding coupling constants of the lattice long-range interactions.

In the lattice model (18) all difference operators have fractional orders. For wide class of nonlocal elastic material the fractional derivatives are important only if short- and long-range particle interactions are present at the same time. It means that the lattice equations should include the difference operators of integer and non-integer orders. For this class of materials, we can use the lattice equation in the form

$$
\begin{aligned}
M\ddot{u}_i(\mathbf{n},t) = {} & A_0^L \sum_{j=1}^{3} {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 2 \\ j \end{bmatrix} \ddot{u}_i(\mathbf{m},t) \\
& + A_1^L \sum_{j=1}^{3} {}^{GL}\mathbb{K}_L^{-,-}\begin{bmatrix} 1 & 1 \\ j & i \end{bmatrix} u_j(\mathbf{m},t) \\
& + A_2^L \sum_{j=1}^{3} {}^{GL}\mathbb{K}_L^{+}\begin{bmatrix} 2 \\ j \end{bmatrix} u_i(\mathbf{m},t) \\
& + B_1^L \sum_{j,m,i}^{3} {}^{GL}\mathbb{K}_L^{-,+,-}\begin{bmatrix} 1 & \alpha & 1 \\ j & m & i \end{bmatrix} u_j(\mathbf{m},t) \\
& + B_2^L \sum_{j,m,i}^{3} {}^{GL}\mathbb{K}_L^{-,+,-}\begin{bmatrix} 1 & \alpha & 1 \\ j & m & j \end{bmatrix} u_i(\mathbf{m},t) \\
& + F_i(\mathbf{n},t),
\end{aligned}
\tag{19}
$$

where the displacement for the lattice is $u_i(\mathbf{m},t) = u_i(m_1, m_2, m_3, t)$, and A_0^L, A_1^L, A_2^L, B_1^L, B_2^L are the coupling constants of the lattice long-range interactions. This three-dimensional lattice model in the continuum limit gives a fractional generalization of the Mindlin model of the first gradient elasticity.

4 Fractional differential equations for nonlocal continuum

4.1 Fractional-order derivatives of the Grünwald–Letnikov type

To describe fractional elasticity of the nonlocal continua, we should use fractional derivatives with respect to space coordinates instead of the lattice operators. Continuum analogs of the fractional-order difference operators of the Grünwald–Letnikov type are the fractional derivatives of Grünwald–Letnikov type. Fractional-order difference operators ${}^{GL}\mathbb{K}_L^{\pm}\begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ defined by (7) are transformed by the continuous limit into the fractional derivative of Grünwald–Letnikov type with respect to coordinate x_j in the form

$$\lim_{a_j\to 0+}\frac{1}{a_j^{\alpha_j}}\left({}^{GL}\mathbb{K}_L^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}u_i(\mathbf{m},t)\right)={}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}u_i(\mathbf{r},t), \tag{20}$$

where ${}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ are the continuum fractional derivatives of the Grünwald–Letnikov type that are defined by

$$^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}=\frac{1}{2}\left({}^{GL}D^{\alpha}_{x_j,+}\pm{}^{GL}D^{\alpha}_{x_j,-}\right), \tag{21}$$

which contain the Grünwald–Letnikov fractional derivatives ${}^{GL}D^{\alpha}_{x_j,\pm}$ with respect to space coordinate x_j that can be written as

$$^{GL}D^{\alpha}_{x_j,\pm}u_i(\mathbf{r},t)=\lim_{a_j\to 0+}\frac{1}{|a_j|^{\alpha}}\times\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\,\Gamma(\alpha+1)}{\Gamma(m_j+1)\Gamma(\alpha-m_j+1)}\,u_i(\mathbf{r}\mp m_j\mathbf{a}_j,t),\quad(\alpha>0). \tag{22}$$

This statement can be proved by analogy with the proof for lattice model with long-range interaction of the Grünwals–Letnikov type suggested in [42].

It is important to note that the Grünwald–Letnikov fractional derivatives coincide with the Marchaud fractional derivatives (see Section 20.3 in [14, 15]) for the functions from the space $L_r(\mathbb{R})$, where $1\leqslant r<\infty$ (see Theorem 20.4 in [14, 15]). Moreover both the Grünwald–Letnikov and Marchaud derivatives have the same domain of definition. The Marchaud fractional derivative is defined by the equation

$$^{M}D^{\alpha,\pm}_{x_j}u_i(\mathbf{r},t)=\frac{1}{a(\alpha,s)}\int_0^{\infty}\frac{\Delta^{s,\pm}_{z_j}u_i(\mathbf{r},t)}{z_j^{\alpha+1}}dz_j,\quad(0<\alpha<s), \tag{23}$$

where $\Delta^{s,\pm}_{z_j}$ is the finite difference of integer order s,

$$\Delta^{s,\pm}_{z_j}u_i(\mathbf{r},t)=\sum_{k=0}^{s}\frac{(-1)^k s!}{(s-k)!k!}u_i(\mathbf{r}-kz_j\mathbf{e}_j,t), \tag{24}$$

and $a(\alpha,s)$ is

$$a(\alpha,s)=\frac{s}{\alpha}\int_0^1\frac{(1-\xi)^{s-1}}{(\ln(1/\xi))^{\alpha}}d\xi. \tag{25}$$

We can note that the derivatives (21) for integer orders $\alpha=n\in\mathbb{N}$ have the forms

$$^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}n\\ j\end{bmatrix}=\frac{1\pm(-1)^n}{2}\frac{\partial^n}{\partial x_j^n}. \tag{26}$$

Therefore the continuum fractional derivatives ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}n\\ j\end{bmatrix}$ are the usual derivatives of integer order n for even values α only, and the continuum operators ${}^{GL}\mathbb{D}_C^{-}\begin{bmatrix}n\\ j\end{bmatrix}$ are the derivatives of integer order n for odd values α only.

For bounden lattices, the fractional-order difference operators ${}^{GL}_{B}\mathbb{K}_L^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ defined by (10) are transformed by the continuous limit

$$\lim_{a_j\to 0+}\frac{1}{a_j^{\alpha_j}}\left({}^{GL}_{B}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}u_i(\mathbf{m},t)\right)={}^{GL}_{B}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha_j\\ j\end{bmatrix}u_i(\mathbf{r},t), \tag{27}$$

into the continuum fractional derivatives of the Grünwald–Letnikov type with respect to space coordinate x_j,

$$^{GL}_{B}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}=\frac{1}{2}\left({}^{GL}_{x_j^1}D^{\alpha}_{x_j,+}\pm{}^{GL}_{x_j^2}D^{\alpha}_{x_j,-}\right), \tag{28}$$

which contain the Grünwald–Letnikov fractional operators defined on the finite interval $[x_j^1,x_j^2]$, where $x_j^1=m_j^1a_j$ and $x_j^1=m_j^2a_j$, in the form

$$^{GL}_{B}D^{\alpha}_{x_j,\pm}f(\mathbf{r},t)=\lim_{a_j\to 0+}\frac{1}{|a_j|^{\alpha}}\times\sum_{m_j=0}^{M_j^{\pm}}\frac{(-1)^{m_j}\,\Gamma(\alpha+1)}{\Gamma(m_j+1)\Gamma(\alpha-m_j+1)}f\left(\mathbf{r}\mp m_j\mathbf{a}_j,t\right), \tag{29}$$

where

$$M_j^{+}=\left[\frac{x_j-x_j^1}{a_j}\right],\quad M_j^{-}=\left[\frac{x_j^2-x_j}{a_j}\right]. \tag{30}$$

The suggested forms of continuum fractional derivatives of the Grünwald–Letnikov type allow us to consider elasticity on bounded areas of nonlocal continuum.

4.2 Three-dimensional continuum models for fractional generalization of Aifantis gradient elasticity

In the continuum limit ($a_j \to 0$), the lattice equations (15) give the continuum equations for the fractional gradient elasticity in the form

$$\rho \frac{\partial^2 u_i(\mathbf{r},t)}{\partial t^2} = \sum_{j,l=1}^{3} A^C_{ijkl} \, {}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} 1 & 1 \\ j & l \end{bmatrix} u_k(\mathbf{r},t) + \sum_{j,m,l=1}^{3} B^C_{ijkl} \, {}^{GL}\mathbb{D}^{-,+,-}_{C} \begin{bmatrix} 1 & \alpha & 1 \\ j & m & l \end{bmatrix} u_k(\mathbf{r},t) + f_i(\mathbf{r},t), \tag{31}$$

where $u_i(\mathbf{r},t)$ are the components of the displacement vector field for continuum, and A^C_{ijkl} and B^C_{ijkl} are the coupling constants for the non-local continuum. We note that the continuum operators, which are used in Eq. (31), can be represented by

$${}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} 1 & 1 \\ j & l \end{bmatrix} = \frac{\partial^2}{\partial x_j \partial x_l}, \tag{32}$$

$${}^{GL}\mathbb{D}^{-,+,-}_{C} \begin{bmatrix} 1 & \alpha & 1 \\ j & m & l \end{bmatrix} = {}^{GL}\mathbb{D}^{-}_{C} \begin{bmatrix} 1 \\ j \end{bmatrix} {}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} \alpha \\ m \end{bmatrix} {}^{GL}\mathbb{D}^{-}_{C} \begin{bmatrix} 1 \\ l \end{bmatrix} = \frac{\partial}{\partial x_j} {}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} \alpha \\ m \end{bmatrix} \frac{\partial}{\partial x_l}. \tag{33}$$

The coupling constants of continuum are defined by the lattice coupling constants A^L_{ijkl} and B^L_{ijkl} by the relations

$$A^C_{ijkl} = \frac{a_l \, a_j \, \rho}{M} A^L_{ijkl}, \quad B^C_{ijkl} = \frac{a_l \, a_j \left(\sum_{m=1}^{3} a_m^{2\alpha}\right) \rho}{M} B^L_{ijkl}. \tag{34}$$

In the case $a_1 = a_2 = a_3 = a$, we get the fourth-order elastic stiffness tensor C_{ijkl} in the form

$$C_{ijkl} = A^C_{ijkl} = \frac{a^2 \rho}{M} A^L_{ijkl}. \tag{35}$$

If $B^L_{ijkl} = g_B \; A^L_{ijkl}$, then the scale parameter l_s^2 is $l_s^2 = 3a^{2\alpha} g_B$, and we have $B^C_{ijkl} = l_\alpha^2 C_{ijkl}$. For isotropic materials, C_{ijkl} are expressed in terms of the Lame constants λ and μ by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \, \delta_{jk}\right). \tag{36}$$

Note that x_k, a_k, l_α^2 are dimensionless values.

If $\alpha = 2$, then Eq. (31) gives the well-known continuum equation of gradient elasticity

$$\rho \ddot{u}_i(\mathbf{r},t) = \sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2 u_k(\mathbf{r},t)}{\partial x_j \partial x_l} \pm l_\alpha^2 \times \sum_{j,k,l,m=1}^{3} C_{ijkl} \frac{\partial^4 u_k(\mathbf{r},t)}{\partial x_j \partial x_m^2 \partial x_l} + f_i(\mathbf{r},t). \tag{37}$$

Let us give the stress–strain constitutive relation for fractional gradient elasticity (31). Equation (31) can be represented in the form

$$\rho \ddot{u}_i(\mathbf{r},t) = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(\mathbf{r},t)}{\partial x_j} + f_i(\mathbf{r},t), \tag{38}$$

where $\sigma_{ij}(\mathbf{r},t)$ is the stress tensor that is connected with the strain tensor

$$\varepsilon_{kl}(\mathbf{r},t) = \frac{1}{2} \left(\frac{\partial u_k(\mathbf{r},t)}{\partial x_l} + \frac{\partial u_l(\mathbf{r},t)}{\partial x_k} \right) \tag{39}$$

by the constitutive relation

$$\sigma_{ij}(\mathbf{r},t) = \sum_{k,l=1}^{3} A^C_{ijkl} \varepsilon_{kl}(\mathbf{r},t) + \sum_{k,l,m=1}^{3} B^C_{ijkl} \, {}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} \alpha \\ m \end{bmatrix} \varepsilon_{kl}(\mathbf{r},t). \tag{40}$$

If we use (35) and assume that

$$B^C_{ijkl} = \pm l_\alpha^2 A^C_{ijkl}, \tag{41}$$

then relation (40) can be rewritten as

$$\sigma_{ij}(\mathbf{r},t) = \sum_{k,l=1}^{3} C_{ijkl} \left(1 \pm l_\alpha^2 \, {}^{GL}\Delta^{\alpha,+}_{C}\right) \varepsilon_{kl}, \tag{42}$$

where ${}^{GL}\Delta^{\alpha,+}_{C}$ is the fractional Laplacian of the Grünwald–Letnikov type of the form

$${}^{GL}\Delta^{\alpha,+}_{C} = \sum_{m=1}^{3} {}^{GL}\mathbb{D}^{+}_{C} \begin{bmatrix} \alpha \\ m \end{bmatrix}. \tag{43}$$

Equation (42) gives the constitutive relation for fractional gradient elasticity. For $\alpha = 2$, relation (42) has the form

$$\sigma_{ij}(\mathbf{r},t) = \sum_{k,l=1}^{3} C_{ijkl} \left(1 \mp l_2^2 \, \Delta\right) \varepsilon_{kl}(\mathbf{r},t). \tag{44}$$

This is the well-known stress–strain constitutive relation for gradient elasticity [9, 62]. If consider the case with $u_x(\mathbf{r},t) = u(x,t), f_x(\mathbf{r},t) = f(x,t)$, where the other components u_y, u_z, f_y, f_z are equal to zero, then we get the one-dimensional fractional elasticity models suggested in [40, 41, 43]. The lattice models (19) and (15) are three-dimensional generalizations of the one-dimensional lattice models proposed in [40, 41, 43]. In addition, the Eq. (15) of lattice with long-range interactions allows us to derive the stress–strain constitutive relations for fractional nonlocal elasticity by using usual law (38).

4.3 Three-dimensional continuum models for fractional generalization of Mindlin gradient elasticity

The three-dimensional lattice model (18) in the continuum limit gives the fractional generalization of Mindlin model of the first gradient elasticity, if the Lame constants λ and μ are defined by the lattice coupling constants

$$\frac{\mu_\alpha}{\rho} = \frac{a^{2\alpha} A_3^L(\alpha)}{M}, \quad \frac{\lambda_\alpha}{\rho} = \frac{a^{2\alpha}}{M}\left(A_1^L(\alpha) - A_3^L(\alpha)\right), \tag{45}$$

and the three additional parameters l_1, l_2, l_3 of the Mindlin model are

$$l_1^2(\alpha) = \frac{a^{2\alpha} A_0^L(\alpha)}{M}, \quad l_2^2(\alpha) = \frac{a^{2\alpha} B_1^L(\alpha)}{A_1^L(\alpha)}, \quad l_3^2(\alpha) = \frac{B_5^L(\alpha)}{A_3^L(\alpha)}, \tag{46}$$

where the coupling constants are not independent

$$A_2^L(\alpha) = A_1^L(\alpha) + A_3^L(\alpha), \quad B_1^L(\alpha) = B_2^L(\alpha) = B_3^L(\alpha) = B_4^L(\alpha), B_5^L(\alpha) = B_6^L(\alpha). \tag{47}$$

In the continuum limit ($a \to 0$), we obtain the equations for fractional non-local continuum model that is a generalization of the Mindlin first gradient elasticity. These equations have the form

$$\begin{aligned}
\rho \ddot{u}_i = {} & \rho l_1^2(\alpha) \sum_{j=1}^{3} {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix} 2\alpha \\ j \end{bmatrix} \ddot{u}_i(\mathbf{r},t) \\
& + (\lambda_\alpha + \mu_\alpha)\left(\sum_{jj\neq i}^{3} {}^{GL}\mathbb{D}_C^{-,-}\begin{bmatrix} \alpha & \alpha \\ j & i \end{bmatrix} u_j(\mathbf{r},t) + {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix} 2\alpha \\ i \end{bmatrix} u_i(\mathbf{r},t) \right) \\
& + \mu_\alpha \sum_{j=1}^{3} {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix} 2\alpha \\ i \end{bmatrix} u_i(\mathbf{r},t) \\
& - (\lambda_\alpha + \mu_\alpha) l_2^2(\alpha) \sum_{jj\neq i}^{3} \left({}^{GL}\mathbb{D}_C^{-,-}\begin{bmatrix} 3\alpha & \alpha \\ j & i \end{bmatrix} u_j(\mathbf{r},t) + {}^{GL}\mathbb{D}_C^{-,-}\begin{bmatrix} \alpha & 3\alpha \\ j & i \end{bmatrix} u_j(\mathbf{r},t) \right) \\
& - (\lambda_\alpha + \mu_\alpha) l_2^2(\alpha) \sum_{jj\neq i}^{3} {}^{GL}\mathbb{D}_C^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ j & i \end{bmatrix} u_i(\mathbf{r},t) \\
& - (\lambda_\alpha + \mu_\alpha) l_2^2(\alpha) \left(\sum_{\substack{k,j \\ j\neq i; j\neq k; k\neq i}}^{3} {}^{GL}\mathbb{D}_C^{+,-,-}\begin{bmatrix} 2\alpha & \alpha & \alpha \\ k & j & i \end{bmatrix} u_i(\mathbf{r},t) + {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix} 4\alpha \\ i \end{bmatrix} u_i(\mathbf{r},t) \right) \\
& - \mu_\alpha l_3^2(\alpha) \left(\sum_{\substack{k,l \\ k\neq l}}^{3} {}^{GL}\mathbb{D}_C^{+,+}\begin{bmatrix} 2\alpha & 2\alpha \\ k & j \end{bmatrix} u_i(\mathbf{r},t) + \sum_{j=1}^{3} {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix} 4\alpha \\ i \end{bmatrix} u_i(\mathbf{r},t) \right) + f_i(\mathbf{r},t),
\end{aligned} \tag{48}$$

where $u_i(\mathbf{r},t)$ are components of the displacement field for the continuum, and $f_i(\mathbf{r},t)$ are the components of the body force.

For $\alpha = 1$, Eq. (48) give the differential equations for gradienl elasticity

$$\begin{aligned}\rho\ddot{u}_i(\mathbf{r},t) &= \rho\, l_1^2 \sum_{j=1}^{3} \frac{\partial^2 \ddot{u}_i(\mathbf{r},t)}{\partial x_j^2} \\ &+ (\lambda+\mu)\left(\sum_{j:j\neq i}^{3} \frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_j\, \partial x_i} + \frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_i^2}\right) + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2} \\ &- (\lambda+\mu)\, l_2^2 \sum_{j:j\neq i}^{3}\left(\frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_j\, \partial x_i^3} + \frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_j^3\, \partial x_i} + \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_j^2\, \partial x_i^2}\right) \\ &- (\lambda+\mu)\, l_2^2 \left(\sum_{\substack{k,j:\\ j\neq i;j\neq k;k\neq i}}^{3} \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\, \partial x_j\, \partial x_i} + \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_i^4}\right) \\ &- \mu l_3^2 \left(\sum_{\substack{k,l\\ k\neq l}}^{3} \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\, \partial x_j^2} + \sum_{j=1}^{3} \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_j^4}\right) + f_i(\mathbf{r},t),\end{aligned} \tag{49}$$

where $\lambda = \lambda_1$, $\mu = \mu_1$, and $l_j = l_j(1)$, where $j = 1,2,3$. In Eq. (49) the derivatives of integer orders with respect to the same spatial coordinates are clearly marked. Equation (49) can be rewrite as the Mindlin equations for displacements components in the form

$$\begin{aligned}\rho\ddot{u}_i(\mathbf{r},t) - \rho l_1^2 \sum_{j=1}^{3} \frac{\partial^2 \ddot{u}_i(\mathbf{r},t)}{\partial x_j^2} &= (\lambda+\mu) \sum_{j=1}^{3} \frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_i \partial x_j} \\ + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2} &- (\lambda+\mu)\, l_2^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_k^2 \partial x_i \partial x_j} \\ - \mu\, l_3^2 \sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\, \partial x_j^2} &+ f_i(\mathbf{r},t),\end{aligned} \tag{50}$$

where $f_i(\mathbf{r},t)$ are the components of the body force, $u_i(\mathbf{r},t)$ are components of the displacement field for the continuum, and

$$l_2^2 = \frac{4\lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 + 3\lambda_5}{2\,(\lambda+\mu)}, \qquad l_3^2 = \frac{\lambda_3 + 2\lambda_4 + \lambda_5}{2\mu}. \tag{51}$$

As a result, continuum equations (50) have two Lame constants and three additional parameters l_1, l_2, l_3. Note that Eq. (50) for Mindlin gradient elasticity model can be obtained [8] by using the expressions of the kinetic density

$$T = \frac{1}{2}\rho\partial_t u_i \partial_t u_i + \frac{1}{2}\rho l_1^2 \dot{u}_{i,j}\dot{u}_{i,j}, \tag{52}$$

the density of the deformation energy in the form

$$\begin{aligned}U = \frac{1}{2}\lambda\, \varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \lambda_1\, \varepsilon_{ik,i}\varepsilon_{jj,k} + \lambda_2\, \varepsilon_{kk,i}\varepsilon_{jj,i} \\ + \lambda_3\varepsilon_{ik,i}\varepsilon_{jk,j} + \lambda_4\varepsilon_{jk,i}\varepsilon_{jk,i} + \lambda_5\varepsilon_{jk,i}\varepsilon_{ij,k},\end{aligned} \tag{53}$$

where λ and μ are the usual Lame constants and the various λ_i $(i = 1,\ldots,5)$ are five additional constitutive coefficients, ρ is the mass density, u_k is the displacement, ε_{ij} is the strain, and $\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$.

If the lattice equations (18) would be written only through even lattice fractional-order differences ${}^{GL}\mathbb{K}_L^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix}$, then the correspondent continuum equations contain the continuum fractional derivatives ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix}$, of orders 1 and 3 that are non-local operators. In this case, we cannot get the usual Mindlin model with derivatives of integer orders. Therefore, we suggest the equations of lattice model that contain two type of lattice fractional derivatives ${}^{GL}\mathbb{K}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}$, in the suggested form (18). It is obvious that we would like to have a fractional generalization of partial differential equations such that to obtain the original equations in the limit case, when the orders of fractional derivatives become equal to initial integer values. This desirable correspondence and the property of the continuum fractional derivatives ${}^{GL}\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ to be the local operators of integer orders α only if we use ${}^{GL}\mathbb{D}_C^{-}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ for the odd values of α, and if we use ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ for the even values of α, allow us to consider equations in the form (18) with the fractional-order differences ${}^{GL}\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix}$ as basic equations of lattices witj lon–range interactions.

The continuum limit for lattice equations (19) gives the continuum equations of the fractional gradient elasticity in the form

$$\rho\ddot{u}_i(\mathbf{r},t) - A_0^C \sum_{j=1}^{3} \frac{\partial^2 \ddot{u}_i(\mathbf{r},t)}{\partial x_j^2} = A_1^C \sum_{j=1}^{3} \frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_j \partial x_i} + A_2^C \sum_{j=1}^{3} \frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2} + B_1^C \sum_{j,m=1}^{3} \frac{\partial}{\partial x_j} {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha \\ m\end{bmatrix} \frac{\partial u_j(\mathbf{r},t)}{\partial x_i} + B_2^C \sum_{j,m=1}^{3} \frac{\partial}{\partial x_j} {}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha \\ m\end{bmatrix} \frac{\partial u_i(\mathbf{r},t)}{\partial x_j} + f_i(\mathbf{r},t), \quad (54)$$

where the constants for continuum are defined by

$$A_i^C = \frac{a^2\rho}{M} A_i^L \quad (i = 0,1,2), \qquad B_j^C = \frac{a^{2+\alpha}\rho}{M} B_j^L \quad (j = 1,2). \quad (55)$$

Note that the definition of the fractional-order difference ${}^{GL}\mathbb{K}_L^{\pm}\begin{bmatrix}\alpha \\ j\end{bmatrix}$ does not include the factor $1/a_j^{\alpha}$. The Lame constants λ and μ are defined by the lattice coupling constants

$$\mu = \frac{a^2\rho}{M} A_2^L, \quad \lambda = \frac{a^2\rho}{M}\left(A_1^L - A_2^L\right). \quad (56)$$

The three additional parameters $l_1, l_2(\alpha), l_3(\alpha)$ of the Mindlin model are

$$l_1^2 = \frac{A_0^L a^2}{M}, \quad l_2^2(\alpha) = \frac{a^{\alpha}|B_1^L|}{|A_1^L|}, \quad l_3^2(\alpha) = \frac{a^{\alpha}|B_2^L|}{|A_2^L|}. \quad (57)$$

Note that x_k, a, l_1^2, $l_2^2(\alpha)$, $l_3^2(\alpha)$ are dimensionless values. Equation (54) can be considered as the fractional Mindlin equations.

For $\alpha = 2$, the suggested three-dimensional lattice model (19) gives the well-known Mindlin equation (50) for the displacement field $u_i = u_i(\mathbf{r},t)$ of the continuum, where we take into account ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}2 \\ m\end{bmatrix} = -\partial^2/\partial x_m^2$. For $\alpha = 1$, Eq. (19) give the differential equations with non-local operator of the first and third orders since the derivatives ${}^{GL}\mathbb{D}_C^{+}\begin{bmatrix}\alpha \\ m\end{bmatrix}$ are non-local operators for odd α.

5 Conclusion

In this paper three-dimensional lattice models with long-range inter-particle interactions are suggested for fractional strain-gradient elasticity of weak nonlocal continuum. The proposed lattice model can be considered as a new microstructural basis of unified description of gradient continuum models. The suggested type of long-range interactions can be considered for integer and non-integer (fractional) values of the parameter α. This allows us to obtain lattice models for the local and nonlocal elasticity theories.

The proposed lattice models are used interactions based on the Grünwald–Letnikov fractional differences and corresponding fractional derivatives. One advantage of such models is a possibility to use the well-known numerical methods developed for this type of fractional derivatives. However, the computer simulation represents a separate volume study, which will be carried out in future, and it will be published in the next paper.

Let us note some possible extensions of the suggested lattice approach to formulate fractional generalizations of nonlocal elasticity theories. We assume that the proposed lattice approach to the elasticity of materials can be used to generalize for different types of Bravais lattices such as monoclinic, triclinic, hexagonal and rhombohedral. We can assume that fractional generalization of the Mindlin nonlocal plate model and correspondent lattice model can be formulated by suggested method. It can be assumed that the proposed three-dimensional lattice model can be modified to describe metamaterials with negative-stiffness phases at the microstructural level. We can assume that the suggested lattice models can be modified to have lattice models for dislocations in the gradient elasticity continuum and in the fractional generalization of nonlocal dislocations. The proposed models of the three-dimensional lattice with long-range interactions can play an important role in the description of nonlocal elastic materials at microscale and nanoscale because at these scales the interatomic interactions can be prevalent in determining the elastic properties of these materials. We also assume that the suggested approach can be generalized for lattice models with the fractal spatial dispersion, which are suggested in [55] (see also [56, 57]), and the continuum limits of these fractal lattice models can give continuum models of fractal material.

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