Lattice model of fractional gradient and integral elasticity: Long-range interaction of Grünwald–Letnikov–Riesz type

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

ABSTRACT

Lattice model with long-range interaction of power-law type that is connected with difference of non-integer order is suggested. The continuous limit maps the equations of motion of lattice particles into continuum equations with fractional Grünwald–Letnikov–Riesz derivatives. The suggested continuum equations describe fractional generalizations of the gradient and integral elasticity. The proposed type of long-range interaction allows us to have united approach to describe of lattice models for the fractional gradient and fractional integral elasticity. Additional important advantages of this approach are the following: (1) It is possible to use this model of long-range interaction in numerical simulations since this type of interactions and the Grünwald–Letnikov derivatives are defined by generalized finite difference; (2) The suggested model of long-range interaction leads to an equation containing the sum of the Grünwald–Letnikov derivatives, which is equal the Riesz's derivative. This fact allows us to get particular analytical solutions of fractional elasticity equations.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Discrete system of long-range interacting particles serve as a model for numerous applications in mechanics and physics (Dyson, 1969, 1971; Frohlich et al., 1978; Nakano and Takahashi, 1994, 1995; Sousa, 2005; Campa et al., 2009; Tarasov, 2006b; Tarasov and Zaslavsky, 2006; Laskin and Zaslavsky, 2006; Zaslavsky et al., 2007). Long-range interactions are important type of interactions for complex media with non-local properties (see references in Tarasov, 2011a).

Using the fractional calculus (Samko et al., 1987; Kilbas et al., 2006), we consider long-range interaction of a special type to describe a fractional generalization of elasticity theory. We transform the set of equations of motion of coupled individual lattice particles into the equation of non-local continuum that contains spatial derivatives of non-integer orders. It allows us to consider different lattice

models for generalization of elasticity theory by applying methods of fractional calculus. The theory of fractional derivatives and integrals has wide applications (Carpinteri and Mainardi, 1997; Hilfer, 2000; Luo and Afraimovich, 2010; Klafter et al., 2011; Mainardi, 2010; Tarasov, 2011b; Uchaikin et al., 2013; Tarasov, 2011a, 2013) and it is a powerful tool for the analysis of different nonlocal continuum models with nonlocality of power-law type.

Non-local continuum mechanics has been treated with two different approaches: the gradient models (weak non-locality) and the integral models (strong non-locality). The correspondent constitutive relations have the form

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} \pm l_s^2 \mathcal{L} \varepsilon_{kl}),$$

where C_{ijkl} is the elasticity tensor, ε_{ij} and σ_{ij} are the strain and stress tensors, respectively. The additional parameter l_s is an internal length scale that can be defined by lattice constant. The operator $\mathcal{L}$ is the integral or differential operator of integer order (Eringen and Edelen, 1972; Askes et al., 2002, 2008; Askes and Sluys, 2002; Askes and Aifantis, 2011). Recently, the problems of non-local continuum are

* Tel.: +7 495 939 03 97.
E-mail address: tarasov@theory.sinp.msu.ru

described by means of fractional calculus. Fractional models of integral non-local elasticity are considered in Lazopoulos (2006), Carpinteri et al. (2009, 2011), Sapora et al. (2013), Cottone et al. (2009,) and Cornetti et al. In fractional elasticity models, the operator $\mathcal{L}$ is the integral or differential operator of non-integer order. To generalize continuum equations by using fractional calculus, we should represent these equation through the dimensionless coordinate variables. Therefore the coordinates x, y, z, the vector $\mathbf{r}$, the value $r = |\mathbf{r}|$, and the parameter l_s^2 are dimensionless in the suggested fractional models.

In this paper we consider one-dimensional lattice model with long-range interactions of Grünwald–Letnikov–Riesz type (Tarasov, 2011a). A feature of suggested long-range interactions is that the interactions terms have the form of fractional differences of non-integer orders. The first advantage of this approach is based on the properties of the Grünwald–Letnikov fractional derivative (Samko et al., 1987; Kilbas et al., 2006). The suggested type of long-range interaction of lattice particles allows us to have united description of lattice models for the fractional gradient and fractional integral elasticity that is characterized by the non-locality of power-law type. The second important advantage of suggested approach is the ability to directly use the model of long-range interaction in numerical simulations since this type of interactions and the Grünwald–Letnikov derivatives are defined by generalized difference of non-integer order. We assume that the suggested form of long-range interaction can be used for different scheme of simulations in fractional gradient and integral elasticity models.

There are some problems with application of the Grünwald–Letnikov derivatives, since there are currently very limited number of analytical solutions for differential equations with fractional Grünwald–Letnikov derivatives in contrast to the equations with derivatives of Riemann–Liouville, Caputo and Riesz types (Kilbas et al., 2006). We note that the suggested model of long-range interaction leads to an equation containing the sum of the Grünwald–Letnikov derivatives, which is equal the Riesz derivative. Using this connection, we derive some particular analytical solutions (Kilbas et al., 2006) for fractional elasticity equations with the Riesz derivatives of non-integer orders.

2. Grünwald–Letnikov fractional derivatives and integrals

2.1. Fractional differences of non-integer orders

The Grünwald–Letnikov derivatives have been introduced by Grünwald in 1867 and independently by Letnikov in 1868. Definition of the Grünwald–Letnikov fractional derivatives are based on a generalization of the usual differentiation of a function $f(x)$ of integer order n of the forms

$$D_x^n f(x) = \lim_{h \to 0} \frac{\Delta_h^n f(x)}{h^n},$$

$$D_x^n f(x) = \lim_{h \to 0} \frac{\nabla_h^n f(x)}{h^n},$$

where Δ_h^n and ∇_h^n are forward and backward finite differences of order n of a function $f(x)$ with a step h and centered at the point x. The nth-order forward and backward differences are respectively given by

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + (n-k)h), \tag{1}$$

$$\nabla_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x - kh). \tag{2}$$

Note that

$$\Delta_h^n f(x) = (-1)^n \nabla_{-h}^n f(x). \tag{3}$$

The difference of a fractional order $\alpha > 0$ is defined by the infinite series (see Section 20 in Samko et al., 1987)

$$\nabla_h^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh), \tag{4}$$

where the binomial coefficients are

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)}.$$

For $h > 0$, the difference (4) is called left-sided fractional difference, and for $h < 0$ it is called a right-sided fractional difference. We note that the series in (4) converges absolutely and uniformly for every bounded function $f(x)$ and $\alpha > 0$.

For the fractional difference, the semigroup property

$$\nabla_h^\alpha \nabla_h^\beta f(x) = \nabla_h^{\alpha+\beta} f(x), \quad (\alpha > 0, \ \beta > 0) \tag{5}$$

is valid for any bounded function $f(x)$ (see Property 2.29 in Kilbas et al., 2006 page 121).

The Fourier transform of the fractional difference is given by

$$\mathcal{F}\{\nabla_h^\alpha f(x)\}(k) = (1 - \exp\{ikh\})^\alpha \mathcal{F}\{f(x)\}(k)$$

for any function $f(x) \in L_1(\mathbb{R})$ (see Property in Kilbas et al., 2006 page 121).

2.2. Grünwald–Letnikov fractional derivatives

The definitions (1) and (2) are used to define the Grünwald–Letnikov fractional derivatives by replacing $n \in \mathbb{N}$ in by $\alpha > 0$. The value h^n is replaced by h^α, while the finite difference ∇_h^n is replaced by the difference ∇_h^α of a fractional order α.

The left- and right-sided Grünwald–Letnikov derivatives of order $\alpha > 0$ are defined by

$$^{GL}D_{x\pm}^\alpha f(x) = \lim_{h \to 0+} \frac{\nabla_{\pm h}^\alpha f(x)}{|h|^\alpha}. \tag{6}$$

Note that the Grünwald–Letnikov derivatives for integer orders $\alpha = n \in \mathbb{N}$ is

$$^{GL}D_{x\pm}^n f(x) = (-1)^n D_x^n. \tag{7}$$

We also note that these derivatives coincide with the Marchaud fractional derivatives of order $\alpha > 0$ for $f(x) \in L_p(\mathbb{R})$, $1 \leqslant p < \infty$ (see Theorem 20.4 in Samko et al., 1987). The properties of the Grünwald–Letnikov fractional

derivatives is described in Section 20 of the book (Samko et al., 1987). Then (6) can be represented by the Marchaud fractional derivatives

$$^{GL}D_{x\pm}^{\alpha}f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x \mp z)}{z^{\alpha+1}} dz,$$

if $f(x) \in L_p(\mathbb{R})$, where $1 < p < 1/\alpha$ and $0 < \alpha < 1$.

2.3. Grünwald–Letnikov fractional integral

It is interesting that series (4) can be used for $\alpha < 0$ (see Section 20 in Samko et al., 1987) and Eq. (6) defines the Grünwald–Letnikov fractional integral if

$$|f(x)| < c(1 + |x|)^{-\mu}, \quad \mu > |\alpha|. \tag{8}$$

The existence of the Grünwald–Letnikov fractional integral means that we have a united definition of fractional derivatives and integrals. It allows us to have a united approach to the fractional gradient and integral elasticity.

2.4. Commutativity and associativity of Grünwald–Letnikov derivatives

In Ortigueira et al. (2012) has been showed that, for analytic functions, the fractional derivatives of the Grünwald–Letnikov type have some nice and useful properties and a semi-group structure. It is connected with the fact that differences of fractional order satisfy the semigroup property (5). Using the semi-group property

$$^{GL}D_{x\pm}^{\alpha} {}^{GL}D_{x\pm}^{\beta} = {}^{GL}D_{x\pm}^{\alpha+\beta}, \quad (\alpha > 0, \ \beta > 0). \tag{9}$$

Ortigueira, Rivero, Trujillo proved (Ortigueira et al., 2012) that the Grünwald–Letnikov derivatives are commutative and associative operators. These properties are very important to application in mechanics, but only a few types of fractional derivatives have them. These properties allow us to represent a single fractional derivative, which appears in the equation of the lattice model, as a product of derivatives, and thus get the fractional gradient and integral elastic constitutive relations.

2.5. Grünwald–Letnikov–Riesz fractional derivative

We can define a fractional derivative of order $\alpha > 0$ by

$$^{GLR}D_x^{\alpha}f(x) = \frac{1}{2\cos(\alpha\pi/2)} \lim_{h \to 0+} \frac{\nabla_h^{\alpha}f(x) + \nabla_{-h}^{\alpha}f(x)}{|h|^{\alpha}}. \tag{10}$$

This derivative coincide (see Section 20.1 in Samko et al., 1987) with the Riesz fractional derivative $^R\mathbf{D}_x^{\alpha}$ of order $\alpha > 0$ if $\alpha \neq 1, 2, 3, \ldots$, i.e. we have the relation

$$^{GLR}D_x^{\alpha}f(x) = {}^R\mathbf{D}_x^{\alpha}f(x), \tag{11}$$

where

$$^R\mathbf{D}_x^{\alpha}f(x) = -\frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)}$$
$$\int_0^{\infty} \frac{f(x+z) - 2f(x) + f(x-z)}{z^{\alpha+1}} dz. \tag{12}$$

Therefore the fractional derivative (10) is called the Grünwald–Letnikov–Riesz derivatives of order $\alpha > 0$ ($\alpha \neq 1, 2, 3, \ldots$) in Samko et al. (1987).

3. Lattice with linear nearest-neighbor interaction

Let us consider equations of motion for particles with the nearest-neighbor interaction of the form

$$M \frac{\partial^2}{\partial t^2} u_n(t) = K \hat{A}_n^k u_k(t) + F_n(t), \tag{13}$$

where

$$\hat{A}_n^m u_m(t) = \sum_{m=0}^{m=1} (-1)^{m+1} \left(u_{n+m}(t) + u_{n-m}(t) \right).$$

For these equations we can give the well-known statement regarding the nearest-neighbor interaction (see for example Maslov, 1976; Tarasov, 2006a,b).

Proposition 1. *In the continuous limit the lattice equations of motion (13) maps into the continuum equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_e^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{\rho}f(x,t), \tag{14}$$

with the mass density $\rho = M/Ah$, the Young's modulus $E = Kh/A$, the force density $f(x,t) = F(x,t)/Ah$, the cross-section area of the medium A and the inter-particle distance h, where

$$C_e^2 = \frac{E}{\rho} = \frac{Kh^2}{M} \tag{15}$$

is a finite parameter.

Proof. To derive the equation for the field $\hat{u}(k,t)$, we multiply Eq. (13) by $\exp(-ikn\Delta x)$, where $\Delta x = h$ and summing over n from $-\infty$ to $+\infty$. Then

$$M \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} \frac{\partial^2 u_n}{\partial t^2} = K \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x}[u_{n+1} - 2u_n + u_{n-1}]$$
$$+ \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} F(n,t). \tag{16}$$

The first term on the right-hand side of (16) is

$$\sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} (u_{n+1} - 2u_n + u_{n-1})$$

$$= \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} u_{n+1} - 2 \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} u_n + \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} u_{n-1}$$

$$= e^{ik\Delta x} \sum_{m=-\infty}^{+\infty} e^{-ikm\Delta x} u_m - 2 \sum_{n=-\infty}^{+\infty} e^{-ikn\Delta x} u_n$$

$$+ e^{-ik\Delta x} \sum_{s=-\infty}^{+\infty} e^{-iks\Delta x} u_s$$

$$= e^{ik\Delta x} \hat{u}(k,t) - 2\hat{u}(k,t) + e^{-ik\Delta x} \hat{u}(k,t)$$

$$= [e^{ik\Delta x} + e^{-ik\Delta x} - 2]\hat{u}(k,t) = 2(\cos(k\Delta x) - 1)\hat{u}(k,t)$$

$$= -4\sin^2\left(\frac{k\Delta x}{2}\right)\hat{u}(k,t).$$

Here we use the definition of $\hat{u}(k,t)$ on $[-k_0/2, k_0/2]$ by the equation

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-ikx_n} = \mathcal{F}_{\Delta}\{u_n(t)\} \tag{17}$$

and $x_n = nh$, where $h = 2\pi/k_0$ is distance between equilibrium positions of the lattice particles.

As a result, we have

$$M \frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = K \hat{J}_\alpha(k\Delta x)\, \hat{u}(k,t) + \mathcal{F}_\Delta\{F_n(t)\}, \tag{18}$$

where

$$\hat{J}_\alpha(k\Delta x) = -4 \sin^2\left(\frac{k\Delta x}{2}\right). \tag{19}$$

For $h = \Delta x \to 0$, the asymptotic behavior of the sine is $\sin(k\Delta x/2) \sim k\Delta x/2$. Then (19) can be represented by

$$\hat{J}_\alpha(k\Delta x) \approx -(k\Delta x)^2.$$

Using the finite parameter $C_e^2 = K h^2/M$, after division by the cross-section area of the medium A and the inter-particle distance h, the transition to the limit $h = \Delta x \to 0$ in Eq. (18) gives

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = -C_e^2 k^2 \tilde{u}(k,t) + \frac{1}{\rho} \mathcal{F}\{f(x,t)\}, \tag{20}$$

where we use $0 < |C_e^2| < \infty$. The inverse Fourier transform $\mathcal{F}^{-1}$ of (20) has the form

$$\frac{\partial^2 \mathcal{F}^{-1}\{\tilde{u}(k,t)\}}{\partial t^2} = -C_e^2 \mathcal{F}^{-1}\{k^2 \tilde{u}(k,t)\} + \frac{1}{\rho} f(x,t).$$

Then we can use the connection between second derivative and its Fourier transform in the form $k^2 \leftrightarrow -\partial^2/\partial x^2$. As a result, we obtain the continuum Eq. (14). □

As a result, we prove that Eq. (13) give the continuum equation with derivatives of second order only.

4. Lattice with Grünwald–Letnikov–Riesz long-range interaction

In this section we describe the type of long-range interaction that is suggested in Tarasov (2011a, see Section 8.19). Let us consider a lattice system of interacting particles, whose displacements from the equilibrium are $u_n(t)$, where $n \in \mathbb{Z}$. We assume that the system is described by the equations of motion

$$M \frac{\partial^2}{\partial t^2} u_n(t) + g \hat{B}_n^m(\alpha) u_m(t) - F_n(t) = 0, \tag{21}$$

where g is the coupling constant for long-range interaction that have the form

$$\hat{B}_n^m(\alpha) u_m(t) = \sum_{m=0}^{+\infty} b_\alpha(m)\,(u_{n+m}(t) + u_{n-m}(t)) \tag{22}$$

and the function $b_\alpha(m)$ is

$$b_\alpha(m) = \frac{(-1)^m}{\Gamma(m+1)\,\Gamma(\alpha-m+1)}. \tag{23}$$

This type of long-range interaction has been called the Grünwald–Letnikov–Riesz interaction (see Section 8.19 in Tarasov, 2011a). Let us give the main statement regarding this interaction.

Proposition 2. *In the limit $h \to 0$ lattice equations (21) and (22) with (23) give the continuum equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} + C(\alpha)\, {}^{GLR}D_x^\alpha u(x,t) - \frac{1}{\rho} f(x,t) = 0,$$
$$(\alpha \in \mathbb{R},\ \alpha \neq \pm 1, \pm 3, \pm 5, \ldots), \tag{24}$$

where

$$C(\alpha) = \frac{2 \cos(\alpha\pi/2)\, g\, h^\alpha}{\Gamma(\alpha+1)\, M} \tag{25}$$

is a finite parameter, and

$${}^{GLR}D_x^\alpha = \frac{1}{2 \cos(\alpha\pi/2)} \left({}^{GL}D_{x+}^\alpha + {}^{GL}D_{x-}^\alpha \right) \tag{26}$$

is the Grünwald–Letnikov–Riesz fractional derivative of order α, and $u(x,t)$ is a smooth function such that $u(nh,t) = u_n(t)$.

Proof. We define smooth functions $u(x,t)$ and $F(x,t)$ such that

$$u(nh,t) = u_n(t), \quad F(nh,t) = F_n(t).$$

Then Eqs. (21) and (22) with (23) can be represented as

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{g h^\alpha}{M} \sum_{m=0}^{+\infty} b_\alpha(m) \frac{1}{h^\alpha}\, (u(x+mh,t) + u(x-mh,t))$$
$$- \frac{1}{M} F(x,t) = 0. \tag{27}$$

After division by the cross-section area of the medium A and the inter-particle distance h it is found that

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{C(\alpha)}{2 \cos(\alpha\pi/2)} \sum_{m=0}^{+\infty} \frac{(-1)^m \Gamma(\alpha+1)}{\Gamma(\alpha-m+1)\,\Gamma(m+1)}$$
$$\times \frac{u(x+mh,t) + u(x-mh,t)}{h^\alpha} - \frac{1}{\rho} f(x,t) = 0 \tag{28}$$

with the mass density $\rho = M/Ah$, the force density $f(x,t) = F(x,t)/Ah$, and $C(\alpha)$ is defined by (25). Using the definitions of the left-sided and right-sided fractional differences and the limit $h \to 0+$, we obtain

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{C(\alpha)}{2 \cos(\alpha\pi/2)} \lim_{h \to 0} \frac{\nabla_h^\alpha u(x,t) + \nabla_{-h}^\alpha u(x,t)}{|h|^\alpha}$$
$$- \frac{1}{\rho} f(x,t) = 0. \tag{29}$$

Using the Grünwald–Letnikov–Riesz derivative (10), Eq. (29) can be rewritten in the form (24). □

5. Fractional gradient and integral elasticity of Grünwald–Letnikov–Riesz type

Let us consider a system of interacting particles, whose displacements from the equilibrium are $u_n(t)$, where $n \in \mathbb{Z}$. We assume that the system is described by the equations of motion

$$M \frac{\partial^2 u_n(t)}{\partial t^2} - K \hat{A}_n^m u_m(t) + g \hat{B}_n^m(\alpha) u_m(t) - F_n(t) = 0. \tag{30}$$

In the limit $h \to 0$ Eq. (30) gives the continuous medium equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - C_e^2 \frac{\partial^2 u(x,t)}{\partial x^2} + C(\alpha) \, {}^{GLR}D_x^\alpha u(x,t) - \frac{1}{\rho} f(x,t) = 0,$$

$$(31)$$

where C_e^2 is defined by (15), $C(\alpha)$ is defined by (25), and

$$\alpha \in \mathbb{R}, \quad \alpha \neq \pm 1, \pm 3, \pm 5, \ldots$$

Let us consider Eq. (31) for two cases: $\alpha > 2$ and $0 < \alpha < 2$.

For $\alpha > 2$, we can use ${}^{GL}D_{x\pm}^1 = \pm D_x^1$, and the semi-group property (9), to represent the Grünwald–Letnikov derivatives in the form

$$^{GL}D_{x\pm}^\alpha = D_x^1 \, {}^{GL}D_{x\pm}^{\alpha-2} D_x^1 \quad (\alpha > 2).$$

$$(32)$$

Therefore, we have

$$^{GL}D_{x+}^\alpha + {}^{GL}D_{x-}^\alpha = D_x^1 \left({}^{GL}D_{x+}^{\alpha-2} + {}^{GL}D_{x-}^{\alpha-2} \right) D_x^1.$$

$$(33)$$

As a result, we can use

$$^{GLR}D_x^\alpha = D_x^1 \, {}^{GLR}D_x^{\alpha-2} D_x^1$$

$$(34)$$

in order to rewrite Eq. (31) in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - C_e^2 D_x^2 u(x,t) + C(\alpha) D_x^1 \, {}^{GLR}D_x^{\alpha-2} D_x^1 u(x,t)$$
$$- \frac{1}{\rho} f(x,t) = 0.$$

$$(35)$$

The correspondent constitutive relation for the continuum equation (35) can be derived by the momentum balance equation

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = D_x^1 \sigma(x) + f(x,t)$$

$$(36)$$

and the strain–displacement relation for small deformations

$$\varepsilon(x) = D_x^1 u(x).$$

$$(37)$$

The commutative and associative properties of fractional Grünwald–Letnikov derivative and relation (32) allow us to represent a single fractional derivative in Eq. (24), as a product of derivatives, and thus get the fractional gradient constitutive relations.

Using (36) and (37), Eq. (35) gives the fractional constitutive relation in the following form

$$\sigma(x,t) = E \left(\varepsilon(x,t) \mp l_s^2(\alpha) \, {}^R\mathbf{D}_x^{\alpha-2} \varepsilon(x,t) \right) \quad (\alpha > 2),$$

$$(38)$$

where E is the Young's modulus, ${}^R\mathbf{D}_x^{\alpha-2}$ is the Riesz fractional derivative of order $\alpha - 2$, which is equivalent by (11) to the Grünwald–Letnikov–Riesz fractional derivative ${}^{GLR}D_x^{\alpha-2}$. As a result, we can state that Eq. (31) with $\alpha > 2$ describes a fractional gradient elasticity. In Eq. (38), we use

$$l_s^2(\alpha) = \frac{|C(\alpha)| \rho}{E} = \frac{2 |g| h^{\alpha-2}}{K} \frac{|\cos(\alpha\pi/2)|}{\Gamma(\alpha+1)}$$

$$(39)$$

is the scale parameter of fractional elasticity. Sign in front of this scale parameter $l_s^2(\alpha)$ in Eq. (38) is determined by the sign of the expression $g \cos(\alpha\pi/2)$, i.e. the sign of the coupling constant g of lattice vibrations and the value of the order α of long-rang interactions. If $g \cos(\alpha\pi/2) > 0$, then we get the minus in front of $l_s^2(\alpha)$ in (39). For the case

$\alpha = 4$ and $g > 0$, we derive the constitutive relations for the gradient elasticity model with the minus in front of l_s^2. We get the equations for phenomenological gradient elasticity model from the lattice model equations. Note that normally it is considered a phenomenological model does not have a corresponding microscopic model (Askes and Aifantis, 2011).

For the case $0 < \alpha < 2$, we cannot use the properties (32) and (34) because semi-group relation (9) holds for positive orders. For $0 < \alpha < 2$, we can use the relation (11) and the Riesz's analytic continuation (see Eq. (17) in Riesz, 1949) of fractional integrals of order α to negative values of orders $\alpha > -p$ in the form

$$^R\mathbf{I}^\alpha = (-1)^p \, {}^R\mathbf{I}^{\alpha+2p} \Delta^p,$$

$$(40)$$

where $p \in \mathbb{N}$ and ${}^R\mathbf{I}^{\alpha+2p}$ is the Riesz fractional integral of order $\alpha + 2p$. Using Eqs. (17), (21) and (22) from the Riesz's review paper Riesz, 1949, we get

$$^R\mathbf{D}_x^\alpha = -{}^R\mathbf{I}^{2-\alpha} \Delta = \Delta \, {}^R\mathbf{I}^{4-\alpha} \Delta.$$

$$(41)$$

For one-dimensional case, we have $\Delta = (D_x^1)^2$, and we can use the relation

$$^R\mathbf{D}_x^\alpha = \Delta \, {}^R\mathbf{I}^{4-\alpha} \Delta = D_x^1 \mathbf{J}_x^{2-\alpha} D_x^1,$$

$$(42)$$

where $\mathbf{J}_x^{2-\alpha}$ is the fractional integral operator of order $(2 - \alpha)$ that is defined by

$$\mathbf{J}_x^{2-\alpha} = D_x^1 \, {}^R\mathbf{I}_x^{4-\alpha} D_x^1.$$

$$(43)$$

As a result, Eq. (31) with $\alpha < 2$ describes a fractional integral elasticity. Using (36), (37) and (42), Eq. (31) gives the fractional constitutive relation in the following form

$$\sigma(x,t) = E \left(\varepsilon(x,t) \mp l_s^2(\alpha) \, {}^R\mathbf{J}_x^{2-\alpha} \varepsilon(x,t) \right) \quad (0 < \alpha < 2),$$

$$(44)$$

where $\mathbf{J}_x^{2-\alpha}$ is the fractional integral (43) of order $0 < 2 - \alpha < 2$, and $l_s^2(\alpha)$ is defined by (39). Sign in front of this scale parameter $l_s^2(\alpha)$ in Eq. (44) is also determined by the sign of the expression $g \cos(\alpha\pi/2)$.

Let us give a remark about the scale parameter $l_s(\alpha)$. Eq. (39) can lead to incorrect conclusion about the behavior of the parameter $l_s^2(\alpha)$ for $h \to 0$ in the case $0 < \alpha < 2$. Using (15), the dimensionless parameter (39) can be written as

$$l_s^2(\alpha) = \frac{2 |g| h^\alpha}{C_e^2 M} \frac{|\cos(\alpha\pi/2)|}{\Gamma(\alpha+1)}.$$

$$(45)$$

Because the value of C_e^2 if finite, then behavior of the parameter $l_s^2(\alpha)$ for $h \to 0$ has the identical type for $\alpha > 2$ and $0 < \alpha < 2$, such that $l_s^2(\alpha)$ is proportion to h^α.

As a result, Eqs. (38) and (44) describe the fractional gradient and fractional integral elasticity of non-local continuum. If $0 < \alpha < 2$, we have a fractional integral elasticity, and if $\alpha > 2$, then equation describes fractional gradient elasticity. It can call the fractional elasticity of Grünwald–Letnikov–Riesz type.

6. Solutions of fractional elasticity equations

Let us consider more general lattice system of interacting particles that is described by the equations of motion

$$M \frac{\partial^2 u_n(t)}{\partial t^2} + K \hat{A}_n^m u_m(t) + \sum_{k=1}^N g_k \hat{B}_n^m(\alpha_k) u_m(t) = F_n(t). \quad (46)$$

The correspondent continuum equation for the fractional elasticity of the Grünwald–Letnikov–Riesz type has the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - C_e^2 D_x^2 u(x,t) + \sum_{k=1}^N C(\alpha_k) \, {}^R\mathbf{D}_x^{\alpha_k} u(x) = \frac{1}{\rho} f(x). \quad (47)$$

Here to get analytical solution of the fractional equations of nonlocal elasticity, we use the equivalence (see Section 20.1 in Samko et al., 1987) of the Grünwald–Letnikov–Riesz derivative ${}^{GLR}D_x^\alpha$ and the Riesz derivative ${}^R\mathbf{D}_x^\alpha$ in the form

$$^{GLR}D_x^\alpha = \frac{1}{2\cos(\alpha\pi/2)} \left({}^{GL}D_{x+}^\alpha + {}^{GL}D_{x-}^\alpha\right) = {}^R\mathbf{D}_x^\alpha. \quad (48)$$

Eq. (47) can be considered as a fractional generalization of the higher-order strain-gradient models (Askes and Sluys, 2002) and the higher-order integral elasticity models. In the static case ($D_t^2 u(x,t) = 0$) the fractional elasticity Eqs. (47) is

$$\sum_{k=1}^N C(\alpha_k) \, {}^R\mathbf{D}_x^{\alpha_k} u(x) - C_e^2 D_x^2 u(x) = \frac{1}{\rho} f(x), \quad (49)$$

where $x \in \mathbb{R}$, $N \in \mathbb{N}$ and $m \neq 1, C(\alpha_k) \in \mathbb{R}$, $\alpha_N > \cdots \alpha_1 > 0$

Eqs. (47) and (49) involve the one-dimensional Riesz fractional derivatives given by

$$^R\mathbf{D}_x^{\alpha_k} u(x) = \frac{1}{d_1(l,\alpha_k)} \int_{-\infty}^\infty \frac{(\Delta_z^m u)(z)}{|z|^{\alpha_k+1}} dz, \quad (50)$$

where $\alpha_k < l$, $k = 1, 2, \ldots, m$, and $(\Delta_z^m f)(z)$ is a finite difference of order m of a function $f(x)$ with a vector step $z \in \mathbb{R}$ and centered at the point $x \in \mathbb{R}$:

$$(\Delta_z^m f)(z) = \sum_{k=0}^m (-1)^k \frac{m!}{k!(m-k)!} f(x - kz).$$

The constant $d_n(m, \alpha)$ is defined by

$$d_1(m,\alpha) = \frac{\pi^{3/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(1/2+\alpha/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = \sum_{j=0}^m (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha.$$

Note that the hypersingular integral $\mathbf{D}_x^\alpha f(x)$ does not depend on the choice of $m > \alpha$.

Eq. (49) are solvable, and it particular solutions are given (see Theorem 5.25 in Kilbas et al., 2006) by the formula

$$u(x) = \frac{1}{\rho} \int_{-\infty}^{+\infty} G_\alpha(x-z) f(z) \, dz, \quad (51)$$

where

$$G_\alpha(x-z) = \frac{1}{\pi} \int_0^\infty \left(\sum_{k=1}^m C(\alpha_k)\lambda^{\alpha_k} + C_e^2 \lambda^2 \right)^{-1} \cos(\lambda|x|) \, d\lambda. \quad (52)$$

Let us consider one-dimensional Thomson (1848) problem (Landau and Lifshitz, 1986). We determine the deformation of an infinite elastic continuum, when a force is applied to a small region in it. We consider one-dimensional elastic media with power-law nonlocality that is described by the equation

$$-C_e^2 D_x^2 u(x) + C(\alpha) \, {}^R\mathbf{D}_x^\alpha u(x) = \frac{1}{\rho} f(x). \quad (53)$$

Note that Eq. (53) coincides with Eq. (31) for static case. If we consider the deformation at positions x, which are larger compare with the size of the region, where the force is applied, we can suppose that the force is applied at a point. In this case, we have

$$f(x) = f_0 \delta(x). \quad (54)$$

Then the deformation, which is a particular solution of Eq. (53), will be described by the equation

$$u(x) = \frac{f_0}{\pi\rho} \int_0^\infty \frac{\cos(\lambda|x|)}{C_e^2 \lambda^2 + C(\alpha)\lambda^\alpha} d\lambda. \quad (55)$$

If $0 < \alpha < 2$, the solution (55) corresponds to the fractional integral elasticity, and if $\alpha > 2$, then the solution (55) corresponds to the fractional gradient elasticity.

We can consider more general model of lattice with long-range interaction in $\mathbb{R}^3$, where all particles are displaced from its equilibrium in one direction, and the displacement of particles is described by a scalar field $u(x)$, where $x = |\mathbf{r}|$ and $\mathbf{r} \in \mathbb{R}^3$. The correspondent continuum equation of the fractional elasticity model is

$$-\Delta u(x) + \frac{C(\alpha)}{C_e^2} \left((-\Delta)^{\alpha/2} u\right)(x) = \frac{1}{C_e^2 \rho} f(x), \quad (56)$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian (Kilbas et al., 2006). The displacement vector $u(x)$ of the point force (54) has the following form

$$u(x) = \frac{1}{4\pi C_e^2 \rho} \frac{f_0}{x} \cdot C_{2,\alpha}(x), \quad (57)$$

where $x = |\mathbf{r}|$, and

$$C_{2,\alpha}(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda x)}{\lambda^2 + (C(\alpha)/C_e^2)\lambda^\alpha} d\lambda. \quad (58)$$

We note that the asymptotic behavior $x = |\mathbf{r}| \to 0$ of the scalar field $u(x)$ does not depend on the parameter α. Using (see Eq. (1) of Section 2.3 in the book Bateman and Erdelyi, 1954), we obtain the asymptotic behavior ($x \to \infty$) for $C_{2,\alpha}(x)$ with $\alpha < 2$ in the form

$$C_{2,\alpha}(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda x)}{\lambda^2 + (C(\alpha)/C_e^2)\lambda^\alpha} d\lambda \approx A_0(\alpha) \frac{1}{x^{2-\alpha}}$$
$$+ \sum_{k=1}^\infty A_k(\alpha) \frac{1}{x^{(2-\alpha)(k+1)}}, \quad (59)$$

where

$$A_0(\alpha) = \frac{2C_e^2}{\pi C(\alpha)} \Gamma(2-\alpha) \sin\left(\frac{\pi}{2}\alpha\right), \quad (60)$$

$$A_k(\alpha) = -\frac{2}{\pi}\left(\frac{C_e^2}{C(\alpha)}\right)^{k+1}\int_0^\infty z^{(2-\alpha)(k+1)-1}\sin(z)\,dz. \qquad (61)$$

As a result, we have in the framework the fractional elasticity, the displacement field of the point force in the infinite media with non-locality of power-law type is given by

$$u(x) \approx \frac{A_0(\alpha)}{4\pi C_e^2 \rho}\cdot\frac{f_0}{x^{3-\alpha}} \quad (0 < \alpha < 2) \qquad (62)$$

for the long distance $x \gg 1$.

In Figs. 1–3 we present some plots of the factor $C_{2,0}(x) = \exp(-x)$ and factors $C_{2,\alpha}(x)$ with $C(\alpha)/C_e^2 = 1$ for different orders of $0 < \alpha < 2$, i.e. for the fractional integral elasticity. The values of the factors $C_{2,\alpha}(x)$ and $C_{2,0}(x)$ are plotted along the Y-axis, and the values of the position $x = |\mathbf{r}|$ are plotted along the X-axis.

For the case $\alpha > 2$, i.e. for the fractional gradient elasticity, the asymptotic behavior $x = |\mathbf{r}| \to \infty$ of $u(x)$ does not depend on the parameter α. The asymptotic behavior of the displacement field $u(|\mathbf{r}|)$ for $|\mathbf{r}| \to 0$ is given by

$$u(x) \approx \frac{f_0\,\Gamma((3-\alpha)/2)}{2^\alpha\,\pi^2\sqrt{\pi}\,\rho\,C(\alpha)\,\Gamma(\alpha/2)}\cdot x^{\alpha-1}, \quad (2 < \alpha < 3), \qquad (63)$$

$$u(x) \approx \frac{f_0}{2\pi\,\alpha\,\rho\,(C_e^2)^{1-3/\alpha}\,C^{3/\alpha}(\alpha)\,\sin(3\pi/\alpha)}, \quad (\alpha > 3). \qquad (64)$$

Note that the function $C_{2,\alpha}$ for the fractional gradient case ($\alpha > 2$) has a maximum. If $\alpha = 4$ then we have the well-known case of gradient elasticity (Askes and Aifantis, 2011).

In Figs. 4–6 we present some plots of the factor $C_{2,0}(x) = \exp(-x)$ and factors $C_{2,\alpha}(x)$ with $C(\alpha)/C_e^2 = 1$ for different orders of $2 < \alpha < 6$, i.e. for the fractional gradient elasticity. The values of the factors $C_{2,\alpha}(x)$ and $C_{2,0}(x)$ are plotted along the Y-axis, and the values of the position $x = |\mathbf{r}|$ are plotted along the X-axis.
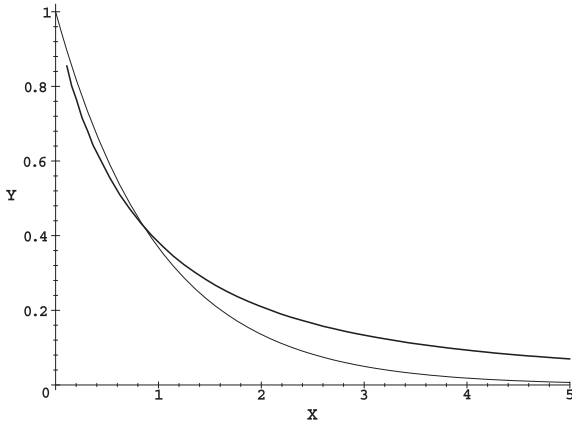

Fig. 1. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 0.7$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.
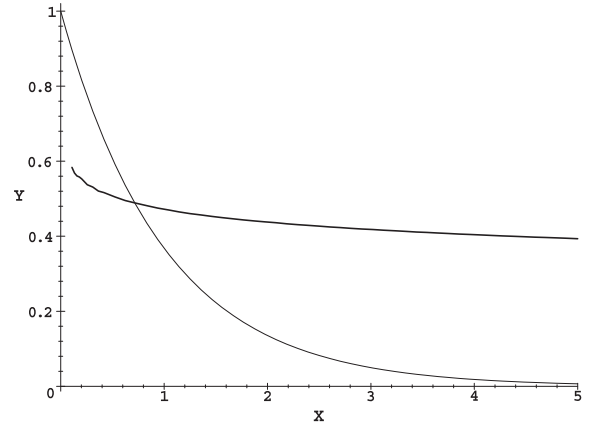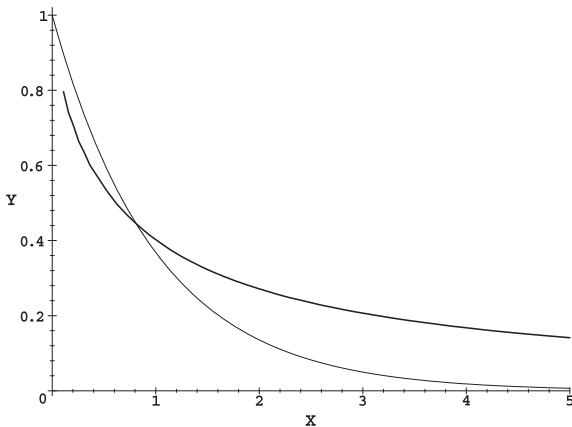

Fig. 3. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 1.8$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.
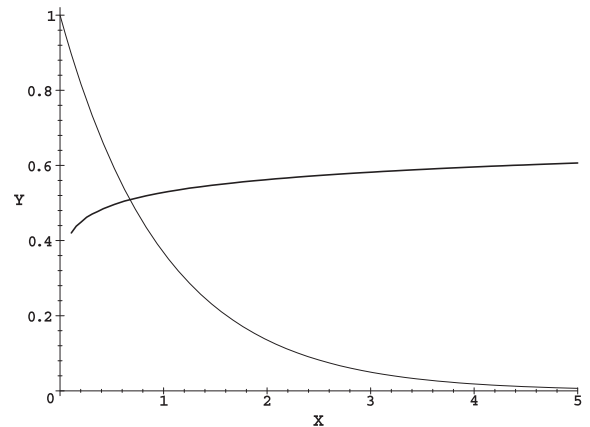

Fig. 2. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 1.1$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.

Fig. 4. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 2.2$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.
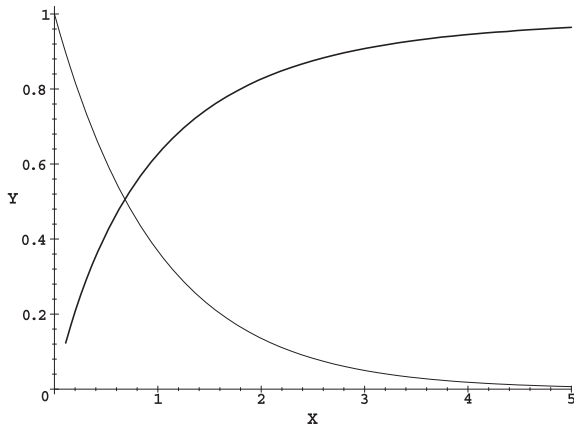
Fig. 5. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 3.6$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.

Fig. 6. Plot of the function $y = C_{2,\alpha}(x)$ with $\alpha = 5.9$ is drawn with black color and the factor $y = C_{2,0}(x) = \exp(-x)$ is drawn by gray color, where $x = |\mathbf{r}|$ and $C(\alpha)/C_e^2 = 1$.
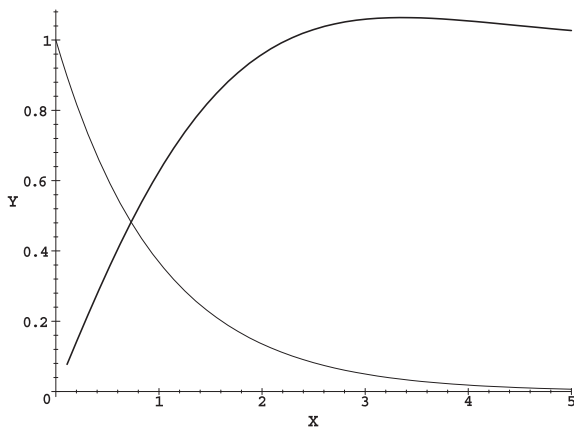
7. Conclusion

A lattice model with long-range interaction of Grünwald–Letnikov–Riesz type is suggested. In the continuum limit we derive continuum equations with spatial derivatives of non-integer order α. The correspondent constitutive relations describe fractional generalization of gradient elasticity for $\alpha > 2$ and fractional integral elasticity for $0 < \alpha < 2$. The suggested lattice model is considered as a microscopic model of the fractional non-local elastic continuum. We can note that a fractional nonlocal continuum model can be obtained from different microscopic or lattice models. The benefits of suggested formulation of fractional elasticity are following. Firstly, the Grünwald–Letnikov–Riesz derivatives in the fractional continuum equations are defined by fractional differences. It can be directly used in numerical simulations of fractional gradient and fractional integral elasticity models. Secondly, the suggested type of long-range interaction for lattice particles allows us to have united lattice model for the fractional

gradient and fractional integral elasticity. We assume that the suggested approach can be generalized for 3-dimensional case, for finite strains and plasticity. An extension of the suggested model for these cases can be realized by the methods suggested in Tarasov (2011a, see Sections 8.2, 8.14, 8.15) with some modifications. In addition, we note that the model of fractional gradient and integral elastic continuum has an analog in the plasma-like dielectric material with power-law spatial dispersion (Tarasov and Trujillo, 2013; Tarasov, 2013). Fractional models of complex material with power-law non-locality allows us to predict unusual properties of materials that are characterized by long-range inter-particle interactions. These materials can demonstrate a common or universal behavior in space by analogy with the universal behavior of low-loss dielectrics in time (Jonscher, 1977, 1996; Tarasov, 2008, 2009).

References

Askes, H., Aifantis, E.C., 2011. Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results. Int. J. Solids Struct. 48 (13), 1962–1990.

Askes, H., Sluys, L.J., 2002. A classification of higher-order strain-gradient models in damage mechanics. Arch. Appl. Mech. 73 (5), 448–465.

Askes, H., Suiker, A.S.J., Sluys, L.J., 2002. A classification of higher-order strain-gradient models – linear analysis. Arch. Appl. Mech. 72, 171–188.

Askes, H., Metrikine, A.V., Pichugin, A.V., Bennett, T., 2008. Four simplified gradient elasticity models for the simulation of dispersive wave propagation. Philos. Mag. 88 (28-29), 3415–3443.

Bateman, H., Erdelyi, A., 1954. Tables of Integral Transforms, vol. 1. McGraw-Hill, New York (Moscow, Nauka, 1969, in Russian).

Campa, A., Dauxois, T., Ruffo, S., 2009. Statistical mechanics and dynamics of solvable models with long-range interactions. Phys. Rep. 480 (3–6), 57–159.

Carpinteri, A., Mainardi, F. (Eds.), 1997. Fractals and Fractional Calculus in Continuum Mechanics. Springer, New York.

Carpinteri, A., Cornetti, P., Sapora, A., 2009. Static-kinematic fractional operators for fractal and non-local solids, Zeitschrift für Angewandte Mathematik und Mechanik. Appl. Math. Mech. 89 (3), 207–217.

Carpinteri, A., Cornetti, P., Sapora, A., 2011. A fractional calculus approach to nonlocal elasticity. Eur. Phys. J. Spec. Top. 193, 193–204.

Cornetti, P., Carpinteri, A., Sapora, A., Di Paola, M., Zingales, M., An explicit mechanical interpretation of Eringen non-local elasticity by means of fractional calculus. <http://www.dipmat.univpm.it/aimeta2009/Atti>.

Cottone, G., Di Paola, M., Zingales, M., 2009. Elastic waves propagation in 1D fractional non-local continuum. Physica E 42 (2), 95–103.

Cottone, G., Di Paola, M., Zingales, M., 2009. Fractional mechanical model for the dynamics of non-local continuum. In: Mastorakis, N., Sakellaris, J. (Eds.), Advances in Numerical Methods, Lecture Notes in Electrical Engineering, vol. 11. Springer, New York, pp. 389–423. http://dx.doi.org/10.1007/978-0-387-76483-2_33.

Dyson, F.J., 1969. Existence of a phase-transition in a one-dimensional Ising ferromagnet. Commun. Math. Phys. 12, 91–107.

Dyson, F.J., 1971. An Ising ferromagnet with discontinuous long-range order. Commun. Math. Phys. 21, 269–283.

Eringen, A.C., Edelen, D.G.B., 1972. On non-local elasticity. Int. J. Eng. Sci. 10, 233–248.

Frohlich, J., Israel, R., Lieb, E.H., Simon, B., 1978. Phase transitions and reflection positivity. I. General theory and long-range lattice model. Commun. Math. Phys. 62, 1–34.

Hilfer, R. (Ed.), 2000. Applications of Fractional Calculus in Physics. World Scientific, Singapore.

Jonscher, A.K., 1977. The universal dielectric response. Nature 267, 673–679.

Jonscher, A.K., 1996. Universal Relaxation Law. Chelsea Dielectrics, London.

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.

Klafter, J., Lim, S.C., Metzler, R. (Eds.), 2011. Fractional Dynamics. Recent Advances. World Scientific, Singapore.

Landau, L.D., Lifshitz, E.M., 1986. Theory of Elasticity. Oxford.

Laskin, N., Zaslavsky, G.M., 2006. Nonlinear fractional dynamics on a lattice with long-range interactions. Physica A 368, 38–54 (nlin.SI/0512010).

Lazopoulos, K.A., 2006. Non-local continuum mechanics and fractional calculus. Mech. Res. Commun. 33 (6), 753–757.

Luo, A.C.J., Afraimovich, V.S. (Eds.), 2010. Long-Range Interaction, Stochasticity and Fractional Dynamics. Springer, Berlin.

Mainardi, F., 2010. Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific, Singapore.

Maslov, V.P., 1976. Operator Methods. Mir, Moscow (Section 8, or Nauka, Moscow, 1973 in Russian, pp. 54–56).

Nakano, H., Takahashi, M., 1994. Quantum Heisenberg model with long-range ferromagnetic interactions. Phys. Rev. B 50, 10331–10334.

Nakano, H., Takahashi, M., 1995. Magnetic properties of quantum Heisenberg ferromagnets with long-range interactions. Phys. Rev. B 52, 6606–6610.

Ortigueira, M.D., Rivero, M., Trujillo, J.J., 2012. The incremental ratio based causal fractional calculus. Int. J. Bifurcation Chaos 22 (4), 1250078 (9p).

Riesz, M., 1949. L'intégrale de Riemann–Liouville et le probléme de Cauchy. Acta Math. 81 (1), 1–222 (in French).

Samko, S.G., Kilbas, A.A., Marichev, O.I., 1987. Integrals and Derivatives of Fractional Order and Applications. Nauka i Tehnika, Minsk (Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach, New York, 1993).

Sapora, A., Cornetti, P., Carpinteri, A., 2013. Wave propagation in nonlocal elastic continua modelled by a fractional calculus approach. Commun. Nonlinear Sci. Numer. Simul. 18 (1), 63–74.

Sousa, J.R., 2005. Phase diagram in the quantum XY model with long-range interactions. Eur. Phys. J. B 43, 93–96.

Tarasov, V.E., 2006a. Map of discrete system into continuous. J. Math. Phys. 47 (9), 092901. Available from: <arXiv:0711.2612>, 24p.

Tarasov, V.E., 2006b. Continuous limit of discrete systems with long-range interaction. J. Phys. A 39 (48), 14895–14910. Available from: <arXiv:0711.0826>.

Tarasov, V.E., 2008. Universal electromagnetic waves in dielectrics. J. Phys: Condens. Matter 20 (17), 175223. Available from: <arXiv:0907.2163>.

Tarasov, V.E., 2009. Fractional integro-differential equations for electromagnetic waves in dielectric media. Theor. Math. Phys. 158 (3), 355–359.

Tarasov, V.E., 2011a. Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, New York.

Tarasov, V.E., 2011b. Theoretical Physics Models with Integro-Differentiation of Fractional Order, IKI, RCD (in Russian).

Tarasov, V.E., 2013. Review of some promising fractional physical models. Int. J. Mod. Phys. B 27 (9), 1330005 (32p).

Tarasov, V.E., 2013. Power-law spatial dispersion from fractional Liouville equation. Phys. Plasmas 20 (10), 102110.

Tarasov, V.E., Trujillo, J.J., 2013. Fractional power-law spatial dispersion in electrodynamics. Ann. Phys. 334, 1–23.

Tarasov, V.E., Zaslavsky, G.M., 2006. Fractional dynamics of coupled oscillators with long-range interaction. Chaos 16 (2), 023110 (nlin.PS/0512013, 13p).

Uchaikin, V.V., 2013. Fractional Derivatives for Physicists and Engineers. Vol. 1. Background and Theory. Vol 2. Application. Springer.

Zaslavsky, G.M., Edelman, M., Tarasov, V.E., 2007. Dynamics of the chain of oscillators with long-range interaction: from synchronization to chaos. Chaos 17 (4), 043124. Available from: <arXiv:0707.3941>, 10p.