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ELECTROMAGNETIC FIELDS ON FRACTALS

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Fractals are measurable metric sets with non-integer Hausdorff dimensions. If electric and magnetic fields are defined on fractal and do not exist outside of fractal in Euclidean space, then we can use the fractional generalization of the integral Maxwell equations. The fractional integrals are considered as approximations of integrals on fractals. We prove that fractal can be described as a specific medium.

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1. Introduction

The theory of integrals and derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov.[1,2] Fractional analysis has found many applications in recent studies in mechanics and physics. The interest in fractional integrals and derivatives has been growing continuously during the last few years because of numerous applications. In a short period of time the list of applications has been expanding. It includes chaotic dynamics,[3,4] physics of fractal and complex media,[5–9] physical kinetics,[3,10–12] plasma physics,[13,14] astrophysics,[15] long-range dissipation,[16,17] non-Hamiltonian mechanics,[18,19] theory of long-range interactions.[20–22]

The new type of problem has increased rapidly in areas in which the fractal features of a process or the medium impose the necessity of using non-traditional tools. In order to use fractional derivatives and fractional integrals for media on fractal, we must use some continuous model.[8] We propose to describe the medium on fractal by a fractional continuous model,[8] where all characteristics and fields are defined everywhere in the volume but they follow some generalized equations, which are derived by using fractional integrals. In many problems the real fractal structure can be disregarded and the medium on fractal can be described by some fractional continuous mathematical model. The order of the fractional integral is equal to the fractal dimension. Fractional integrals can be considered as approximations of integrals on fractals.[23,24] In Ref. 24, authors proved that integrals on a net of fractals can be approximated by fractional integrals. In Ref. 18, we proved that fractional integrals can be considered as integrals over the space with non-integer dimension up to numerical factor. This interpretation follows from the well-known formulas for dimensional regularizations.[29]

In Sec. 2, a brief review of Hausdorff measure, Hausdorff dimension and integration on fractals suggested to fix notation and provide a convenient reference. The connection integration on fractals and fractional integration is discussed. In Sec. 3, the fractional electrodynamics on fractals is considered. Fractional generalization of the integral Maxwell equations is suggested. Finally, a short conclusion is given in Sec. 4.

2. Integration on Fractal and Fractional Integration

2.1. *Hausdorff measure and Hausdorff dimension*

Fractals are measurable metric sets with fractal Hausdorff dimension. The main property of fractal is non-integer Hausdorff dimension. Let us consider a brief review of Hausdorff measure and Hausdorff dimension to fix notation and provide a convenient reference.

Consider a measurable metric set (W, μ_H) with $W \subset \mathbb{R}^n$. The elements of W are denoted by $x, y, z, \ldots$, and represented by n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$ such that W is embedded in $\mathbb{R}^n$. The set W is restricted by the conditions: (a) W is closed; (b) W is unbounded; (c) W is regular (homogeneous, uniform) with its points randomly distributed.

The diameter of a subset $E \subset W \subset \mathbb{R}^n$ is

$$d(E) = \mathrm{diam}(E) = \sup\{d(x,y) : x, y \in E\}\,,$$

where $d(x,y)$ is a metric function of two points: x and $y \in W$.

Let us consider a set $\{E_i\}$ of subsets E_i such that $\dim(E_i) < \varepsilon\ \forall i$, and $W \subset \bigcup_{i=1}^{\infty} E_i$. Then, we define

$$\xi(E_i, D) = \omega(D)[\mathrm{diam}(E_i)]^D = \omega(D)[d(E_i)]^D \tag{1}$$

for non-empty subsets E_i of W. The factor $\omega(D)$ depends on the geometry of E_i, used for covering W. If $\{E_i\}$ is the set of all (closed or open) balls in W, then

$$\omega(D) = \frac{\pi^{D/2} 2^{-D}}{\Gamma(D/2+1)}\,. \tag{2}$$

The Hausdorff dimension D of a subset $E \subset W$ is defined[25–28] by

$$D = \dim_H(E) = \sup\{d \in R : \mu_H(E,d) = \infty\} = \inf\{d \in R : \mu_H(E,d) = 0\}\,. \tag{3}$$

The Hausdorff measure μ_H of a subset $E \subset W$ is defined[25–28] by

$$\mu_H(E,D) = \omega(D) \lim_{d(E_i)\to 0} \inf_{\{E_i\}} \sum_{i=1}^{\infty} [d(E_i)]^D\,. \tag{4}$$

Note that $\mu_H(\lambda E, D) = \lambda^D \mu_H(E,D)$, where $\lambda > 0$ and $\lambda E = \{\lambda x, x \in E\}$.

2.2. *Function and integrals on fractal*

Let us consider the functions on W:

$$f(x) = \sum_{i=1}^{\infty} \beta_i \chi_{E_i}(x) \,, \tag{5}$$

where χ_E is the characteristic function of E:

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \,. \end{cases}$$

The Lebesgue–Stieltjes integral for (5) is defined by

$$\int_W f \, d\mu = \sum_{i=1}^{\infty} \beta_i \mu_H(E_i) \,. \tag{6}$$

Therefore

$$\begin{aligned} \int_W f(x) d\mu_H(x) &= \lim_{d(E_i)\to 0} \sum_{E_i} f(x_i) \xi(E_i, D) \\ &= \omega(D) \lim_{d(E_i)\to 0} \sum_{E_i} f(x_i) [d(E_i)]^D \,. \end{aligned} \tag{7}$$

It is possible to divide $\mathbb{R}^n$ into parallelepipeds

$$\begin{aligned} E_{i_1 \cdots i_n} = \{(x_1, \ldots, x_n) \in W : x_j = (i_j - 1)\Delta x_j \\ + \alpha_j, 0 \leq \alpha_j \leq \Delta x_j, j = 1, \ldots, n\} \,. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} d\mu_H(x) &= \lim_{d(E_{i_1 \cdots i_n})\to 0} \xi(E_{i_1 \cdots i_n}, D) \\ &= \lim_{d(E_{i_1 \cdots i_n})\to 0} \prod_{j=1}^{n} (\Delta x_j)^{D/n} = \prod_{j=1}^{n} d^{D/n} x_j \,. \end{aligned} \tag{9}$$

The range of integration W can be parametrized by polar coordinates with $r = d(x, 0)$ and angle Ω. Then $E_{r,\Omega}$ can be thought of as spherically symmetric covering around a center at the origin. In the limit, the function $\xi(E_{r,\Omega}, D)$ gives

$$d\mu_H(r, \Omega) = \lim_{d(E_{r,\Omega})\to 0} \xi(E_{r,\Omega}, D) = d\Omega^{D-1} r^{D-1} dr \,. \tag{10}$$

Let us consider $f(x)$ that is symmetric with respect to some center $x_0 \in W$, i.e. $f(x) = \text{const.}$ for all x such that $d(x, x_0) = r$ for arbitrary values of r. Then the transformation

$$W \to W' : x \to x' = x - x_0 \tag{11}$$

can be performed to shift the center of symmetry. Since W is not a linear space, (11) need not be a map of W onto itself. The map (11) is measure-preserving. Then the integral over a D-dimensional metric space is

$$\int_W f\, d\mu_H = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty f(r) r^{D-1}\, dr\,. \tag{12}$$

This integral is known in the theory of the fractional calculus.[1] The right Riemann–Liouville fractional integral is

$$I_-^D f(z) = \frac{1}{\Gamma(D)} \int_z^\infty (x-z)^{D-1} f(x) dx\,. \tag{13}$$

Equation (12) is reproduced by

$$\int_W f\, d\mu_H = \frac{2\pi^{D/2}\Gamma(D)}{\Gamma(D/2)} I_-^D f(0)\,. \tag{14}$$

Relation (14) connects the integral on fractal with integral of fractional order. This result permits to apply different tools of the fractional calculus[1] for the fractal medium. As a result, the fractional integral can be considered as an integral on fractal (fractional Hausdorff dimension set) up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$.

Note that the interpretation of fractional integration is connected with fractional dimension.[18] This interpretation follows from the well-known formulas for dimensional regularizations.[29,30] The fractional integral can be considered as an integral in the fractional dimension space up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$. In Ref. 23 was proved that the fractal spacetime approach is technically identical with dimensional regularization.

The integral defined in (7) satisfies the translational invariance property:

$$\int_W f(x+x_0) d\mu_H(x) = \int_W f(x) d\mu_H(x) \tag{15}$$

since $d\mu_H(x-x_0) = d\mu_H(x)$ as a consequence of homogeneity. The integral (7) satisfies the scaling property:

$$\int_W f(\lambda x) d\mu_H(x) = \lambda^{-D} \int_W f(x) d\mu_H(x) \tag{16}$$

since $d\mu_H(x/\lambda) = \lambda^{-D}\, d\mu_H(x)$.

2.3. *Multi-variable integration on fractal*

The integral in (12) is defined for a single variable but not multiple variables. It is only useful for integrating spherically symmetric functions. We consider multiple variables by using the product spaces and product measures.

Let us consider a collection of $n = 3$ measure spaces (W_k, μ_k, D) with $k = 1, 2, 3$, and form a Cartesian product of the sets W_k producing the space $W = W_1 \times$

$W_2 \times W_3$. The definition of product measures and application of Fubini's theorem provides a measure for the product set $W = W_1 \times W_2 \times W_3$ as

$$(\mu_1 \times \mu_2 \times \mu_3)(W) = \mu_1(W_1)\mu_2(W_2)\mu_3(W_3)\,. \tag{17}$$

The integration over a function f on the product space is

$$\int f(\mathbf{r})d(\mu_1 \times \mu_2 \times \mu_3) = \iiint f(x_1, x_2, x_3)d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3)\,. \tag{18}$$

In this form, the single-variable measure from (12) may be used for each coordinate x_k, which has an associated dimension α_k:

$$d\mu_k(x_k) = \frac{2\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)}|x_k|^{\alpha_k-1}dx_k\,, \qquad k = 1, 2, 3\,. \tag{19}$$

The total dimension of $W = W_1 \times W_2 \times W_3$ is $D = \alpha_1 + \alpha_2 + \alpha_3$.

Let us reproduce the result for the single-variable integration (12), from the product space $W_1 \times W_2 \times W_3$. We take a spherically symmetric function $f(\mathbf{r}) = f(x_1, x_2, x_3) = f(r)$, where $r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$ and to perform the integration in spherical coordinates (r, ϕ, θ), we use

$$\int_0^{\pi/2} \sin^{\mu-1} x \, \cos^{\nu-1} x \, dx = \frac{\Gamma(\mu/2)\Gamma(\nu/2)}{2\Gamma((\mu+\nu)/2)} \tag{20}$$

where $\mu > 0$, $\nu > 0$. Then Eq. (18) becomes

$$\int d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3)f(r) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\int f(r)r^{D-1}\,dr\,. \tag{21}$$

This equation describes integration over a spherically symmetric function in a D-dimensional space and reproduces the result (12).

3. Electrodynamics on Fractal

3.1. *Electric charge of fractal set*

Let us consider the electric charge that is distributed on the measurable metric set W with the fractional Hausdorff dimension D. Suppose that the density of charge distribution is described by the function $\rho(\mathbf{r}, t)$. In this case, the total charge is defined by

$$Q_D(W) = \int_W \rho(\mathbf{r}, t)dV_D\,, \quad dV_D = d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3) = c_3(D, \mathbf{r})dV_3\,, \tag{22}$$

where $dV_3 = dx\,dy\,dz$ for Cartesian coordinates, $\dim_H(W) = D = \alpha_1 + \alpha_2 + \alpha_3$, and

$$c_3(D, \mathbf{r}) = \frac{8\pi^{D/2}|x|^{\alpha_1-1}|y|^{\alpha_2-1}|z|^{\alpha_3-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\,. \tag{23}$$

As a result, we get Riemann–Liouville fractional integral[1] up to numerical factor $8\pi^{D/2}$. Note that the final equations that relate the physical variables have the form that are independent of numerical factor in the function $c_3(D,\mathbf{r})$. However, the dependence of $\mathbf{r}$ is important to these equations.

Equation (22) describes the charge that is distributed in the volume and has the fractal dimension D by fractional integrals. There are many different definitions of fractional integrals.[1] For the Riemann–Liouville fractional integral, the function $c_3(D,\mathbf{r})$ is

$$c_3(D,\mathbf{r}) = \frac{|x|^{\alpha_1-1}|y|^{\alpha_2-1}|z|^{\alpha_3-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\,, \tag{24}$$

where x, y, z are Cartesian's coordinates and $D = \alpha_1 + \alpha_2 + \alpha_3$, $0 < D \leq 3$. Note that for $D = 2$, we have the distribution in the volume. In general, this case is not equivalent to the distribution on the two-dimensional surface. For $\rho(\mathbf{r}) = \rho(|\mathbf{r}|)$, we can use the fractional integrals with

$$c_3(D,\mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}|\mathbf{r}|^{D-3}\,. \tag{25}$$

If we consider the ball region $W = \{\mathbf{r} : |\mathbf{r}| \leq R\}$, and stationary spherically symmetric distribution of charged particles ($\rho(\mathbf{r},t) = \rho(r)$), then

$$Q_D(R) = 4\pi\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}\int_0^R \rho(r)r^{D-1}\,dr\,.$$

For the homogeneous case, $\rho(r,t) = \rho_0$, and

$$Q_D(R) = 4\pi\rho_0\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}\,\frac{R^D}{D} \sim R^D\,.$$

The distribution of charged particles is homogeneous if all regions W and W' with equal volumes $V_D(W) = V_D(W')$ have the same total charges on these regions $Q_D(W) = Q_D(W')$.

For charged particles that are distributed with a constant density over a fractal with Hausdorff dimension D, the electric charge Q satisfies the scaling law $Q(R) \sim R^D$, whereas for a regular n-dimensional Euclidean object we have $Q(R) \sim R^n$.

3.2. *Electric current for fractal*

For charged particles with density $\rho(\mathbf{r},t)$ flowing with velocity $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$, the resulting current density $\mathbf{J}(\mathbf{r},t)$ is

$$\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)\,.$$

The electric current $I(S)$ is defined as the flux of electric charge. Measuring the field $\mathbf{J}(\mathbf{r},t)$ passing through a surface $S = \partial W$ gives

$$I(S) = \Phi_J(S) = \int_S (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_2)\,, \tag{26}$$

where $d\mathbf{S_2} = dS_2\mathbf{n}$ is a differential unit of area pointing perpendicular to the surface S, and the vector $\mathbf{n} = n_k\mathbf{e}_k$ is a vector of normal. The fractional generalization of (26) is

$$I(S) = \int_S (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_d)\,, \tag{27}$$

where

$$d\mathbf{S}_d = c_2(d,\mathbf{r})d\mathbf{S}_2\,, \qquad c_2(d,\mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)}|\mathbf{r}|^{d-2}\,. \tag{28}$$

Note that $c_2(2,\mathbf{r}) = 1$ for $d = 2$. The boundary ∂W has dimension d. In general, the dimension d is not equal to 2 and is not equal to $(D-1)$.

3.3. *Charge conservation for fractal*

The electric charge has a fundamental property established by numerous experiments: the velocity of charge change in region W bounded by the surface $S = \partial W$ is equal to the flux of charge through this surface. This is known as the law of charge conservation:

$$\frac{dQ(W)}{dt} = -I(S)\,,$$

or, in the form

$$\frac{d}{dt}\int_W \rho(\mathbf{r},t)dV_D = -\oint_{\partial W}(\mathbf{J}(\mathbf{r},t), d\mathbf{S}_d)\,. \tag{29}$$

In particular, when the surface $S = \partial W$ is fixed, we can write

$$\frac{d}{dt}\int_W \rho(\mathbf{r},t)dV_D = \int_W \frac{\partial\rho(\mathbf{r},t)}{\partial t}dV_D\,. \tag{30}$$

Using the fractional generalization of the Gauss's theorem (see the Appendix), we get

$$\oint_{\partial W}(\mathbf{J}(\mathbf{r},t), d\mathbf{S}_d) = \int_W c_3^{-1}(D,\mathbf{r})\frac{\partial}{\partial x_k}(c_2(d,\mathbf{r})J_k(\mathbf{r},t))dV_D\,. \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (29) gives

$$c_3(D,\mathbf{r})\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \frac{\partial}{\partial x_k}(c_2(d,\mathbf{r})J_k(\mathbf{r},t)) = 0\,. \tag{32}$$

As a result, we obtain the law of charge conservation in differential form (32). This equation can be considered as a continuity equation for fractal.

3.4. *Electric field and Coulomb's law*

For a continuous stationary distribution $\rho(\mathbf{r}')$, the electric field at a point $\mathbf{r}$ is defined by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_W \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \rho(\mathbf{r}') dV_3' \,, \tag{33}$$

where ε_0 is a fundamental constant called the permittivity of free space. For Cartesian's coordinates $dV_3' = dx'dy'dz'$. The fractional generalization of (33) is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_W \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \rho(\mathbf{r}') dV_D' \,, \tag{34}$$

where $dV_D' = c_3(D, \mathbf{r}')dV_3'$. Equation (34) can be considered as Coulomb's law for a stationary distribution of electric charges on fractal.

3.5. *Gauss's law for fractal*

The Gauss's law tells us that the total flux $\Phi_E(S)$ of the electric field $\mathbf{E}$ through a closed surface $S = \partial W$ is proportional to the total electric charge $Q(W)$ inside the surface:

$$\Phi_E(\partial W) = \frac{1}{\varepsilon_0} Q(W) \,. \tag{35}$$

The electric flux for the surface $S = \partial W$ is

$$\Phi_E(S) = \int_S (\mathbf{E}, d\mathbf{S}_2) \,,$$

where $\mathbf{E}(\mathbf{r}, t)$ is the electric field vector, and $d\mathbf{S}_2$ is a differential unit of area pointing perpendicular to the surface S.

For the distribution on fractal, the Gauss's law (35) states

$$\int_S (\mathbf{E}, d\mathbf{S}_2) = \frac{1}{\varepsilon_0} \int_W \rho(\mathbf{r}, t) dV_D \,, \tag{36}$$

where $\rho(\mathbf{r}, t)$ is the density of electric charge that is distributed on fractal, $dV_D = c_3(D, \mathbf{r})dV_3$, and ε_0 is the permittivity of free space.

If $\rho(\mathbf{r}, t) = \rho(r)$, and $W = \{\mathbf{r} : |\mathbf{r}| \le R\}$, then

$$Q(W) = 4\pi \int_0^R \rho(r) c_3(D, \mathbf{r}) r^2 \, dr = 4\pi \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R \rho(r) r^{D-1} \, dr \,. \tag{37}$$

For the sphere $S = \partial W = \{\mathbf{r} : |\mathbf{r}| = R\}$,

$$\Phi_E(\partial W) = 4\pi R^2 E(R) \,. \tag{38}$$

Substituting (37) and (38) into (35), we get

$$E(R) = \frac{2^{3-D}\Gamma(3/2)}{\varepsilon_0 R^2 \Gamma(D/2)} \int_0^R \rho(r) r^{D-1} \, dr \,.$$

For homogeneous ($\rho(\mathbf{r}) = \rho$) distribution,

$$E(R) = \rho \frac{2^{3-D}\Gamma(3/2)}{\varepsilon_0 D \Gamma(D/2)} R^{D-2} \sim R^{D-2} \,.$$

3.6. *Magnetic field and Biot–Savart law*

The Biot–Savart law relates magnetic fields to the currents, which are their sources. For a continuous distribution, the law is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_W \frac{[\mathbf{J}(\mathbf{r}'), \mathbf{r} - \mathbf{r}']}{|\mathbf{r} - \mathbf{r}'|^3} dV_3' \,, \tag{39}$$

where $[,]$ is a vector product, $\mathbf{J}$ is the current density, μ_0 is the permeability of free space. The fractional generalization of Eq. (39) is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_W \frac{[\mathbf{J}(\mathbf{r}'), \mathbf{r} - \mathbf{r}']}{|\mathbf{r} - \mathbf{r}'|^3} dV_D' \,. \tag{40}$$

This equation is Biot–Savart law written for a steady current with fractal distribution of electric charges. The law (40) can be used to find the magnetic field produced by any distribution of steady currents on fractal.

3.7. *Ampere's law for fractal*

The magnetic field in space around an electric current is proportional to the electric current, which serves as its source. In the case of static electric field, the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. The Ampere's law is equivalent to the steady state of the integral Maxwell equation in free space, and relates the spatially varying magnetic field $\mathbf{B}(\mathbf{r})$ to the current density $\mathbf{J}(\mathbf{r})$.

Note that, as mentioned in Ref. 13, Liouville, who was one of the pioneers in developing fractional calculus, was inspired by the problem of fundamental force law in Ampere's electrodynamics and used fractional differential equation in that problem.

The Ampere's law states that the line integral of the magnetic field $\mathbf{B}$ along the closed path L around a current given in MKS by

$$\oint_L (\mathbf{B}, d\mathbf{l}) = \mu_0 I(S) \,,$$

where $d\mathbf{l}$ is the differential length element. For the distribution of particles on the fractal, $I(S)$ is defined by (27). For the cylindrically symmetric distribution,

$$I(S) = 2\pi \int_0^R J(r) c_2(d, \mathbf{r}) r \, dr = 4\pi \frac{2^{2-d}}{\Gamma(d/2)} \int_0^R J(r) r^{d-1} \, dr \,,$$

where we use $c_2(d, \mathbf{r})$ from Eq. (28). For the circle $L = \partial W = \{\mathbf{r} : |\mathbf{r}| = R\}$, we get

$$\oint_L (\mathbf{B}, d\mathbf{l}) = 2\pi R B(R) \,.$$

As a result,

$$B(R) = \frac{\mu_0 2^{2-d}}{R\Gamma(d/2)} \int_0^R J(r) r^{d-1} \, dr \,.$$

For the homogeneous case, $J(r) = J_0$ and

$$B(R) = J_0 \frac{\mu_0 2^{2-d}}{d\Gamma(d/2)} R^{d-1} \sim R^{d-1} .$$

3.8. *Fractional integral Maxwell equations*

Let us consider the fractional integral Maxwell equations. The Maxwell equations are the set of fundamental equations for electric and magnetic fields. The equations can be expressed in integral form known as Gauss's law, Faraday's law, the absence of magnetic monopoles, and Ampere's law with displacement current. In MKS, these become

$$\oint_S (\mathbf{E}, d\mathbf{S}_2) = \frac{1}{\varepsilon_0} \int_W \rho \, dV_D \,,$$

$$\oint_L (\mathbf{E}, d\mathbf{l}_1) = -\frac{\partial}{\partial t} \int_S (\mathbf{B}, d\mathbf{S}_2) \,,$$

$$\oint_S (\mathbf{B}, d\mathbf{S}_2) = 0 \,,$$

$$\oint_L (\mathbf{B}, d\mathbf{l}_1) = \mu_0 \int_S (\mathbf{J}, d\mathbf{S}_d) + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S (\mathbf{E}, d\mathbf{S}_2) \,.$$

Let us consider the fields that are defined on fractal only. The hydrodynamic and thermodynamics fields can be defined in the fractal media.[8,9] Suppose that the electromagnetic field can be defined on fractal as an approximation of some real case with fractal medium. If the electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic fields $\mathbf{B}(\mathbf{r}, t)$ can be defined on fractal and do not exist outside of fractal in Euclidean space E^3, then we must use the fractional generalization of the integral Maxwell equations in the form:

$$\oint_S (\mathbf{E}, d\mathbf{S}_d) = \frac{1}{\varepsilon_0} \int_W \rho \, dV_D \,, \tag{41}$$

$$\oint_L (\mathbf{E}, d\mathbf{l}_\gamma) = -\frac{\partial}{\partial t} \int_S (\mathbf{B}, d\mathbf{S}_d) \,, \tag{42}$$

$$\oint_S (\mathbf{B}, d\mathbf{S}_d) = 0 \,, \tag{43}$$

$$\oint_L (\mathbf{B}, d\mathbf{l}_\gamma) = \mu_0 \int_S (\mathbf{J}, d\mathbf{S}_d) + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S (\mathbf{E}, d\mathbf{S}_d) \,. \tag{44}$$

The fractional integrals are considered as approximation of integrals on fractals.[23,24]

Using the fractional generalization of Stokes's and Gauss's theorems (see the Appendix), we can rewrite Eq. (44) in the form

$$\int_W c_3^{-1}(D, \mathbf{r}) \operatorname{div}(c_2(d, \mathbf{r})\mathbf{E}) dV_D = \frac{1}{\varepsilon_0} \int_W \rho \, dV_D \,, \tag{45}$$

$$\int_S c_2^{-1}(d,\mathbf{r})(\mathrm{curl}(c_1(\gamma,\mathbf{r})\mathbf{E}), d\mathbf{S}_d) = -\frac{\partial}{\partial t}\int_S (\mathbf{B}, d\mathbf{S}_d)\,, \tag{46}$$

$$\int_W c_3^{-1}(D,\mathbf{r})\,\mathrm{div}(c_2(d,\mathbf{r})\mathbf{B})dV_d = 0\,, \tag{47}$$

$$\int_S c_2^{-1}(d,\mathbf{r})(\mathrm{curl}(c_1(\gamma,\mathbf{r})\mathbf{B}), d\mathbf{S}_d) = \mu_0\int_S (\mathbf{J}, d\mathbf{S}_d) + \varepsilon_0\mu_0\frac{\partial}{\partial t}\int_S (\mathbf{E}, d\mathbf{S}_d)\,. \tag{48}$$

As a result, we obtain

$$\mathrm{div}(c_2(d,\mathbf{r})\mathbf{E}) = \frac{1}{\varepsilon_0}c_3(D,\mathbf{r})\rho\,, \tag{49}$$

$$\mathrm{curl}(c_1(\gamma,\mathbf{r})\mathbf{E}) = -c_2(d,\mathbf{r})\frac{\partial}{\partial t}\mathbf{B}\,, \tag{50}$$

$$\mathrm{div}(c_2(d,\mathbf{r})\mathbf{B}) = 0\,, \tag{51}$$

$$\mathrm{curl}(c_1(\gamma,\mathbf{r})\mathbf{B}) = \mu_0 c_2(d,\mathbf{r})\mathbf{J} + \varepsilon_0\mu_0 c_2(d,\mathbf{r})\frac{\partial \mathbf{E}}{\partial t}\,. \tag{52}$$

Note that the law of absence of magnetic monopoles for the fractal leads us to $\mathrm{div}(c_2(d,\mathbf{r})\mathbf{B}) = 0$. It can be rewritten as

$$\mathrm{div}\,\mathbf{B} = -(\mathbf{B},\, \mathrm{grad} c_2(d,\mathbf{r}))\,.$$

In the general case ($d \neq 2$), the vector $\mathrm{grad}(c_2(d,\mathbf{r}))$ is not equal to zero and the magnetic field satisfies $\mathrm{div}\,\mathbf{B} \neq 0$. If $d = 2$, we have $\mathrm{div}(\mathbf{B}) \neq 0$ only for non-solenoidal field $\mathbf{B}$. Therefore the magnetic field on the fractal is similar to the non-solenoidal field. As a result, the field on fractal can be considered as a field with some "fractional magnetic monopole" $q_m \sim (\mathbf{B}, \nabla c_2)$.

3.9. *Fractal as effective medium*

The Maxwell equations (49)–(52) on fractal can be considered as the equations for medium

$$\mathrm{div}(\mathbf{D}) = \rho^{\mathrm{eff}}_{\mathrm{free}}\,, \tag{53}$$

$$\mathrm{curl}(\mathbf{E}^{\mathrm{eff}}) = -\frac{\partial}{\partial t}\mathbf{B}^{\mathrm{eff}}\,, \tag{54}$$

$$\mathrm{div}(\mathbf{B}^{\mathrm{eff}}) = 0\,, \tag{55}$$

$$\mathrm{curl}(\mathbf{H}) = \mathbf{J}^{\mathrm{eff}} + \frac{\partial \mathbf{D}}{\partial t}\,. \tag{56}$$

The effective Maxwell equations (53)–(56) prove that fractal creates some polarization and magnetization. In the equations, we use some effective fields

$$\mathbf{E}^{\mathrm{eff}}(\mathbf{r},t) = c_1(\gamma,\mathbf{r})\mathbf{E}(\mathbf{r},t)\,, \qquad \mathbf{B}^{\mathrm{eff}}(\mathbf{r},t) = c_2(d,\mathbf{r})\mathbf{B}(\mathbf{r},t)\,. \tag{57}$$

The fields $\mathbf{E}^{\mathrm{eff}}$ and $\mathbf{B}^{\mathrm{eff}}$ mean that electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of particles are changed by fractal.

Equations (53)–(56) have the effective free charge and current densities:

$$\rho^{\text{eff}}_{\text{free}}(\mathbf{r},t) = c_3(D,\mathbf{r})\rho(\mathbf{r},t), \qquad \mathbf{J}^{\text{eff}}(\mathbf{r},t) = c_2(d,\mathbf{r})\mathbf{J}(\mathbf{r},t)\,. \tag{58}$$

We can interpret the existence of $\rho^{\text{eff}}_{\text{free}}$ and $\mathbf{J}^{\text{eff}}$ in the equations as an effect of change of the free electric charge and current densities by fractal. This change exists in addition to the effect of appearance the dipole charges and polarization or magnetization currents. The fractal can be considered as a medium that has the electrical and magnetic permittivities in the form

$$\varepsilon = c_2(d,\mathbf{r})c_1^{-1}(\gamma,\mathbf{r}), \qquad \mu = c_2(d,\mathbf{r})c_1^{-1}(\gamma,\mathbf{r})\,. \tag{59}$$

The fields $\mathbf{D}$ and $\mathbf{H}$ are related to $\mathbf{E}^{\text{eff}}$ and $\mathbf{B}^{\text{eff}}$ by the usual equations:

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}^{\text{eff}}, \qquad \mathbf{H} = \frac{1}{\mu\mu_0}\mathbf{B}^{\text{eff}}\,. \tag{60}$$

Note that the continuity equation (32) for fractal can be presented by

$$\frac{\partial\rho^{\text{eff}}(\mathbf{r},t)}{\partial t} + \operatorname{div}(\mathbf{J}^{\text{eff}}(\mathbf{r},t)) = 0\,. \tag{61}$$

As a result, the fractal can be considered as a specific medium that changes the fields, free charges and currents in addition to the creation of polarization and magnetization.

4. Conclusion

Fractals are measurable metric sets with non-integer Hausdorff dimensions. We consider the electric and magnetic fields that are defined on fractal and does not exist outside of fractal in Euclidean space. For charged particles that are distributed with a constant density over a fractal with Hausdorff dimension D, the electric charge Q satisfies the scaling law $Q(R) \sim R^D$, whereas for a regular n-dimensional Euclidean object we have $Q(R) \sim R^n$. This property can be used to measure the fractal Hausdorff dimension D.

The fractional integrals can be used to describe electromagnetic fields on fractals. These integrals are considered as approximations of integrals on fractals. The fractional generalizations of integral Maxwell equations for fractal set are derived. The magnetic field on fractal can be considered as a field with some "fractional magnetic monopole".

We can interpret the equations for electromagnetic fields on fractal as an effect of creation of some polarization and magnetization by fractal. Moreover, the electromagnetic fields are also changed by fractal. From the generalized Maxwell equations, we can see the effect of change of the free electric charge and current densities by fractal. This change exists in addition to the effect of appearance the dipole charges and polarization or magnetization currents. The electrical permittivity ε and the magnetic permittivity μ of fractal are defined by the Hausdorff measure and dimension of fractal.

Appendix A. Fractional Gauss's Theorem

Let us derive the fractional generalization of the Gauss's theorem

$$\int_{\partial W} (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_2) = \int_W \operatorname{div}(\mathbf{J}(\mathbf{r},t)) dV_3 \,, \tag{A.1}$$

where the vector $\mathbf{J}(\mathbf{r},t) = J_k \mathbf{e}_k$ is a field, and

$$\operatorname{div}(\mathbf{J}) = \partial \mathbf{J}/\partial \mathbf{r} = \partial J_k / \partial x_k \,.$$

Here, we mean the sum on the repeated index k from 1 to 3. Using

$$d\mathbf{S}_d = c_2(d,\mathbf{r}) d\mathbf{S}_2 \,, \qquad c_2(d,\mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}|^{d-2} \,,$$

we get

$$\int_{\partial W} (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_d) = \int_{\partial W} c_2(d,\mathbf{r}) (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_2) \,.$$

Note that $c_2(2,\mathbf{r}) = 1$ for $d = 2$. Using (A.1), we get

$$\int_{\partial W} c_2(d,\mathbf{r}) (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_2) = \int_W \operatorname{div}(c_2(d,\mathbf{r}) \mathbf{J}(\mathbf{r},t)) dV_3 \,.$$

The relation $dV_3 = c_3^{-1}(D,\mathbf{r}) dV_D$ allows us to derive the fractional generalization of the Gauss's theorem:

$$\int_{\partial W} (\mathbf{J}(\mathbf{r},t), d\mathbf{S}_d) = \int_W c_3^{-1}(D,\mathbf{r}) \operatorname{div}(c_2(d,\mathbf{r}) \mathbf{J}(\mathbf{r},t)) dV_D \,.$$

Analogously, we can get the fractional generalization of Stokes's theorem in the form

$$\oint_L (\mathbf{E}, d\mathbf{l}_\gamma) = \int_S c_2^{-1}(d,\mathbf{r}) (\operatorname{curl}(c_1(\gamma,\mathbf{r}) \mathbf{E}), d\mathbf{S}_d) \,,$$

where

$$c_1(\gamma,\mathbf{r}) = \frac{2^{1-\gamma} \Gamma(1/2)}{\Gamma(\gamma/2)} |\mathbf{r}|^{\gamma-1} \,.$$

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