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MULTIPOLE MOMENTS OF FRACTAL DISTRIBUTION OF CHARGES

VASILY E. TARASOV

Skobeltsyn Institute of Nuclear Physics,
Moscow State University, Moscow 119992, Russia

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In this paper we consider the electric multipole moments of fractal distribution of charges. To describe fractal distribution, we use the fractional integrals. The fractional integrals are considered as approximations of integrals on fractals. In the paper we compute the electric multipole moments for homogeneous fractal distribution of charges.

Keywords: Multipole moment; fractal distribution; fractional integrals.

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1. Introduction

Integrals and derivatives of fractional order have found many applications in recent studies in science. The interest in fractals and fractional analysis has been growing continually in the last few years. Fractional derivatives and integrals have numerous applications: kinetic theories;[1–3] statistical mechanics;[4–6] dynamics in complex media;[7–11] electrodynamics[12–15] and many others. The new type of problem has increased rapidly in areas in which the fractal features of a process or the medium imposes the necessity of using non-traditional tools in smooth physical models. In order to use fractional derivatives and fractional integrals for fractal distribution, we must use some continuous medium model.[7,8] We propose to describe the fractal distribution by a fractional continuous medium,[7,8] where all characteristics and fields are defined everywhere in the volume but they follow some generalized equations which are derived by using fractional integrals. In many problems the real fractal structure of a medium can be disregarded and the fractal distribution can be replaced by some fractional continuous mathematical model. By smoothing of microscopic characteristics over the physically infinitesimal volume, we transform the initial fractal distribution into a fractional continuous model[7,8] that uses the fractional integrals. The order of the fractional integral is equal to the fractal dimension of distribution. The fractional integrals allow us to take into account the fractality of the media.[7] Fractional integrals are considered as approximations of integrals on fractals.[16] It was proved that integrals on the net of fractals can be approximated by fractional integrals.[16] In Refs. 4 and 5, we proved that fractional integrals can be considered as integrals over the space with the fractional dimension up to a numerical factor.

In this paper, we consider electric multipole moments of the fractal distribution of charges. Fractal distribution is described by the fractional continuous model.[7–10] In the general case, the fractal distribution cannot be considered as a continuous distribution. There are domains that are not filled by particles. We suggest[7] to consider the fractal distribution as a special (fractional) continuous distribution. We use the procedure of replacement of the distribution with fractal dimension by some continuous model that uses fractional integrals. The suggested procedure can be considered as a fractional generalization of the Christensen approach.[17]

In Sec. 2, the density of electric charge for fractal distribution is considered. In Sec. 3, we consider the electric multipole expansion. In Sec. 4, the examples of electric dipole moment for fractal distribution of charges are derived. In Sec. 5, we consider the electric quadrupole moment of fractal distribution of charges. In Sec. 6, the examples of electric quadrupole moments of charged fractal parallelepiped are computed. In Sec. 7, the examples of electric quadrupole moments of charged fractal ellipsoid are computed. Finally, a short conclusion is given in Sec. 8.

2. Electric Charge of Fractal Distribution

Let us consider a fractal distribution of charges. For example, we can assume that charged particles with a constant density are distributed over the fractal. In this case, the number of particles $N(R)$ enclosed in a volume with characteristic size R satisfies the scaling law

$$N(R) \sim R^D , \tag{1}$$

whereas for a regular n-dimensional Euclidean object we have $N(R) \sim R^n$.

For charged particles with number density $n(\mathbf{r}, t)$, we have that the charge density can be defined by

$$\rho(\mathbf{r}, t) = qn(\mathbf{r}, t) , \tag{2}$$

where q is the charge of particle (for example, q is an electron charge), and $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. The total charge of region W is then given by the integral

$$Q(W) = \int_W \rho(\mathbf{r}, t) dV_3 = q \int_W n(\mathbf{r}, t) dV_3 , \tag{3}$$

i.e., $Q(W) = qN(W)$, where $N(W)$ is a number of particles in the region W. The fractional generalization of this equation can be written in the following form:

$$Q(W) = \int_W \rho(\mathbf{r}, t) dV_D = q \int_W n(\mathbf{r}, t) dV_D , \tag{4}$$

where D is a fractal dimension of the distribution, and dV_D is an element of D-dimensional volume such that

$$dV_D = C_3(D, \mathbf{r}) dV_3 . \tag{5}$$

For the Riesz definition of the fractional integral,[18] the function $C_3(D, \mathbf{r})$ is defined by the relation

$$C_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}|\mathbf{r}|^{D-3}. \tag{6}$$

The initial points of the fractional integral[18] are set to zero. The numerical factor in Eq. (6) has this form in order to derive the usual integral in the limit $D \to (3-0)$. Note that the usual numerical factor $\gamma_3^{-1}(D) = \Gamma(1/2)/(2^D\pi^{3/2}\Gamma(D/2))$, which is used in Ref. 18, leads to $\gamma_3^{-1}(3-0) = \Gamma(1/2)/(2^3\pi^{3/2}\Gamma(3/2)) = 1/(4\pi^{3/2})$ in the limit $D \to (3-0)$.

For the Riemann–Liouville fractional integral,[18] the function $C_3(D, \mathbf{r})$ is defined by

$$C_3(D, \mathbf{r}) = \frac{|xyz|^{D/3-1}}{\Gamma^3(D/3)}. \tag{7}$$

Here we use Cartesian's coordinates x, y, and z. In order to have the usual dimensions of the physical values, we can use vector $\mathbf{r}$, and coordinates x, y, z as dimensionless values.

Note that the interpretation of fractional integration is connected with the noninteger dimension.[4,5] This interpretation follows from the well known formulas for dimensional regularizations. The fractional integral can be considered as an integral in the noninteger dimension space up to the numerical factor $\Gamma(D/2)/(2\pi^{D/2}\Gamma(D))$.

If we consider the ball region $W = \{\mathbf{r} : |\mathbf{r}| \le R\}$, Riesz fractional integral (6), and spherically symmetric distribution of charged particles ($\rho(\mathbf{r}, t) = \rho(r)$), then we have

$$Q(W) = 4\pi\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}\int_0^R \rho(r)r^{D-1}dr. \tag{8}$$

For the homogeneous ($\rho(r) = \rho_0$) fractal distribution, we get

$$Q(W) = \frac{4\pi\rho_0}{D}\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}R^D \sim R^D. \tag{9}$$

If $D = 3$, we have $Q(W) = (4\pi/3)\rho_0 R^3$.

If we consider the Riemann–Liouville fractional integral (7) for the ball region W, and spherically symmetric distribution ($\rho(\mathbf{r}, t) = \rho(r)$), then we have

$$Q(W) = \frac{2\Gamma^3(D/6)}{\Gamma^3(D/3)\Gamma(D/2)}\int_0^R \rho(r)r^{D-1}dr. \tag{10}$$

For the homogeneous ($\rho(r) = \rho_0$) fractal distribution, we get

$$Q(W) = \frac{2\rho_0\Gamma^3(D/6)}{D\Gamma^3(D/3)\Gamma(D/2)}R^D \sim R^D. \tag{11}$$

If $D = 3$, we get the usual expression $Q(W) = (4\pi/3)\rho_0 R^3$.

The fractal distribution of charged particles is called a homogeneous fractal distribution if the power law $Q(R) \sim R^D$ does not depend on the translation of the region. The homogeneity property of the distribution can be formulated in the following form: For all regions W and W' such that the volumes are equal $V(W) = V(W')$, we have that the number of particles in these regions are equal $N(W) = N(W')$. Note that the wide class of the fractal media satisfies the homogeneous property. In Refs. 7 and 8, the continuous medium model for the fractal distribution was suggested.

3. Electric Multipole Expansion for Fractal Distribution of Charges

It is known that a multipole expansion is a series expansion of the effect produced by a given system in terms of an expansion parameter which becomes small as the distance away from the system increases. Therefore, the leading one of the terms in a multipole expansion are generally the strongest. The first-order behavior of the system at large distances can therefore be obtained from the first terms of this series, which is generally much easier to compute than the general solution. Multipole expansions are most commonly used in problems involving the electric and magnetic fields of charge and current distributions, and the propagation of electromagnetic waves.

To compute one particular case of a multipole expansion, let $\mathbf{R} = X_k \mathbf{e}_k$ be the vector from a fixed reference point to the observation point; $\mathbf{r} = x_k \mathbf{e}_k$ be the vector from the reference point to a point in the distribution; and $\mathbf{s} = \mathbf{R} - \mathbf{r}$ be the vector from a point in the distribution to the observation point. The law of cosine then yields

$$s^2 = r^2 + R^2 - 2rR \cos\theta \,, \tag{12}$$

where $s = |\mathbf{s}|$, $r = |\mathbf{r}|$, $R = |\mathbf{R}|$, and θ is the polar angle, defined such that

$$\cos\theta = (\mathbf{r}, \mathbf{R})/(rR) \,. \tag{13}$$

Using Eq. (12), we get

$$s = R\sqrt{1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}} \,. \tag{14}$$

Now define $\epsilon = r/R$, and $\xi = \cos\theta$, then

$$\frac{1}{s} = \frac{1}{R}(1 - 2\epsilon\xi + \epsilon^2)^{-1/2} \,. \tag{15}$$

But the right hand side of Eq. (15) is the generating function for Legendre polynomials $P_n(\xi)$ by the following relation:

$$(1 - 2\epsilon\xi + \epsilon^2)^{-1/2} = \sum_{n=0}^{\infty} \epsilon^n P_n(\xi) \,, \tag{16}$$

so, we have the equation

$$\frac{1}{s} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta) \,. \tag{17}$$

Any physical potential that obeys a $(1/s)$ law can therefore be expressed as a multipole expansion

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos\,\theta)\rho(\mathbf{r})dV_D\,. \tag{18}$$

The $n = 0$ term of this expansion, called the monopole term, can be pulled out by noting that $P_0(x) = 1$, so

$$V = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\int_W \rho(\mathbf{r})dV_D + \frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{R^{n+1}}\int_W r^n P_n(\cos\,\theta)\rho(\mathbf{r})dV_D\,. \tag{19}$$

The nth term

$$V_n = \frac{1}{4\pi\varepsilon_0}\frac{1}{R^{n+1}}\int_W r^n P_n(\cos\,\theta)\rho(\mathbf{r})dV_D \tag{20}$$

is usually named according to the following: n – multipole, 0 – monopole, 1 – dipole, 2 – quadrupole.

4. Electric Dipole Moment of Fractal Distribution of Charges

An electric multipole expansion is a determination of the voltage V due to a collection of charges obtained by performing a multipole expansion. This corresponds to a series expansion of the charge density $\rho(\mathbf{r})$ in terms of its moments, normalized by the distance to a point $\mathbf{R}$ far from the charge distribution. In MKS, the electric multipole expansion is given by Eq. (18):

$$V = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos\,\theta)\rho(\mathbf{r})dV_D\,, \tag{21}$$

where $P_n(\cos\,\theta)$ is a Legendre polynomial.

The first term arises from $P_0(\xi) = 1$, while all further terms vanish as a result of $P_n(\xi)$ being a polynomial in ξ for $n \geq 1$, giving $P_n(0) = 0$ for all $n \geq 1$.

Set up the coordinate system so that θ measures the angle from the charge-charge line with the midpoint of this line being the origin. Then the $n = 1$ term is given by

$$\begin{aligned} V_1 &= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^2}\int_W rP_1(\cos\,\theta)\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0 R^2}\int_W r\,\cos\,\theta\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0 R^2}\int_W \frac{(\mathbf{r},\mathbf{R})}{R}\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0 R^3}\int_W (\mathbf{r},\mathbf{R})\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0 R^3}\left(\int_W \mathbf{r}\rho(\mathbf{r})dV_D, \mathbf{R}\right)\,. \end{aligned} \tag{22}$$

For a continuous charge distribution, the electric dipole moment is given by

$$\mathbf{p}^{(3)} = \int_W \mathbf{r}\rho(\mathbf{r})dV_3\,, \tag{23}$$

where $\mathbf{r}$ points from positive to negative. Defining the dipole moment for the fractal distribution by the equation

$$\mathbf{p}^{(D)} = \int_W \mathbf{r}\rho(\mathbf{r})dV_D\,, \tag{24}$$

then gives

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{p}^{(D)},\mathbf{R})}{R^3} = \frac{1}{4\pi\varepsilon_0}\frac{p^{(D)}\cos\alpha}{R^2}\,, \tag{25}$$

where we use

$$\cos\alpha = (\mathbf{p}^{(D)},\mathbf{R})/(p^{(D)}R)\,, \qquad p^{(D)} = \sqrt{(p_x^{(D)})^2 + (p_y^{(D)})^2 + (p_z^{(D)})^2}\,. \tag{26}$$

Let us consider the example of electric dipole moment for the homogeneous ($\rho(\mathbf{r}) = \rho_0$) fractal distribution of electric charges in the parallelepiped region W:

$$0 \le x \le A\,, \qquad 0 \le y \le B\,, \qquad 0 \le z \le C\,. \tag{27}$$

In the case of the Riemann–Liouville fractional integral, we have $p_x^{(D)}$ in the form:

$$p_x^{(D)} = \frac{\rho_0}{\Gamma^3(a)}\int_0^A dx \int_0^B dy \int_0^C dz x^a y^{a-1} z^{a-1} = \frac{\rho_0 (ABC)^a}{\Gamma^3(a)}\frac{A}{a^2(a+1)}\,, \tag{28}$$

where $a = D/3$. The electric charge of parallelepiped region (27) is defined by

$$Q(W) = \rho_0 \int_W dV_D = \frac{\rho_0(ABC)^a}{a^3\Gamma^3(a)}\,. \tag{29}$$

Therefore, we have the dipole moments for fractal distribution in parallelepiped in the form

$$p_x^{(D)} = \frac{a}{a+1}Q(W)A\,. \tag{30}$$

By analogy with these equation, we can derive

$$p_y^{(D)} = \frac{a}{a+1}Q(W)B\,, \qquad p_z^{(D)} = \frac{a}{a+1}Q(W)C\,. \tag{31}$$

Using $a/(a+1) = D/(D+3)$, we get

$$p^{(D)} = \frac{2D}{D+3}p^{(3)}\,, \tag{32}$$

where $p^{(3)} = |\mathbf{p}^{(3)}|$ are the dipole moment for the usual 3-dimensional homogeneous distribution. For example, the relation $2 \le D \le 3$ leads us to the following inequality

$$0.8 \le p^{(D)}/p^{(3)} \le 1\,. \tag{33}$$

5. Electric Quadrupole Moment of Fractal Distribution of Charges

There are also higher-order terms in the multipole expansion that become smaller as R becomes large. The electric quadrupole term in MKS is given by

$$\begin{aligned} V_2 &= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^3}\int_W r^2 P_2(\cos\theta)\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^3}\int_W r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho(\mathbf{r})dV_D \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{2R^3}\int_W \left(\frac{3}{R^2}(\mathbf{R},\mathbf{r})^2 - r^2\right)\rho(\mathbf{r})dV_D\,. \end{aligned} \tag{34}$$

The electric quadrupole is the third term in an electric multipole expansion, and can be defined in MKS by

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{2R^3}\sum_{k,l=1}^{3}\frac{X_k X_l}{R^2}Q_{kl}\,, \tag{35}$$

where ε_0 is the permittivity of free space, R is the distance from the fractal distribution of charges, and Q_{kl} is the electric quadrupole moment, which is a tensor. Note that X_k are Cartesian's coordinates of the vector $\mathbf{R}$, and x_k are coordinates of the vector $\mathbf{r}$.

The electric quadrupole moment is defined by the equation

$$Q_{kl} = \int_W (3x_k x_l - r^2\delta_{kl})\rho(\mathbf{r})dV_D\,, \tag{36}$$

where $x_k = x, y$, or z. From this definition, it follows that

$$Q_{kl} = Q_{lk}\,, \quad \text{and} \quad \sum_{k=1}^{3} Q_{kk} = 0\,. \tag{37}$$

Therefore, we have $Q_{zz} = -Q_{xx} - Q_{yy}$. In order to compute the values

$$Q^{(D)}_{xx} = \int_W (2x^2 - y^2 - z^2)\rho(\mathbf{r})dV_D\,, \tag{38}$$

$$Q^{(D)}_{yy} = \int_W (-x^2 + 2y^2 - z^2)\rho(\mathbf{r})dV_D\,, \tag{39}$$

$$Q^{(D)}_{zz} = \int_W (-x^2 - y^2 + 2z^2)\rho(\mathbf{r})dV_D\,, \tag{40}$$

we consider the following expression

$$Q(\alpha,\beta,\gamma) = \int_W (\alpha x^2 + \beta y^2 + \gamma z^2)\rho(\mathbf{r})dV_D\,. \tag{41}$$

Using Eq. (41), we have

$$Q^{(D)}_{xx} = Q(2,-1,-1)\,, \qquad Q^{(D)}_{xx} = Q(-1,2,-1)\,, \qquad Q^{(D)}_{zz} = Q(-1,-1,2)\,. \tag{42}$$

The example of electric quadrupole moment for the parallelepiped and ellipsoid regions are considered in Secs. 6 and 7.

6. Quadrupole Moment of Charged Fractal Parallelepiped

Let us consider the example of electric quadrupole moment for the homogeneous ($\rho(\mathbf{r}) = \rho_0$) fractal distribution of electric charges in the parallelepiped region (27). If we consider the region W in the form (27), then we get

$$Q(\alpha,\beta,\gamma) = \frac{\rho_0(ABC)^a}{(a+2)a^2\Gamma^3(a)}[\alpha A^2 + \beta B^2 + \gamma C^2], \qquad (43)$$

where we use the Riemann–Liouville fractional integral,[18] and the function $C_3(D,\mathbf{r})$ in the form

$$C_3(D,\mathbf{r}) = \frac{|xyz|^{a-1}}{\Gamma^3(a)}, \qquad a = D/3. \qquad (44)$$

The electric charge of the region W is

$$Q(W) = \rho_0 \int_W dV_D = \frac{\rho_0(ABC)^a}{a^3\Gamma^3(a)}. \qquad (45)$$

If $D = 3$, we have $Q(W) = \rho_0 ABC$. Using Eqs. (43) and (45), we get the following equations:

$$Q(\alpha,\beta,\gamma) = \frac{a}{a+2} Q(W)(\alpha A^2 + \beta B^2 + \gamma C^2). \qquad (46)$$

If $D = 3$, then we have $a/(a+2) = 1/3$. As the result, we have electric quadrupole moments $Q^{(D)}_{kk}$ of fractal distribution in the region W:

$$Q^{(D)}_{kk} = \frac{3D}{D+6} Q^{(3)}_{kk}, \qquad (47)$$

where $Q^{(3)}_{kk}$ are moments for the usual homogeneous distribution ($D = 3$). By analogy with these equations, we can derive $Q^{(D)}_{kl}$ for the case $k \neq l$. These electric quadrupole moments are

$$Q^{(D)}_{kl} = \frac{4D^2}{(D+3)^2} Q^{(3)}_{kl}, \qquad (k \neq l). \qquad (48)$$

Using inequality $2 < D < 3$, we get the relations for diagonal elements

$$0.75 < Q^{(D)}_{kk}/Q^{(3)}_{kk} \leq 1, \qquad (49)$$

and nondiagonal elements

$$0.64 < Q^{(D)}_{kl}/Q^{(3)}_{kl} \leq 1, \qquad (50)$$

where $k \neq l$.

7. Quadrupole Moment of Charged Fractal Ellipsoid

Let us consider the example of electric quadrupole moment for the homogeneous ($\rho(\mathbf{r}) = \rho_0$) fractal distribution in the ellipsoid region W:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \leq 1 . \tag{51}$$

If we consider the region W in the form (51), then we get expression (41) in the form

$$Q(\alpha,\beta,\gamma) = \frac{8\rho_0 (ABC)^a}{(3a+2)\Gamma^3(a)} \left[\alpha A^2 Z_1(a) + \beta B^2 Z_2(a) + \gamma C^2 Z_3(a)\right], \tag{52}$$

where $a = D/3$, and $Z_i(a)$, $i = 1,2,3$ are defined by

$$Z_1(a) = S(a+1, a-1) S(a-1, 2a+1), \tag{53}$$

$$Z_2(a) = S(a-1, a+1) S(a-1, 2a+1), \tag{54}$$

$$Z_3(a) = S(a-1, a-1) S(a+1, 2a-1). \tag{55}$$

Here we use the following function:

$$S(n,m) = \int_0^{\pi/2} dx \cos^n(x) \sin^m(x) = \frac{\Gamma(n/2+1/2)\Gamma(m/2+1/2)}{2\Gamma(n/2+m/2+1)} . \tag{56}$$

Note that $Z_1(a) = Z_2(a) = Z_3(a)$. Using these equations, we get the following relation:

$$Q(\alpha,\beta,\gamma) = \frac{2\rho_0 (ABC)^a}{(3a+2)} \frac{\Gamma^2(a/2)\Gamma(a/2+1)}{\Gamma^3(a)\Gamma(3a/2+1)} (\alpha A^2 + \beta B^2 + \gamma C^2) . \tag{57}$$

Using $\Gamma(\beta+1) = \beta\Gamma(\beta)$, we have

$$Q(\alpha,\beta,\gamma) = \frac{2\rho_0 (ABC)^a}{3(3a+2)} \frac{\Gamma^3(a/2)}{\Gamma^3(a)\Gamma(3a/2)} (\alpha A^2 + \beta B^2 + \gamma C^2) . \tag{58}$$

If $D = 3$, we obtain

$$Q(\alpha,\beta,\gamma) = \frac{4\pi\rho_0 ABC}{15} (\alpha A^2 + \beta B^2 + \gamma C^2) . \tag{59}$$

The total charge of the ellipsoid region W is defined by

$$Q(W) = \rho_0 \int_W dV_D = \rho_0 (ABC)^a \frac{2\Gamma^3(a/2)}{3a\Gamma^3(a)\Gamma(3a/2)} . \tag{60}$$

If $D = 3$, we have the total charge $Q(W) = (4\pi/3)\rho_0 ABC$. Here we use $\Gamma(1/2) = \sqrt{\pi}$.

Using Eqs. (58) and (60), we can derive the electric quadrupole moments (42) for fractal ellipsoid. As the result, we have

$$Q(\alpha,\beta,\gamma) = \frac{D}{3D+6} Q(W) (\alpha A^2 + \beta B^2 + \gamma C^2) . \tag{61}$$

If $D = 3$, then we have the well known relation:

$$Q(\alpha, \beta, \gamma) = (1/5)Q(W)(\alpha A^2 + \beta B^2 + \gamma C^2). \tag{62}$$

If $2 < D < 3$, then we have

$$\frac{5}{6} < Q_{kk}^{(D)}/Q_{kk}^{(3)} < 1. \tag{63}$$

The nondiagonal elements of electric quadrupole moment are defined by the following equations:

$$Q_{xy} = 3\rho_0 \int_W xy dV_D, \qquad Q_{xz} = 3\rho_0 \int_W xz dV_D, \qquad Q_{yz} = 3\rho_0 \int_W yz dV_D. \tag{64}$$

Using these equations, we can derive the nondiagonal elements in the forms:

$$Q_{xy}^{(D)} = \frac{6\rho_0 (ABC)^a}{3a+2} \frac{\Gamma(a/2)\Gamma^2(a/2+1/2)}{\Gamma^3(a)\Gamma(3a/2+1)} AB, \tag{65}$$

$$Q_{xz}^{(D)} = \frac{6\rho_0 (ABC)^a}{3a+2} \frac{\Gamma(a/2)\Gamma^2(a/2+1/2)}{\Gamma^3(a)\Gamma(3a/2+1)} AC, \tag{66}$$

$$Q_{yz}^{(D)} = \frac{6\rho_0 (ABC)^a}{3a+2} \frac{\Gamma(a/2)\Gamma^2(a/2+1/2)}{\Gamma^3(a)\Gamma(3a/2+1)} BC. \tag{67}$$

Using $\Gamma(\beta+1) = \beta\Gamma(\beta)$ and Eq. (60), we get the following equations:

$$Q_{xy}^{(D)} = \frac{6Q(W)}{3a+2} \frac{\Gamma^2(a/2+1/2)}{\Gamma^2(a/2)} AB, \tag{68}$$

$$Q_{xz}^{(D)} = \frac{6Q(W)}{3a+2} \frac{\Gamma^2(a/2+1/2)}{\Gamma^2(a/2)} AC, \tag{69}$$

$$Q_{yz}^{(D)} = \frac{6Q(W)}{3a+2} \frac{\Gamma^2(a/2+1/2)}{\Gamma^2(a/2)} BC. \tag{70}$$

As the result, we have

$$Q_{kl}^{(D)} = \frac{5\pi}{D+2} \frac{\Gamma^2(D/6+1/2)}{\Gamma^2(D/6)} Q_{kl}^{(3)}, \tag{71}$$

where $k \neq l$. Here we use $\Gamma(1/2) = \sqrt{\pi}$. If we consider $2 < D < 3$, then we get

$$0.6972 < Q_{kl}^{(D)}/Q_{kl}^{(3)} < 1. \tag{72}$$

8. Conclusion

In this paper, we use the fractional continuous model for fractal distribution of electric charges. The fractional continuous models for fractal distribution of particles can have wide applications. This is due in part to the relatively small number of parameters that define a fractal distribution of great complexity and rich structure. In many cases, the real fractal structure of matter can be disregarded and the distribution of particles can be replaced by some fractional continuous model.[7,8] In order to describe the distribution with noninteger dimension, we must use the fractional calculus. Smoothing of the microscopic characteristics over the physically infinitesimal volume transforms the initial fractal distribution into the fractional continuous model that uses the fractional integrals. The order of the fractional integral is equal to the fractal dimension of the distribution. The fractional continuous model for the fractal distribution allows us to describe the dynamics of a wide class fractal media.[3,8,9] One of the dynamical equation of physics is a Liouville equation. Note that the Liouville equation is a cornerstone of statistical mechanics. The fractional generalization of the Liouville equation was suggested in Refs. 4 and 6. The fractional generalization of the Liouville equation allows us to derive the fractional generalization of the Bogoliubov equations.[5] Using the fractional analog of the Liouville equation[4] and Bogoliubov equations,[5,6] we can derive the description of a fractal distribution as a fractional system. Fractional systems can be considered as a special case of non-Hamiltonian systems.[4,5] Note that non-Hamiltonian systems can have stationary states of the Hamiltonian systems.[19–24]

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