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THE FRACTIONAL CHAPMAN–KOLMOGOROV EQUATION

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The Chapman–Kolmogorov equation with fractional integrals is derived. An integral of fractional order is considered as an approximation of the integral on fractal. Fractional integrals can be used to describe the fractal media. Using fractional integrals, the fractional generalization of the Chapman–Kolmogorov equation is obtained. From the fractional Chapman–Kolmogorov equation, the Fokker–Planck equation is derived.

Keywords: Chapman–Kolmogorov equation; fractional integral; fractal.

1. Introduction

Integrals and derivatives of the fractional order goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov.[1] Fractional analysis has found many applications in recent studies in mechanics and physics. The interest in fractional integrals and derivatives has been growing continually during the last few years because of numerous applications. In a fairly short period of time, the list of such applications has become long. It includes chaotic dynamics,[2,3] mechanics of fractal and complex media,[4–6] physical kinetics,[2,7,8] plasma physics,[9–11] astrophysics,[12] long-range dissipation,[13,14] non-Hamiltonian mechanics,[15,16] and long-range interaction.[17–19]

The natural question arises: What could be the physical meaning of the fractional integration? This physical meaning can be following: the fractional integration can be considered as an integration in some noninteger-dimensional space. If we use the well-known formulas for dimensional regularizations,[20] then we get that the fractional integration can be considered as an integration in the fractional dimension space[15] up to the numerical factor $\Gamma(\alpha/2)/[2\pi^{\alpha/2}\Gamma(\alpha)]$. This interpretation was suggested in Ref. 15. Fractional integrals can be considered as approximations of integrals on fractals.[21,22] In Ref. 22, authors proved that integrals on a net of fractals can be approximated by fractional integrals. Using fractional integrals, we derive the fractional generalization of the Chapman–Kolmogorov equation.[23,24] In this paper, the generalization of the Fokker–Planck equation for fractal media is derived from the fractional Chapman–Kolmogorov equation.

In Sec. 2, a brief review of the Hausdorff measure, the Hausdorff dimension and integration on fractals is carried out to fix notation and provide a convenient reference. The connection of integration on fractals and fractional integration is discussed. We derive the fractional generalization of the average values equation. In Sec. 3, the fractional Chapman–Kolmogorov equation is derived by using fractional integration. In Sec. 4, the fractional Fokker–Planck equation for the fractal media is derived from the suggested fractional Chapman–Kolmogorov equation. The stationary solutions of the Fokker–Planck equation for fractal media are derived.

2. Integration on Fractal and Fractional Integration

Fractals are measurable metric sets with a non-integer Hausdorff dimension. Let us consider a brief review of the Hausdorff measure and the Hausdorff dimension in order to fix notation and provide a convenient reference.

2.1. *Hausdorff measure and Hausdorff dimension*

Consider a measurable metric set (W, μ_H). The elements of W are denoted by $x, y, z, \ldots$, and represented by n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$ such that W is embedded in R^n. The set W is restricted by the conditions: (i) W is closed; (ii) W is unbounded; (iii) W is regular (homogeneous, uniform) with its points randomly distributed.

The metric $d(x, y)$ as a function of two points x and $y \in W$ can be defined by

$$d(x, y) = \sum_{i=1}^{n} |y_i - x_i| \,. \tag{1}$$

The diameter of a subset $E \subset W \subset R^n$ is

$$d(E) = \mathrm{diam}(E) = \sup\{d(x, y) : x, y \in E\} \,,$$

Let us consider a set $\{E_i\}$ of non-empty subsets E_i such that $\dim(E_i) < \varepsilon$, $\forall i$, and $W \subset \bigcup_{i=1}^{\infty} E_i$. Then, we define

$$\xi(E_i, D) = \omega(D)[\mathrm{diam}(E_i)]^D = \omega(D)[d(E_i)]^D \,. \tag{2}$$

The factor $\omega(D)$ depends on the geometry of E_i, used for covering W. If $\{E_i\}$ is the set of all (closed or open) balls in W, then

$$\omega(D) = \frac{\pi^{D/2} 2^{-D}}{\Gamma(D/2 + 1)} \,. \tag{3}$$

The Hausdorff dimension D of a subset $E \subset W$ is defined[25, 26] by

$$D = \dim_H(E) = \{\sup\{d \in R : \mu_H(E, d) = \infty\} = \inf\{d \in R : \mu_H(E, d) = 0\} \,. \tag{4}$$

From (4), we obtain $\mu_H(E, d) = 0$ for $d > D$; and $\mu_H(E, d) = \infty$ for $d < D$.

The Hausdorff measure μ_H of a subset $E \subset W$ is:[25, 26]

$$\mu_H(E, D) = \lim_{\varepsilon \to 0} \inf_{\{E_i\}} \left\{ \sum_{i=1}^{\infty} \xi(E_i, D) : E \subset \bigcup_i E_i, d(E_i) < \varepsilon \, \forall i \right\} \,, \tag{5}$$

or

$$\mu_H(E,D) = \omega(D) \lim_{d(E_i)\to 0} \inf_{\{E_i\}} \sum_{i=1}^{\infty} [d(E_i)]^D \,. \tag{6}$$

If $E \subset W$ and $\lambda > 0$, then $\mu_H(\lambda E, D) = \lambda^D \mu_H(E,D)$, where $\lambda E = \{\lambda x, x \in E\}$.

2.2. *Function and integrals on fractal*

Let us consider the functions

$$f(x) = \sum_{i=1}^{\infty} \beta_i \chi_{E_i}(x)\,, \tag{7}$$

where χ_E is the characteristic function of E: $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

The Lebesgue–Stieltjes integral for (7) is defined by

$$\int_W f d\mu = \sum_{i=1}^{\infty} \beta_i \mu_H(E_i)\,. \tag{8}$$

Therefore

$$\begin{aligned} \int_W f(x) d\mu_H(x) &= \lim_{d(E_i)\to 0} \sum_{E_i} f(x_i) \xi(E_i, D) \\ &= \omega(D) \lim_{d(E_i)\to 0} \sum_{E_i} f(x_i) [d(E_i)]^D \,. \end{aligned} \tag{9}$$

It is always possible to divide R^n into parallelepipeds:

$$\begin{aligned} E_{i_1 \dots i_n} &= \{(x_1, \dots, x_n) \in W : x_j \\ &= (i_j - 1)\Delta x_j + \alpha_j,\, 0 \le \alpha_j \le \Delta x_j,\, j = 1, \dots, n\}\,. \end{aligned} \tag{10}$$

Then

$$d\mu_H(x) = \lim_{d(E_{i_1 \cdots i_n})\to 0} \xi(E_{i_1 \cdots i_n}, D) = \lim_{d(E_{i_1 \cdots i_n})\to 0} \prod_{j=1}^{n} (\Delta x_j)^{D/n} = \prod_{j=1}^{n} d^{D/n} x_j \,. \tag{11}$$

The range of integration W may also be parametrized by polar coordinates with $r = d(x, 0)$ and angle Ω. Then $E_{r,\Omega}$ can be thought of as spherically symmetric covering around a center at the origin. In the limit, the function $\xi(E_{r,\Omega}, D)$ gives

$$d\mu_H(r, \Omega) = \lim_{d(E_{r,\Omega})\to 0} \xi(E_{r,\Omega}, D) = d\Omega^{D-1} r^{D-1} dr\,. \tag{12}$$

Let us consider $f(x)$ that is symmetric with respect to some point $x_0 \in W$, i.e. $f(x) = \text{const}$ for all x such that $d(x, x_0) = r$ for arbitrary values of r. Then the transformation

$$W \to W' : x \to x' = x - x_0 \tag{13}$$

can be performed to shift the center of symmetry. Since W is not a linear space, the transformation (13) need not be a map of W onto itself, and (13) is measure-preserving. Then the integral over a D-dimensional metric space is

$$\int_W f d\mu_H = \lambda(D) \int_0^\infty f(r) r^{D-1} dr \,, \tag{14}$$

where $\lambda(D) = 2\pi^{D/2}/\Gamma(D/2)$. This integral is known in the theory of the fractional calculus.[1] The right Riemann–Liouville fractional integral is

$$(I_-^D f)(z) = \frac{1}{\Gamma(D)} \int_z^\infty (x-z)^{D-1} f(x) dx \,. \tag{15}$$

Then Eq. (14) is reproduced by

$$\int_W f d\mu_H = \frac{2\pi^{D/2}\Gamma(D)}{\Gamma(D/2)} (I_-^D f)(0) \,. \tag{16}$$

Equation (16) connects the integral on the fractal with the integral of fractional order. This result permits us to apply different tools of the fractional calculus[1] for the fractal medium. As a result, the fractional integral can be considered as an integral on fractal up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$.

Note that the interpretation of fractional integration is connected with fractional dimension.[15] This interpretation follows from the well-known formulas for dimensional regularizations.[20] The fractional integral can be considered as an integral in the fractional dimension space up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$. In Ref. 21, it was proved that the fractal space-time approach is technically identical to the dimensional regularization.

2.3. *Properties of integrals*

The integral defined in Eq. (9) satisfies the properties:

(i) Linearity:

$$\int_W (af_1 + bf_2) d\mu_H = a \int_W f_1 d\mu_H + b \int_W f_2 d\mu_H \,, \tag{17}$$

where f_1 and f_2 are arbitrary functions; a and b are arbitrary constants.

(ii) Translational invariance:

$$\int_W f(x + x_0) d\mu_H(x) = \int_W f(x) d\mu_H(x) \,, \tag{18}$$

since $d\mu_H(x - x_0) = d\mu_H(x)$ as a consequence of homogeneity (uniformity).

(iii) Scaling property:

$$\int_W f(\lambda x) d\mu_H(x) = \lambda^{-D} \int_W f(x) d\mu_H(x) \,, \tag{19}$$

since $d\mu_H(x/\lambda) = \lambda^{-D} d\mu_H(x)$.

It is well-known[20,27] that conditions (17)–(19) define the integral of the function $f(x) = \exp(-ax^2 + bx)$ up to normalization:

$$\int_W \exp(-ax^2 + bx) d\mu_H(x) = \pi^{D/2} a^{-D/2} \exp(b^2/4a) . \tag{20}$$

For $b = 0$, Eq. (20) is identical to the result that can be derived from Eq. (16) and is obtained directly without conditions (17)–(19).

2.4. *Fractional average values*

The usual average value

$$\langle A \rangle_1 = \int_{-\infty}^{+\infty} A(x)\rho(x)dx \tag{21}$$

can be written as

$$\langle A \rangle_1 = \int_{-\infty}^{y} A(x)\rho(x)dx + \int_{y}^{\infty} A(x)\rho(x)dx . \tag{22}$$

Using

$$(I_+^\alpha A)(y) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{y} \frac{A(x)dx}{(y-x)^{1-\alpha}} , \tag{23}$$

$$(I_-^\alpha A)(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\infty} \frac{A(x)dx}{(x-y)^{1-\alpha}} , \tag{24}$$

the average value (22) can be present by

$$\langle A \rangle_1 = (I_+^1 A\rho)(y) + (I_-^1 A\rho)(y) . \tag{25}$$

The fractional generalization of Eq. (25) is

$$\langle A \rangle_\alpha(y) = (I_+^\alpha A\rho)(y) + (I_-^\alpha A\rho)(y) . \tag{26}$$

Equation (26) can be rewritten as

$$\langle A \rangle_\alpha(y) = \int_0^\infty [(A\rho)(y-x) + (A\rho)(y+x)] d\mu_\alpha(x) , \tag{27}$$

where

$$d\mu_\alpha(x) = \frac{|x|^{\alpha-1}dx}{\Gamma(\alpha)} = \frac{dx^\alpha}{\alpha\Gamma(\alpha)} . \tag{28}$$

Here, we use

$$x^\alpha = \beta(x)(x)^\alpha = \mathrm{sgn}(x)|x|^\alpha , \tag{29}$$

where $\beta(x) = (\mathrm{sgn}(x))^{\alpha-1}$. The function $\mathrm{sgn}(x)$ is equal to $+1$ for $x \geq 0$, and -1 for $x < 0$.

To have the symmetric limits of the integral, we consider Eq. (27) in the form

$$\langle A \rangle_\alpha(y) = \frac{1}{2} \int_{-\infty}^{+\infty} [(A\rho)(y-x) + (A\rho)(y+x)] d\mu_\alpha(x) . \tag{30}$$

If $\alpha = 1$, then we have the usual equation for the average value.

Let us introduce some notations to simplify Eq. (30). We define the integral operators

$$\hat{I}^{\alpha}_{x} f(x) = \frac{1}{2} \int_{-\infty}^{+\infty} [f(x) + f(-x)] d\mu_{\alpha}(x) \,. \tag{31}$$

Then Eq. (30) has the form

$$\langle A \rangle_{\alpha} = \hat{I}^{\alpha}_{x} A(x) \rho(x) \,. \tag{32}$$

We will use the initial points that are set to zero ($y = 0$). Note that the fractional normalization condition is a special case of this definition of average values: $\langle 1 \rangle_{\alpha} = 1$.

3. Fractional Chapman–Kolmogorov (FCK) Equation

The Chapman–Kolmogorov equation[23,24] may be interpreted as the condition of consistency of distribution functions of different orders. Kolmogorov[23,24] derived a kinetic equation using a special scheme and conditions that are important for kinetics. Let $W(x, t; x_0, t_0)$ be a probability density of having a particle at the position x at time t if the particle was at the position x_0 at time $t_0 \leq t$.

Denote by $\rho(x, t)$ the distribution functions for the given time t. Let us consider two well-known identities

$$\rho(x, t) = \int_{-\infty}^{+\infty} dx' W(x, t|x', t') \rho(x', t'), \qquad \int_{-\infty}^{+\infty} \rho(x, t) = 1 \,. \tag{33}$$

Using the notation (31), we can rewrite (33) in the form

$$\rho(x, t) = \hat{I}^{1}_{x'} W(x, t|x', t') \rho(x', t'), \qquad \hat{I}^{1}_{x} \rho(x, t) = 1 \,.$$

Then the fractional generalization of (33) is

$$\rho(x, t) = \hat{I}^{\alpha}_{x'} W(x, t|x', t') \rho(x', t') \,. \tag{34}$$

This equation is the definition of the conditional distribution function $W(x, t|x', t')$ referring to different time instants. The normalization conditions for the functions $W(x, t|x', t')$ and $\rho(x, t)$ are

$$\hat{I}^{\alpha}_{x} W(x, t|x', t') = 1, \qquad \hat{I}^{\alpha}_{x} \rho(x, t) = 1 \,. \tag{35}$$

Substituting into the right-hand side of Eq. (34) the value of $\rho(x', t')$ expressed via the distribution $\rho(x_0, t_0)$ at an earlier time,

$$\rho(x', t') = \hat{I}^{\alpha}_{x_0} W(x', t'|x_0, t_0) \rho(x_0, t_0) \,, \tag{36}$$

we obtain the integral relation which includes the intermediate point x',

$$\rho(x, t) = \hat{I}^{\alpha}_{x'} \hat{I}^{\alpha}_{x_0} W(x, t|x', t') W(x', t'|x_0, t_0) \rho(x_0, t_0) \,. \tag{37}$$

Using Eq. (37), and Eq. (34) in the form

$$\rho(x, t) = \hat{I}^{\alpha}_{x_0} W(x, t|x_0, t_0) \rho(x_0, t_0) \,, \tag{38}$$

we derive a closed equation for transition probabilities

$$\hat{I}^{\alpha}_{x_0} W(x,t|x_0,t_0)\rho(x_0,t_0) = \hat{I}^{\alpha}_{x'} \hat{I}^{\alpha}_{x_0} W(x,t|x',t')W(x',t'|x_0,t_0)\rho(x_0,t_0)\,.$$

Since the equation holds for arbitrary $\rho(x_0,t_0)$, we may equate the integrand. As the result, we obtain the fractional Chapman–Kolmogorov (FCK) equation

$$W(x,t|x_0,t_0) = \hat{I}^{\alpha}_{x'} W(x,t|x',t')W(x',t'|x_0,t_0)\,. \tag{39}$$

This equation can be used to describe the Markov-type process in the fractal medium that is described by the continuous medium model.[5]

4. Fokker–Planck Equation from FCK Equation

4.1. *Derivations of the Fokker–Planck equation*

Let us consider the fractional average value (32). Using Eq. (34) in the form

$$\rho(x,t) = \hat{I}^{\alpha}_{x_0} W(x,t|x_0,t_0)\rho(x_0,t_0)\,, \tag{40}$$

we get

$$\langle A\rangle_{\alpha} = \hat{I}^{\alpha}_{x} A(x) \hat{I}^{\alpha}_{x_0} W(x,t|x_0,t_0)\rho(x_0,t_0)\,. \tag{41}$$

We can rewrite Eq. (41) as

$$\langle A\rangle_{\alpha} = \hat{I}^{\alpha}_{x_0} \rho(x_0,t_0) \hat{I}^{\alpha}_{x} A(x) W(x,t|x_0,t_0)\,. \tag{42}$$

We assume that $A = A(x^{\alpha})$, and use the Taylor expansion

$$\begin{aligned} A(x^{\alpha}) &= A(x_0^{\alpha} + \Delta x^{\alpha}) \\ &= A(x_0^{\alpha}) + \left(\frac{\partial A(x^{\alpha})}{\partial x^{\alpha}}\right)_{x_0} \Delta x^{\alpha} + \frac{1}{2}\left(\frac{\partial^2 A(x^{\alpha})}{(\partial x^{\alpha})^2}\right)_{x_0} (\Delta x^{\alpha})^2 + \cdots\,, \end{aligned} \tag{43}$$

where $x^{\alpha} = \mathrm{sgn}(x)|x|^{\alpha}$ is defined by Eq. (29), $\Delta x^{\alpha} = x^{\alpha} - x_0^{\alpha}$, and

$$\frac{\partial}{\partial x^{\alpha}} = \frac{|x|^{1-\alpha}}{\alpha}\frac{\partial}{\partial x}\,. \tag{44}$$

If we use the usual Taylor expansion, then the integration by parts in Eq. (42) is more complicated. For the expansion (43), the integration by parts in (42) can be realized in the simple form,

$$\begin{aligned} \hat{I}^{\alpha}_{x} B(x)\frac{\partial A(x^{\alpha})}{\partial x^{\alpha}} &= \int_{-\infty}^{+\infty} \frac{dx^{\alpha}}{\alpha\Gamma(\alpha)} B(x)\frac{\partial A(x^{\alpha})}{\partial x^{\alpha}} \\ &= (B(x)A(x))_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{dx^{\alpha}}{\alpha\Gamma(\alpha)} A(x^{\alpha})\frac{\partial B(x)}{\partial x^{\alpha}}\,. \end{aligned}$$

Substituting Eq. (43) in Eq. (42), we get

$$\langle A\rangle_\alpha = \hat{I}^\alpha_{x_0} A(x_0^\alpha)\rho(x_0,t_0)\hat{I}^\alpha_x W(x,t|x_0,t_0)$$
$$+\hat{I}^\alpha_{x_0}\left(\frac{\partial A(x^\alpha)}{\partial x^\alpha}\right)_{x_0}\rho(x_0,t_0)\hat{I}^\alpha_x \Delta x^\alpha W(x,t|x_0,t_0)$$
$$+\frac{1}{2}\hat{I}^\alpha_{x_0}\left(\frac{\partial^2 A(x^\alpha)}{(\partial x^\alpha)^2}\right)_{x_0}\rho(x_0,t_0)\hat{I}^\alpha_x (\Delta x^\alpha)^2 W(x,t|x_0,t_0)+\cdots. \quad (45)$$

Let us introduce the function:

$$P_n(x_0,t,t_0) = \hat{I}^\alpha_x (\Delta x^\alpha)^n W(x,t|x_0,t_0)\,. \quad (46)$$

Using Eqs. (46) and (35), Eq. (45) gives

$$\langle A\rangle_\alpha = \hat{I}^\alpha_{x_0} A(x_0^\alpha)\rho(x_0,t_0) + \hat{I}^\alpha_{x_0}\left(\frac{\partial A(x^\alpha)}{\partial x^\alpha}\right)_{x_0}\rho(x_0,t_0)P_1(x_0,t,t_0)$$
$$+\frac{1}{2}\hat{I}^\alpha_{x_0}\left(\frac{\partial^2 A(x^\alpha)}{(\partial x^\alpha)^2}\right)_{x_0}\rho(x_0,t_0)P_2(x_0,t,t_0)+\cdots. \quad (47)$$

Substitution of Eq. (32) in the form

$$\langle A\rangle_\alpha = \hat{I}^\alpha_{x_0} A(x_0^\alpha)\rho(x_0,t)\,,$$

into Eq. (47) gives

$$\hat{I}^\alpha_{x_0} A(x_0)\left(\rho(x_0,t)-\rho(x_0,t_0)\right) = \hat{I}^\alpha_{x_0}\left(\frac{\partial A(x)}{\partial x^\alpha}\right)_{x_0}\rho(x_0,t_0)P_1(x_0,t,t_0)$$
$$+\frac{1}{2}\hat{I}^\alpha_{x_0}\left(\frac{\partial^2 A(x)}{(\partial x^\alpha)^2}\right)_{x_0}\rho(x_0,t_0)P_2(x_0,t,t_0)+\cdots. \quad (48)$$

Then we use the so-called Kolmogorov condition,[23,24] and assume that the following finite limits exist:

$$\lim_{\Delta t\to 0}\frac{P_1(x,t,t_0)}{\Delta t} = a(x,t_0),\quad \lim_{\Delta t\to 0}\frac{P_2(x,t,t_0)}{\Delta t} = b(x,t_0),\quad \lim_{\Delta t\to 0}\frac{P_n(x,t,t_0)}{\Delta t} = 0\,,$$

where $n = 3,4,\ldots$, and $\Delta t = t - t_0$. It is due to the Kolmogorov conditions that irreversibility appears at the final equation. Multiplying both sides of Eq. (48) by $1/\Delta t$ and considering the limit $\Delta t \to 0$, we obtain

$$\hat{I}^\alpha_{x_0} A(x_0^\alpha)\left(\frac{\partial \rho(x_0,t)}{\partial t}\right)_{t_0} = \hat{I}^\alpha_{x_0}\left(\frac{\partial A(x^\alpha)}{\partial x^\alpha}\right)_{x_0}\rho(x_0,t_0)a(x_0,t_0)$$
$$+\frac{1}{2}\hat{I}^\alpha_{x_0}\left(\frac{\partial^2 A(x^\alpha)}{(\partial x^\alpha)^2}\right)_{x_0}\rho(x_0,t_0)b(x_0,t_0)\,.$$

Integrating by parts, we obtain

$$\hat{I}_x^\alpha \frac{\partial A(x^\alpha)}{\partial x^\alpha}\rho(x,t)a(x,t) = -\hat{I}_x^\alpha A(x^\alpha)\frac{\partial(\rho(x,t)a(x,t))}{\partial x^\alpha}\,, \tag{49}$$

$$\hat{I}_x^\alpha \frac{\partial^2 A(x^\alpha)}{(\partial x^\alpha)^2}\rho(x,t)b(x,t) = \hat{I}_x^\alpha A(x^\alpha)\frac{\partial^2(\rho(x,t)b(x,t))}{(\partial x^\alpha)^2}\,. \tag{50}$$

Here, we use

$$\lim_{x\to\pm\infty}\rho(x,t) = 0\,.$$

Then

$$\hat{I}_x^\alpha A(x^\alpha)\left(\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial(\rho(x,t)a(x))}{\partial x^\alpha} - \frac{1}{2}\frac{\partial^2(\rho(x,t)b(x))}{(\partial x^\alpha)^2}\right) = 0\,.$$

Since the function $A = A(x^\alpha)$ is an arbitrary function, we then have

$$\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial(\rho(x,t)a(x,t))}{\partial x^\alpha} - \frac{1}{2}\frac{\partial^2(\rho(x,t)b(x,t))}{(\partial x^\alpha)^2} = 0\,, \tag{51}$$

that is the Fokker–Planck equation that corresponds to the FCK equation. This equation is derived from the fractional generalization of the average value and fractional normalization condition, which uses the fractional integrals.

4.2. *Stationary solutions*

For the stationary case, the Fokker–Planck equation (51) is

$$\frac{\partial(\rho(x,t)a(x,t))}{\partial x^\alpha} - \frac{1}{2}\frac{\partial^2(\rho(x,t)b(x,t))}{(\partial x^\alpha)^2} = 0\,. \tag{52}$$

This equation can be rewritten as

$$\frac{\partial}{\partial x^\alpha}\left(\rho(x,t)a(x,t) - \frac{1}{2}\frac{\partial(\rho(x,t)b(x,t))}{\partial x^\alpha}\right) = 0\,. \tag{53}$$

Then

$$\rho(x,t)a(x,t) - \frac{1}{2}\frac{\partial(\rho(x,t)b(x,t))}{\partial x^\alpha} = \mathrm{const}\,. \tag{54}$$

Supposing that the constant is equal to zero, we get

$$\frac{\partial(\rho(x,t)b(x,t))}{\partial x^\alpha} = \frac{2a(x,t)}{b(x,t)}(\rho(x,t)b(x,t))\,. \tag{55}$$

The solution of Eq. (55) is

$$\ln(\rho(x,t)b(x,t)) = \int \frac{2a(x,t)}{b(x,t)}dx^\alpha + \mathrm{const}\,. \tag{56}$$

As the result, we obtain

$$\rho(x,t) = \frac{N}{b(x,t)}\exp 2\int \frac{a(x,t)}{b(x,t)}dx^\alpha\,, \tag{57}$$

where the coefficient N is defined by the normalization condition.

Let us consider the special cases of the solution (57).

(i) If $a(x) = k$ and $b(x) = -D$, then the Fokker–Planck equation (51) has the form

$$\frac{\partial \rho(x,t)}{\partial t} + k\frac{\partial \rho(x,t)}{\partial x^\alpha} + \frac{D}{2}\frac{\partial^2 \rho(x,t)}{(\partial x^\alpha)^2} = 0\,, \tag{58}$$

and the stationary solution is

$$\rho(x,t) = N_1 \exp\left(-\frac{2k|x|^\alpha}{D}\right)\,. \tag{59}$$

(ii) If $a(x) = k|x|^\beta$ and $b(x) = -D$, then

$$\frac{\partial \rho(x,t)}{\partial t} + k\frac{\partial |x|^\beta \rho(x,t)}{\partial x^\alpha} + \frac{D}{2}\frac{\partial^2 \rho(x,t)}{(\partial x^\alpha)^2} = 0\,. \tag{60}$$

The stationary solution is

$$\rho(x,t) = N_2 \exp\left(-\frac{2\alpha k|x|^{\alpha+\beta}}{(\alpha+\beta)D}\right)\,. \tag{61}$$

If $\alpha + \beta = 2$, we have

$$\rho(x,t) = N_2 \exp\left(-\frac{\alpha k}{D}x^2\right)\,. \tag{62}$$

(iii) If

$$a(x) = \frac{\partial U(x)}{\partial x^\alpha} = \frac{|x|^{1-\alpha}}{\alpha}\frac{\partial U(x)}{\partial x}\,,$$

and $b = -D$, then

$$\rho(x,t) = N_4 \exp\left(-\frac{U(x)}{D}\right)\,.$$

Let us consider Eq. (51) with $a(x) = k|x|^\alpha$ and $b = -D$. The general solution can be presented as

$$\rho(x,t) = \sum_{n=0}^{+\infty}\sqrt{\frac{k}{2^n n!\pi D}}e^{-kx^{2\alpha}/D}H_n(x^\alpha\sqrt{k/D})e^{-nkt}A_n\,,$$

where

$$A_n = \sqrt{\frac{1}{2^n n!}}\hat{I}^\alpha_x p(x,0)H_n(x^\alpha\sqrt{k/D})\,.$$

The stationary solution is

$$\rho(x) = \left(\frac{k}{\pi D}\right)^{1/2} e^{-kx^{2\alpha}/D}\,.$$

5. Conclusion

The concept of fractional integration provides an approach to describe the fractal media. The fractional integrals can be used in order to formulate the dynamical equations in the fractal media. The fractional integration approach is potentially more useful for the physics of fractal media than traditional methods that use the integer integration. Using fractional integrals, we derive the fractional generalization of the Chapman–Kolmogorov equation. This equation can be used to describe the Markov-type process in the fractal medium that is described by the continuous medium model.[5] The fractional Chapman–Kolmogorov equation can have a wide application since it uses a relatively small number of parameters that can define a fractal medium of great complexity and rich structure. In this paper, we derive the Fokker–Planck equations for the fractal media from the suggested fractional Chapman–Kolmogorov equation.

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