

LIOUVILLE AND BOGOLIUBOV EQUATIONS WITH FRACTIONAL DERIVATIVES

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The Liouville equation, first Bogoliubov hierarchy and Vlasov equations with derivatives of non-integer order are derived. Liouville equation with fractional derivatives is obtained from the conservation of probability in a fractional volume element. This equation is used to obtain Bogoliubov hierarchy and fractional kinetic equations with fractional derivatives. Statistical mechanics of fractional generalization of the Hamiltonian systems is discussed. Fractional kinetic equation for the system of charged particles are considered.

Keywords: Liouville equation; Bogoliubov equation; fractional derivatives; fractional kinetics.

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1. Introduction

Fractional equations¹ are equations that contain derivatives of non-integer order.^{2,3} The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, and Letnikov.³ Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. In a short period of time the list of applications have become long. For example, it includes chaotic dynamics,^{4,5} mechanics of fractal media,^{6–8} physical kinetics,^{4,9–12} plasma physics,^{13–15} astrophysics,¹⁶ long-range dissipation,^{17–19} mechanics of non-Hamiltonian systems,^{20,21} theory of long-range interaction,^{22–24} and many others physical topics.

In this paper, we derive Liouville equation with fractional derivatives with respect to coordinates and momenta. To derive the fractional Liouville equation, we consider the conservation of probability to find a system in the fractional differential volume element. Using the fractional Liouville equation, we derive the fractional generalization of the Bogoliubov hierarchy equations. These equations can be used to derive fractional kinetic equations.^{4,9,10,12} A linear fractional kinetic equation for the system of charged particles is suggested.

In Sec. 2, we derive the Liouville equation with fractional derivatives from the conservation of probability to find a system in the fractional volume element of the phase space. In Sec. 3, we obtain the first Bogoliubov hierarchy equation with

fractional derivatives in the phase space from the fractional Liouville equation. In Sec. 4, the Vlasov equation with fractional derivatives in phase space is considered. In Sec. 5, a linear fractional kinetic equation for the system of charged particles is suggested. Finally, a short conclusion is given in Sec. 6.

2. Liouville Equation with Fractional Derivatives

A basic principle of statistical mechanics is the conservation of probability. The Liouville equation is an expression of this basic principle in a convenient form for the analysis. In this section, we derive the Liouville equation with fractional derivatives from the conservation of probability in a fractional volume element.

For the phase space R^{2n} with coordinates $(x^1, \dots, x^{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n)$, we consider a fractional differential volume element

$$d^\alpha V = d^\alpha x_1 \cdots d^\alpha x_{2n}. \quad (1)$$

Here, d^α is a fractional differential.²⁵ For the function $f(x)$,

$$d^\alpha f(x) = \sum_{k=1}^{2n} D_{x_k}^\alpha f(x) (dx_k)^\alpha, \quad (2)$$

where $D_{x_k}^\alpha$ is a fractional derivative³ of order α with respect to x_k . The Caputo derivative^{6,26} is defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(z)}{(x-z)^{\alpha+1-m}} dz, \quad (3)$$

where $m-1 < \alpha < m$, $f^{(m)}(\tau) = d^m f(\tau)/d\tau^m$, and $\Gamma(z)$ is the Euler gamma-function. For Caputo and Riesz³ fractional derivatives, we have $D_{x_k}^\alpha 1 = 0$, and $D_{x_k}^\alpha x_l = 0 (k \neq l)$. Using Eq. (2), we obtain

$$d^\alpha x_k = D_{x_k}^\alpha x_k (dx_k)^\alpha, \quad (\alpha > 0). \quad (4)$$

Then

$$(dx_k)^\alpha = (D_{x_k}^\alpha x_k)^{-1} d^\alpha x_k. \quad (5)$$

For Caputo derivatives,

$$D_{x_k}^\alpha x_k^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x_k^{\beta-\alpha}, \quad (\beta > \alpha), \quad (6)$$

we get

$$(dx_k)^\alpha = (D_{x_k}^\alpha x_k)^{-1} d^\alpha x_k = \Gamma(2-\alpha) x_k^{\alpha-1} d^\alpha x_k. \quad (7)$$

The conservation of probability to find a many-particle system in the phase-space volume element $d^\alpha V$ may be expressed as

$$-dV \frac{\partial \rho(t, x)}{\partial t} = d[\rho(t, x)(\mathbf{u}, d\mathbf{S})], \quad (8)$$

for the usual volume element ($\alpha = 1$), and

$$-d^\alpha V \frac{\partial \rho(t, x)}{\partial t} = d^\alpha [\rho(t, x)(\mathbf{u}, d^\alpha \mathbf{S})], \tag{9}$$

for non-integer α . Here, $\rho = \rho(t, x)$ is the density of probability to find a many-particle system in the phase-space volume element; $\mathbf{u} = \mathbf{u}(t, x)$ is the velocity vector field, $d^\alpha \mathbf{S}$ is a surface element, and the brackets $(,)$ is a scalar product of vectors:

$$\mathbf{u} = \sum_{k=1}^{2n} u_k \mathbf{e}_k, \quad d^\alpha \mathbf{S} = \sum_{k=1}^{2n} d^\alpha S_k \mathbf{e}_k, \quad (\mathbf{u}, d^\alpha \mathbf{S}) = \sum_{k=1}^{2n} u_k d^\alpha S_k, \tag{10}$$

where \mathbf{e}_k are the basic vectors of Cartesian coordinate system, and

$$d^\alpha S_k = d^\alpha x_1 \cdots d^\alpha x_{k-1} d^\alpha x_{k+1} \cdots d^\alpha x_{2n}. \tag{11}$$

The functions $u_k = u_k(t, x)$ define x_k components of the velocity vector field $\mathbf{u}(t, x)$, the rate at which probability density is transported through the area element $d^\alpha S_k$. In the usual case ($\alpha = 1$), the outflow of probability in the x_k direction is defined by

$$d[\rho u_k] dS_k = D_{x_k} [\rho u_k] dx_k dS_k = D_{x_k} [\rho u_k] dV. \tag{12}$$

The fractional generalization of Eq. (12) is

$$\begin{aligned} d^\alpha [\rho u_k] d^\alpha S_k &= D_{x_k}^\alpha [\rho u_k] (dx)^\alpha d^\alpha S_k \\ &= (D_{x_k}^\alpha x_k)^{-1} D_{x_k}^\alpha [\rho u_k] d^\alpha x_k d^\alpha S_k = (D_{x_k}^\alpha x_k)^{-1} D_{x_k}^\alpha [\rho u_k] d^\alpha V. \end{aligned} \tag{13}$$

Here, we use Eqs. (11), (1) and (5). Substitution of Eq. (13) into Eq. (9) gives

$$-d^\alpha V \frac{\partial \rho}{\partial t} = d^\alpha V \sum_{k=1}^{2n} (D_{x_k}^\alpha x_k)^{-1} D_{x_k}^\alpha [\rho u_k]. \tag{14}$$

As the result, we obtain

$$\frac{\partial \rho}{\partial t} = - \sum_{k=1}^{2n} (D_{x_k}^\alpha x_k)^{-1} D_{x_k}^\alpha [\rho u_k]. \tag{15}$$

For Caputo derivatives,

$$\frac{\partial \rho}{\partial t} = -\Gamma(2 - \alpha) \sum_{k=1}^{2n} x_k^{\alpha-1} D_{x_k}^\alpha [\rho u_k]. \tag{16}$$

Equation (15) is a Liouville equation that contains the derivatives of fractional order α . Fractional Liouville equation (15) describes the probability conservation to find a system in the fractional volume element (1) of the phase space.

For the coordinates $(x^1, \dots, x^{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n)$, Eq. (15) is

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^n ((D_{q_k}^\alpha q_k)^{-1} D_{q_k}^\alpha [\rho V_k]) + \sum_{k=1}^n ((D_{p_k}^\alpha p_k)^{-1} D_{p_k}^\alpha [\rho F_k]) = 0, \tag{17}$$

where $V_k = u_k$, and $F_k = u_{k+n}$ ($k = 1, \dots, n$). The functions $V_k = V_k(t, q, p)$ are the components of velocity field, and $F_k = F_k(t, q, p)$ are the components of force field. In general,

$$D_{p_k}^\alpha [\rho F_k] \neq \rho D_{p_k}^\alpha F_k + F_k D_{p_k}^\alpha \rho. \tag{18}$$

Suppose that F_k does not depend on p_k , and the k th component V_k of the velocity field does not depend on k th component q_k of coordinates. In this case, Eq. (17) gives

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^n V_k (D_{q_k}^\alpha q_k)^{-1} D_{q_k}^\alpha \rho + \sum_{k=1}^n F_k (D_{p_k}^\alpha p_k)^{-1} D_{p_k}^\alpha \rho = 0. \tag{19}$$

If we consider the fractional generalization of Hamiltonian system,²¹ then V_k and F_k can be represented as fractional derivatives of some function $H(q, p)$:

$$V_k = D_{p_k}^\alpha H(q, p), \quad F_k = -D_{q_k}^\alpha H(q, p). \tag{20}$$

For $\alpha = 1$, we have the usual Hamiltonian system.

The classical system that is defined by the equations

$$\frac{dq_k}{dt} = V_k(t, q, p), \quad \frac{dp_k}{dt} = F_k(t, q, p) \tag{21}$$

is called a Hamiltonian system, if the right-hand sides of equations (21) satisfy the Helmholtz conditions

$$\frac{\partial V_i}{\partial p_j} - \frac{\partial V_j}{\partial p_i} = 0, \quad \frac{\partial V_j}{\partial q_i} + \frac{\partial F_i}{\partial p_j} = 0, \quad \frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} = 0. \tag{22}$$

In this case, Eq. (21) can be presented in the form

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \tag{23}$$

that is uniquely defined by the Hamiltonian H . The fractional Hamiltonian systems are defined in Ref. 20.

For Eq. (20), Eq. (19) is

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^n (D_{q_k}^\alpha q_k)^{-1} D_{p_k}^\alpha H D_{q_k}^\alpha \rho - \sum_{k=1}^n (D_{p_k}^\alpha p_k)^{-1} D_{q_k}^\alpha H D_{p_k}^\alpha \rho = 0. \tag{24}$$

We can define

$$\{A, B\}_\alpha = \sum_{k=1}^n ((D_{q_k}^\alpha q_k)^{-1} D_{q_k}^\alpha A D_{p_k}^\alpha B - (D_{p_k}^\alpha p_k)^{-1} D_{q_k}^\alpha B D_{p_k}^\alpha A). \tag{25}$$

For $\alpha = 1$, Eq. (25) defines Poisson brackets. Note that the brackets (25) satisfy the relations

$$\{A, B\}_\alpha = -\{B, A\}_\alpha, \quad \{1, A\}_\alpha = 0.$$

In general, the Jacoby identity cannot be satisfied. The property $\{1, A\}_\alpha = 0$ is satisfied only for Caputo and Riesz fractional derivatives ($D_x^\alpha 1 = 0$). For the Riemann–Liouville derivative, $D_x^\alpha 1 \neq 0$. Using Eq. (25), we get Eq. (24) in the form

$$\frac{\partial \rho}{\partial t} + \{\rho, H\}_\alpha = 0. \quad (26)$$

As the result, we have the Liouville equation for the fractional generalization of Hamiltonian systems²¹ that are defined by Eq. (20). For $\alpha = 1$, Eq. (26) is the usual Liouville equation for Hamiltonian systems.

3. Bogoliubov Equation with Fractional Derivatives

Let us consider a classical system with a fixed number N of identical particles. Suppose the k th particle is described by the generalized coordinates q_{ks} and generalized momenta p_{ks} , where $s = 1, \dots, m$. We use the notations $\mathbf{q}_k = (q_{k1}, \dots, q_{km})$ and $\mathbf{p}_k = (p_{k1}, \dots, p_{km})$. The state of this system is described by the N -particle density of probability ρ_N in the $2mN$ -dimensional phase space:

$$\rho_N(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_N, \mathbf{p}_N, t).$$

The fractional Liouville equation is

$$\frac{\partial \rho_N}{\partial t} = - \sum_{k=1}^N (\mathbf{D}_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) + \mathbf{D}_{\mathbf{p}_k}^\alpha (\mathbf{F}_k \rho_N)), \quad (27)$$

where

$$\mathbf{D}_{\mathbf{q}_k}^\alpha \mathbf{V}_k = (D_{\mathbf{q}_k}^\alpha \mathbf{q}_k)^{-1} D_{\mathbf{q}_k}^\alpha \mathbf{V}_k = \sum_{s=1}^m (D_{q_{ks}}^\alpha q_{ks})^{-1} D_{q_{ks}}^\alpha V_{ks}, \quad (28)$$

$$\mathbf{D}_{\mathbf{p}_k}^\alpha \mathbf{F}_k = (D_{\mathbf{p}_k}^\alpha \mathbf{p}_k)^{-1} D_{\mathbf{p}_k}^\alpha \mathbf{F}_k = \sum_{s=1}^m (D_{p_{ks}}^\alpha p_{ks})^{-1} D_{p_{ks}}^\alpha F_{ks}. \quad (29)$$

The one-particle reduced density of probability ρ_1 can be defined by

$$\rho_1(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}[2, \dots, N] \rho_N(\mathbf{q}, \mathbf{p}, t), \quad (30)$$

where $\hat{I}[2, \dots, N]$ is an integration with respect to variables $\mathbf{q}_2, \dots, \mathbf{q}_N, \mathbf{p}_2, \dots, \mathbf{p}_N$. Obviously, that one-particle density of probability satisfies the normalization condition

$$\hat{I}[1] \rho_1(\mathbf{q}, \mathbf{p}, t) = 1. \quad (31)$$

The Bogoliubov hierarchy equations^{27–30} describe the evolution of the reduced density of probability. They can be derived from the Liouville equation. Let us derive the first Bogoliubov equation with fractional derivatives from the fractional Liouville equation (27). Differentiation of Eq. (30) with respect to time gives

$$\frac{\partial \rho_1}{\partial t} = \frac{\partial}{\partial t} \hat{I}[2, \dots, N] \rho_N = \hat{I}[2, \dots, N] \frac{\partial \rho_N}{\partial t}.$$

Using Eq. (27), we get

$$\frac{\partial \rho_1}{\partial t} = -\hat{I}[2, \dots, N] \sum_{k=1}^N (\mathbf{D}_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) + \mathbf{D}_{\mathbf{p}_k}^\alpha (\mathbf{F}_k \rho_N)). \quad (32)$$

Let us consider the integration over \mathbf{q}_k and \mathbf{p}_k for the k -particle term of Eq. (32). Since the coordinates and momenta are independent variables, we derive

$$\hat{I}[\mathbf{q}_k] \mathbf{D}_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) = \hat{I}[\mathbf{q}_k] (D_{\mathbf{q}_k}^\alpha \mathbf{q}_k)^{-1} D_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) \sim (\mathbf{V}_k \rho_N)_{-\infty}^{+\infty} = 0. \quad (33)$$

For example, the Caputo derivatives give

$$\begin{aligned} \hat{I}[\mathbf{q}_k] \mathbf{D}_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) &= \Gamma(2 - \alpha) \hat{I}[\mathbf{q}_k] \mathbf{q}_k^{\alpha-1} D_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) \\ &= \Gamma(\alpha) \Gamma(2 - \alpha) \hat{I}^\alpha[\mathbf{q}_k] D_{\mathbf{q}_k}^\alpha (\mathbf{V}_k \rho_N) = \Gamma(\alpha) \Gamma(2 - \alpha) (\mathbf{V}_k \rho_N)_{-\infty}^{+\infty} = 0, \end{aligned} \quad (34)$$

where $\hat{I}^\alpha[\mathbf{q}_k]$ is a fractional integration with respect to variables \mathbf{q}_k . In Eq. (33), we use the fact that the density of probability ρ_N in the limit $\mathbf{q}_k \rightarrow \pm\infty$ is equal to zero. It follows from the normalization condition. If the limit is not equal to zero, then the integration over the phase space is equal to infinity. Similarly, we obtain

$$\hat{I}[\mathbf{p}_k] \mathbf{D}_{\mathbf{p}_k}^\alpha (\mathbf{F}_k \rho_N) \sim (\mathbf{F}_k \rho_N)_{-\infty}^{+\infty} = 0.$$

Then all terms in Eq. (32) with $k = 2, \dots, N$ are equal to zero. We have only the term with $k = 1$. Therefore Eq. (32) has the form

$$\frac{\partial \rho_1}{\partial t} = -\hat{I}[2, \dots, N] (\mathbf{D}_{\mathbf{q}_1}^\alpha (\mathbf{V}_1 \rho_N) + \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_1 \rho_N)). \quad (35)$$

Since the variable \mathbf{q}_1 is an independent of $\mathbf{q}_2, \dots, \mathbf{q}_N$ and $\mathbf{p}_2, \dots, \mathbf{p}_N$, the first term in Eq. (35) can be written as

$$\hat{I}[2, \dots, N] \mathbf{D}_{\mathbf{q}_1}^\alpha (\mathbf{V}_1 \rho_N) = \mathbf{D}_{\mathbf{q}_1}^\alpha \mathbf{V}_1 \hat{I}[2, \dots, N] \rho_N = \mathbf{D}_{\mathbf{q}_1}^\alpha (\mathbf{V}_1 \rho_1).$$

The force \mathbf{F}_1 acts on the first particle. It is a sum of the internal forces

$$\mathbf{F}_{1k} = \mathbf{F}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_k, \mathbf{p}_k, t),$$

and the external force $\mathbf{F}_1^e = \mathbf{F}^e(\mathbf{q}_1, \mathbf{p}_1, t)$. In the case of binary interaction, we have

$$\mathbf{F}_1 = \mathbf{F}_1^e + \sum_{k=2}^N \mathbf{F}_{1k}. \quad (36)$$

Using Eq. (36), the second term in Eq. (35) can be rewritten in the form

$$\begin{aligned} \hat{I}[2, \dots, N] \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_1 \rho_N) &= \hat{I}[2, \dots, N] (\mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_1^e \rho_N) + \sum_{k=2}^N \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_{1k} \rho_N)) \\ &= \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_1^e \rho_1) + \sum_{k=2}^N \mathbf{D}_{\mathbf{p}_1}^\alpha \hat{I}[2, \dots, N] (\mathbf{F}_{1k} \rho_N). \end{aligned} \quad (37)$$

We assume that ρ_N is invariant under the permutations of identical particles:³¹

$$\rho_N(\dots, \mathbf{q}_k, \mathbf{p}_k, \dots, \mathbf{q}_l, \mathbf{p}_l, \dots, t) = \rho_N(\dots, \mathbf{q}_l, \mathbf{p}_l, \dots, \mathbf{q}_k, \mathbf{p}_k, \dots, t).$$

In this case, ρ_N is a symmetric function, and all $(N - 1)$ terms of sum (37) are identical. Therefore the sum can be replaced by one term with the multiplier $(N - 1)$:

$$\sum_{k=2}^N \hat{I}[2, \dots, N] \mathbf{D}_{\mathbf{p}_{1s}}^\alpha (\mathbf{F}_{1k} \rho_N) = (N - 1) \hat{I}[2, \dots, N] \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_{12} \rho_N). \quad (38)$$

Using $\hat{I}[2, \dots, N] = \hat{I}[2] \hat{I}[3, \dots, N]$, we rewrite the right-hand side of (38) in the form

$$\hat{I}[2] \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_{12} \hat{I}[3, \dots, N] \rho_N) = \mathbf{D}_{\mathbf{p}_1}^\alpha \hat{I}[2] (\mathbf{F}_{12} \rho_2), \quad (39)$$

where ρ_2 is two-particle density of probability that is defined by the fractional integration of the N -particle density of probability over all \mathbf{q}_k and \mathbf{p}_k , except $k = 1, 2$:

$$\rho_2 = \rho(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \hat{I}[3, \dots, N] \rho_N(\mathbf{q}, \mathbf{p}, t). \quad (40)$$

Since \mathbf{p}_1 is independent of $\mathbf{q}_2, \mathbf{p}_2$, we can change the order of the integrations and the differentiations:

$$\hat{I}[2] \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_{12} \rho_2) = \mathbf{D}_{\mathbf{p}_1}^\alpha \hat{I}[2] \mathbf{F}_{12} \rho_2.$$

Finally, we obtain

$$\frac{\partial \rho_1}{\partial t} + \mathbf{D}_{\mathbf{q}_1}^\alpha (\mathbf{V}_1 \rho_1) + \mathbf{D}_{\mathbf{p}_1}^\alpha (\mathbf{F}_1^e \rho_1) = I(\rho_2). \quad (41)$$

Here, $I(\rho_2)$ is defined by

$$I(\rho_2) = -(N - 1) \mathbf{D}_{\mathbf{p}_1}^\alpha \hat{I}[2] \mathbf{F}_{12} \rho_2. \quad (42)$$

The term $I(\rho_2)$ describes a velocity of particle number change in $4m$ -dimensional two-particle elementary phase volume. This change is caused by the interactions between particles.

Equation (41) is the fractional generalization of the first Bogoliubov equation. If $\alpha = 1$, then we have the first Bogoliubov equation for non-Hamiltonian systems.³⁶ For Hamiltonian systems,

$$\mathbf{F}_1 = -\frac{\partial H(q_1, p_1)}{\partial q_1}, \quad \mathbf{V}_1 = \frac{\partial H(q_1, p_1)}{\partial p_1}, \quad (43)$$

and Eq. (41) has the well-known form.²⁷⁻³⁰

4. Vlasov Equation with Fractional Derivatives

Let us consider the particles as statistical independent systems.^{32,33} Then

$$\rho_2(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \rho_1(\mathbf{q}_1, \mathbf{p}_1, t) \rho_1(\mathbf{q}_2, \mathbf{p}_2, t). \quad (44)$$

Substitution of Eq. (44) into Eq. (42) gives

$$I(\rho_2) = -\mathbf{D}_{\mathbf{p}_1}^\alpha \rho_1 \hat{I}[2] \mathbf{F}_{12} \rho_1(\mathbf{q}_2, \mathbf{p}_2, t), \quad (45)$$

where $\rho_1 = \rho_1(\mathbf{q}_1, \mathbf{p}_1, t)$. As the result, the effective force is

$$\mathbf{F}^{\text{eff}}(\mathbf{q}_1, \mathbf{p}_1, t) = \hat{I}[2] \mathbf{F}_{12} \rho_1(\mathbf{q}_2, \mathbf{p}_2, t).$$

In this case, we can rewrite Eq. (45) in the form

$$I(\rho_2) = -\mathbf{D}_{\mathbf{p}_1}^\alpha (\rho_1 \mathbf{F}^{\text{eff}}). \quad (46)$$

Substituting of Eq. (46) into Eq. (41), we obtain

$$\frac{\partial \rho_1}{\partial t} + \mathbf{D}_{\mathbf{q}_1}^\alpha (\mathbf{V}_1 \rho_1) + \mathbf{D}_{\mathbf{p}_1}^\alpha ((\mathbf{F}_1^e + (N-1)\mathbf{F}^{\text{eff}})\rho_1) = 0. \quad (47)$$

This equation is a closed equation for the one-particle density of probability with the external force \mathbf{F}_1^e and the effective force \mathbf{F}^{eff} . Equation (47) is the fractional generalization of the Vlasov equation that has coordinate derivatives of non-integer order. For $\alpha = 1$, we get the Vlasov equation for the non-Hamiltonian systems that is described by non-potential fields. For Hamiltonian systems (43), Eq. (47) has the usual form.^{32,33}

5. Linear Fractional Kinetic Equation for a System of Charged Particles

Let us consider fractional kinetic equation (41) with $I(\rho_2) = 0$, $\mathbf{V}_1 = \mathbf{p}/m = \mathbf{v}$, and $\mathbf{F}^e = e\mathbf{E}$, $\mathbf{B} = 0$. In this case, Eq. (41) has the form

$$\frac{\partial f}{\partial t} + (\mathbf{v}, \mathbf{D}_{\mathbf{q}}^\alpha f) + e(\mathbf{E}, \mathbf{D}_{\mathbf{p}}^\alpha f) = 0, \quad (48)$$

where $f = \rho_1$ is the one-particle density of probability, and

$$(\mathbf{v}, \mathbf{D}_{\mathbf{q}}^\alpha f) = \sum_{s=1}^m (v_s, \mathbf{D}_{\mathbf{q}_s}^\alpha f). \quad (49)$$

If we take into account the magnetic field ($\mathbf{B} \neq 0$), then we must use the fractional generalization of Leibnitz rules:

$$\mathbf{D}_{\mathbf{p}}^\alpha (fg) = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} (\mathbf{D}_{\mathbf{p}}^{\alpha-r} f) D_{p_g}^r g, \quad (50)$$

where r are integer numbers. In this case, Eq. (48) has the addition term

$$\begin{aligned} \frac{e}{mc} \mathbf{D}_{\mathbf{p}}^\alpha ([\mathbf{p}, \mathbf{B}]f) &= \frac{e}{mc} \sum_{klm} \mathbf{D}_{p_k}^\alpha (\varepsilon_{klm} p_l B_m f) = \frac{e}{mc} \sum_{klm} \varepsilon_{klm} B_m \mathbf{D}_{p_k}^\alpha (p_l f) \\ &= \frac{e}{mc} \sum_{klm} \varepsilon_{klm} B_m \sum_{r=0}^1 \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} [\mathbf{D}_{p_k}^{\alpha-r} f] \delta_{kl} p_l^r \\ &= \frac{e}{mc} \sum_{klm} \varepsilon_{klm} B_m ([\mathbf{D}_{p_k}^\alpha f] p_l + \alpha [\mathbf{D}_{p_k}^{\alpha-1} f] \delta_{kl}) \\ &= \frac{e}{mc} \sum_{klm} \varepsilon_{klm} B_m p_l [\mathbf{D}_{p_k}^\alpha f] = \frac{e}{mc} ((\mathbf{D}_{p_k}^\alpha f), [\mathbf{p}, \mathbf{B}]). \end{aligned} \quad (51)$$

Let us consider the perturbation^{34,35} of the density of probability f_0 in the form

$$f(t, q, p) = f_0 + \delta f(t, q, p), \quad (52)$$

where f_0 is a homogeneous stationary density of probability that satisfies Eq. (48) for $\mathbf{E} = 0$. Substituting of Eq. (52) into (48), we get for the first perturbation

$$\frac{\partial \delta f}{\partial t} + (\mathbf{v}, \mathbf{D}_{\mathbf{q}}^\alpha \delta f) + e(\mathbf{E}, \mathbf{D}_{\mathbf{p}}^\alpha f_0) = 0. \tag{53}$$

Equation (53) is the linear fractional kinetic equation for the first perturbation of the density of probability. Solutions of fractional linear kinetic equations of type (53) are considered in Ref. 10. For $\mathbf{E} = 0$, the function δf is described by

$$(g_{st})^{-1/\alpha} L_\alpha [q_s (g_{st})^{-1/\alpha}], \tag{54}$$

where $g_s = v_s (D_{q_s}^\alpha q_s)^{-1}$, and

$$L_\alpha [x] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{-ikx} e^{-|k|^\alpha} \tag{55}$$

is the Levy stable density of probability.⁴⁰

For $\alpha = 1$, the function (55) gives the Cauchy distribution

$$L_1 [x] = \frac{1}{\pi} \frac{1}{x^2 + 1}, \tag{56}$$

and Eq. (54) is

$$\frac{1}{\pi} \frac{(g_{st})^{-1}}{q_s^2 (g_{st})^{-2} + 1}. \tag{57}$$

For $\alpha = 2$, we get the Gauss distribution:

$$L_2 [x] = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}, \tag{58}$$

and the function (54) is

$$(g_{st})^{-1/2} \frac{1}{2\sqrt{\pi}} e^{-q_s^2/(4g_{st})}. \tag{59}$$

For $1 < \alpha \leq 2$, the function $L_\alpha [x]$ can be presented as the expansion

$$L_\alpha [x] = -\frac{1}{\pi x} \sum_{n=1}^{\infty} (-x)^n \frac{\Gamma(1 + n/\alpha)}{n!} \sin(n\pi/2). \tag{60}$$

The asymptotic ($x \rightarrow \infty, 1 < \alpha < 2$) is given by

$$L_\alpha [x] \sim -\frac{1}{\pi x} \sum_{n=1}^{\infty} (-1)^n x^{-n\alpha} \frac{\Gamma(1 + n\alpha)}{n!} \sin(n\pi/2). \tag{61}$$

As the result, we arrive at the asymptotic of the solution, which exhibits power-like tails for $x \rightarrow \infty$. The tail is the important property for the solutions of fractional equations.

6. Conclusion

In this paper, we consider equations with derivatives of non-integer order that can be used in statistical mechanics and kinetic theory. We derive the Liouville, Bogoliubov and Vlasov equations with fractional derivatives with respect to coordinates and momenta. To derive the fractional Liouville equation, we consider the conservation of probability to find a system in the fractional differential volume element. Using the fractional Liouville equation, we obtain the fractional generalization of the Bogoliubov hierarchy equations. Fractional Bogoliubov equations can be used to derive fractional kinetic equations.^{4,9,10} The fractional kinetic is related to equations that have derivatives of non-integer orders. Fractional equations appear in the description of chaotic dynamics and fractal media. For the fractional linear oscillator, the physical meaning of the derivative of order $\alpha < 2$ is dissipation. Note that fractional derivatives with respect to coordinates can be connected with the long-range power-law interaction of the systems.^{22–24}

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