Modern Physics Letters B
Vol. 30, No. 10 (2016) 1650132 (11 pages)

DOI: 10.1142/S0217984916501323

Electric field in media with power-law spatial dispersion

Vasily E. Tarasov
Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University,
Moscow 119991, Russia
tarasov@theory.sinp.msu.ru

Received 17 July 2015
Accepted 10 January 2016
Published 5 April 2016

In this paper, we consider electric fields in media with power-law spatial dispersion (PLSD). Spatial dispersion means that the absolute permittivity of the media depends on the wave vector. Power-law type of this dispersion is described by derivatives and integrals of non-integer orders. We consider electric fields of point charge and dipole in media with PLSD, infinite charged wire, uniformly charged disk, capacitance of spherical capacitor and multipole expansion for PLSD-media.

Keywords: Electrodynamics; spatial dispersion; nonlocal media; fractional derivative; fractional integral; fractional dynamics; fractional electrodynamics.

1. Introduction

Spatial dispersion is a dependence of the absolute permittivity tensor of a medium on the wave vector.[1–3] This dependence leads to a number of phenomena, such as the rotation of the plane of polarization, anisotropy of cubic crystals and other (see Refs. 4–13). The spatial dispersion is caused by nonlocal connection between the electric induction $\mathbf{D}$ and the electric field $\mathbf{E}$. The vector $\mathbf{D}$ at any point $\mathbf{r}$ of the medium is not uniquely defined by the values of $\mathbf{E}$ at this point. It also depends on the values of $\mathbf{E}$ at neighboring points $\mathbf{r}'$, located near the point $\mathbf{r}$. Spatial dispersion medium can be considered as a medium, where the moving free charges create electric and magnetic fields, which significantly distorts the external field and the motion of the charges themselves.[1–3]

The fractional calculus[18–22] as a theory of differentiation and integrations of non-integer orders have a wide application in different areas of physics.[26–28] The fractional-order derivatives and integrals have a lot of unusual properties.[23–25] The mathematical tool, which is based on these operators, allows us to describe the behavior of media and fields that are characterized by power-law nonlocality and memory (hereditarity).

Power-law spatial dispersion (PLSD) of the media can be described by using derivatives and integrals of non-integer order.[14–16] The fractional-order differential equations for electric fields in media with PLSD are derived in Ref. 14. The particular solutions of these equations for the electric field of point charge are also considered in Ref. 14.

As it was shown in Refs. 31 and 32 the equations with fractional-order derivatives are directly connected with microstructural models with long-range interactions. A connection between the dynamics of system with long-range interactions and the fractional fields or continuum equations is proved by using the transform operation.[31–33] A microscopic model in the framework of fractional kinetics to describe spatial dispersion of power-law type[14] has been suggested in Ref. 17. The Liouville equation with the spatial Caputo derivatives of non-integer order is used to get the power-law dependence of the absolute permittivity on the wave vector for PLSD-media.

In this paper, we use the theory, which is proposed in Ref. 14, to describe the electric fields of point charge and dipole in media with PLSD, the fields of the charged infinite wire and uniformly charged disk, the capacitance of spherical capacitor and the multipole expansion for PLSD-media.

2. Power-Law Spatial Dispersion in Electromagnetism

In this section, we briefly describe the spatial dispersion in electrodynamics to fix notation for further consideration. For more details, can refer to Refs. 1–4 and 14.

The behavior of electric fields $(\mathbf{E}, \mathbf{D})$, magnetic fields $(\mathbf{B}, \mathbf{H})$, charge density ρ, and current density $\mathbf{j}$ is described by the Maxwell's equations.[1–3] For electromagnetic fields which are changed slowly in a linear medium, we have the constitutive equations in the form $D_i(t, \mathbf{r}) = \varepsilon_{ij} E_j(t, \mathbf{r})$, $B_i(t, \mathbf{r}) = \mu_{ij} H_j(t, \mathbf{r})$, where ε_{ij} and μ_{ij} are second-rank tensors. For fields varying in space rapidly, we should consider the influence of the field at remote points $\mathbf{r}'$ on the electromagnetic properties of the medium at a given point $\mathbf{r}$. The field at a given point $\mathbf{r}$ of the medium will be determined not only the value of the field at this point, but the field in the areas of environment, where the influence of the field is transferred. For example, it can be caused by the transport processes in the medium. Therefore, we should use nonlocal space relations that take into account spatial dispersion. In linear electrodynamics, these constitutive relation has the form

$$D_i(t, \mathbf{r}) = \int_{\mathbb{R}^3} \hat{\varepsilon}_{ij}(\mathbf{r}, \mathbf{r}') E_j(t, \mathbf{r}') d\mathbf{r}' \,. \tag{1}$$

For unbounded and homogeneous media, the kernel $\hat{\varepsilon}_{ij}(\mathbf{r}, \mathbf{r}')$ is a function of the difference $\mathbf{r} - \mathbf{r}'$ in the form $\hat{\varepsilon}_{ij}(\mathbf{r}, \mathbf{r}') = \hat{\varepsilon}_{ij}(\mathbf{r} - \mathbf{r}')$.

For an isotropic medium, the dependence of the tensor of the absolute permittivity

$$\varepsilon_{ij}(\mathbf{k}) = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\mathbf{r}} \hat{\varepsilon}_{ij}(\mathbf{r}) d\mathbf{r}' \tag{2}$$

depends on the wave vector $\mathbf{k}$ in the form

$$\varepsilon_{ij}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right)\varepsilon_{\perp}(|\mathbf{k}|) + \frac{k_i k_j}{|\mathbf{k}|^2}\varepsilon_{\parallel}(|\mathbf{k}|), \tag{3}$$

where $\varepsilon_{\perp}(|\mathbf{k}|)$ is the transverse permittivity, and $\varepsilon_{\parallel}(|\mathbf{k}|)$ is the longitudinal permittivity.

In the static case, we have an electric field $\mathbf{E}(t,\mathbf{r}) = \mathbf{E}(\mathbf{r})$, which is given by the equation

$$\mathbf{E}(\mathbf{r}) = -\mathrm{grad}\,\Phi(\mathbf{r})\,, \tag{4}$$

where $\Phi(\mathbf{r})$ is a scalar potential of electric field. The electrostatic potential of the point charge is described by delta-distribution

$$\rho(\mathbf{r}) = Q\delta^{(3)}(\mathbf{r})\,. \tag{5}$$

In the isotropic medium, we have

$$\Phi(\mathbf{r}) = \frac{Q}{(2\pi)^3}\int_{\mathbb{R}^3} e^{+i\mathbf{k}\mathbf{r}}\frac{1}{|\mathbf{k}|^2\varepsilon_{\parallel}(|\mathbf{k}|)}d^3\mathbf{k}\,, \tag{6}$$

where $\Phi(\mathbf{r})$ is the electric potential created by a point charge Q at a distance $|\mathbf{r}|$ from the charge.

For simplest case $\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0$, where ε_0 is the vacuum permittivity, the equation for the field $\Phi(\mathbf{r})$ has the form

$$\Delta\Phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r})\,, \tag{7}$$

where Δ is the Laplace operator. The electrostatic potential of the point charge (5) has the Coulomb's form

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}|}\,. \tag{8}$$

For the media with the absolute permittivity,

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0\left(1 + \frac{1}{r_D^2|\mathbf{k}|^2}\right)\,, \tag{9}$$

the corresponding equation for electric potential is

$$\Delta\Phi(\mathbf{r}) - \frac{1}{r_D^2}\Phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r})\,. \tag{10}$$

The solution of this equation gives the screened potential of the point charge (5) in the Debye's form:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}|}\exp\left(-\frac{|\mathbf{r}|}{r_D}\right)\,, \tag{11}$$

where r_D is the Debye's radius of screening. It is easy to see that the Debye's potential differs from the Coulomb's potential by factor $C_D(|\mathbf{r}|) = \exp(-|\mathbf{r}|/r_D)$. Such factor is a decay factor for the Coulomb's law, where the parameter r_D defines the distance over which significant charge separation can occur. The Debye's sphere with the radius r_D is a region, outside of which the electric charges are screened.

In Ref. 14, we consider media with a simple form of PLSD. The suggested model is described by the deformation of two terms in Eq. (9) for permittivity in the from

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0 \left(|\mathbf{k}|^{\alpha-2} + \frac{1}{r_D^2 |\mathbf{k}|^{2-\beta}} \right) . \tag{12}$$

The parameter α characterizes the deviation from Coulomb's law due to nonlocal properties of the medium. The parameter β characterizes the deviation from Debye's screening due to non-integer power-law type of nonlocality of the medium. The suggested simple models allow us to consider new possible types of an anomalous behavior of media with power-law type of nonlocality. Substitution of (12) into (6) gives the equation for scalar potential

$$((-\Delta)^{\alpha/2}\Phi)(\mathbf{r}) + a_\beta((-\Delta)^{\beta/2}\Phi)(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}), \tag{13}$$

where $(-\Delta)^{\alpha/2}$ and $(-\Delta)^{\beta/2}$ are the Riesz fractional Laplacian,[18] and $a_\beta = r_D^{-2}$. Note that $\mathbf{r}$ and r_D are dimensionless variables.

Let us consider particular solutions of the fractional differential equation (13) where $1 < \alpha$, $0 < \beta < \alpha$, $\beta < 3$. Equation (13) has the following particular solution

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} G_{\alpha,\beta}(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d^3\mathbf{r}' \,, \tag{14}$$

where the Green type function $G_{\alpha,\beta}(\mathbf{r})$ is given by

$$G_{\alpha,\beta}(\mathbf{r}) = \frac{|\mathbf{r}|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty \left(|\lambda|^\alpha + a_\beta|\lambda|^\beta\right)^{-1} \lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)d\lambda \,. \tag{15}$$

We can distinguish two special cases of spatial dispersion media described by Eq. (13): (1) Nonlocal deformation of Coulomb's law corresponds to the case $\beta = 0$; (2) Nonlocal deformation of Debye's screening corresponds to the case $\alpha = 2$.

Let us give a short description of the electric field for these special cases.

(1) Fractional model of nonlocal deformation of Coulomb's law corresponds to the case $\beta = 0$ and $\alpha \neq 2$, $\alpha > 1$. In this case Eq. (12) has the form

$$((-\Delta)^{\alpha/2}\Phi)(\mathbf{r}) + \frac{1}{r_D^2}\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}) \,. \tag{16}$$

This model allows us to describe a possible deviation from the Coulomb's law in the media with nonlocal properties defined by power-law type of spatial dispersion. The electrostatic potential of the point charge in a media with this type of spatial dispersion has the form $\Phi(\mathbf{r}) \sim |\mathbf{r}|^{\alpha-3}$ for $1 < \alpha < 2$ and $2 < \alpha < 3$ on small distances $|\mathbf{r}| \to 0$. In the case $\alpha > 3$, we have the constant value of the potential for $|\mathbf{r}| \to 0$.

(2) Fractional model of nonlocal deformation of Debye's screening is defined by $\alpha = 2$ and $0 < \beta < 2$. In this case Eq. (12) has the form

$$-\Delta\Phi(\mathbf{r}) + \frac{1}{r_D^2}((-\Delta)^{\beta/2}\Phi)(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}) \,. \tag{17}$$

Such model allows us to describe a possible deviation from the Debye's screening by nonlocal properties of the media with the generalized power-law type of spatial dispersion. The generalized nonlocal properties deform the Debye's screening such that the exponential decay is replaced by the fractional power-law, and the electrostatic potential of the point charge in the media with this type of spatial dispersion is given by $\Phi(\mathbf{r}) \sim |\mathbf{r}|^{\beta-3}$ for $0 < \beta < 2$ on the long distance $|\mathbf{r}| \to \infty$.

The behavior of electrostatic potentials for fractional differential models described by Eqs. (16) and (17) is considered with details in Ref. 14.

3. Electrostatic Field of Point Charge in Media with PLSD

For the point charge (5) in the PLSD-media, the electrostatic potential (14) has the form

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}C_{\alpha,\beta}(r)\,, \tag{18}$$

where $\mathbf{r}$ is the radius-vector from the charge to the point, $r = |\mathbf{r}|$, and the function

$$C_{\alpha,\beta}(r) = \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin(\lambda r)}{\lambda^\alpha + a_\beta\lambda^\beta}d\lambda \tag{19}$$

describes the difference of the Coulomb's potential.

For $\alpha = 2$ and $a_\beta = 0$, we have $C_{\alpha,\beta}(r) = 1$ and the electrostatic potential of the point charge in the vacuum is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\,. \tag{20}$$

Electric field $\mathbf{E}$ is defined as a gradient of the potential $\mathbf{E} = -\nabla\Phi$. Using

$$\frac{\partial}{\partial r}\left(\frac{\sin(\lambda r)}{r}\right) = \frac{(\lambda r)\cos(\lambda r) - \sin(\lambda r)}{r^2}, \tag{21}$$

where $r = |\mathbf{r}|$, we obtain the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\cdot B_{\alpha,\beta}(r)\,, \tag{22}$$

where

$$B_{\alpha,\beta}(r) = \frac{2}{\pi}\int_0^\infty \frac{\lambda(\sin(\lambda r) - (\lambda r)\cos(\lambda r))}{\lambda^\alpha + a_\beta\lambda^\beta}d\lambda\,. \tag{23}$$

The function $B_{\alpha,\beta}(r)$ describes the nonlocal properties of media with PLSD. For $\alpha = 2$ and $a_\beta = 0$, we get well-known equation in the form (22) with $B_{\alpha,\beta}(r) = 1$.

4. Electric Dipole in Media with PLSD

The electric dipole is a combination of the two equal in magnitude of opposite point charges, located at some distance from each other.

The electrostatic potential of dipole in the PLSD-media is

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \left(\frac{C_{\alpha,\beta}(r_+)}{r_+} - \frac{C_{\alpha,\beta}(r_-)}{r_-} \right), \tag{24}$$

where $\mathbf{r}_\pm$ are vectors from point charges of signs $\pm$ to a given observation point.

The vectors $\mathbf{r}_\pm$ are connected by the equation $\mathbf{r}_- = \mathbf{r}_+ + \mathbf{l}$, where $\mathbf{l}$ is the vector from the point with a negative charge to a point with a positive charge.

Substitution of (19) into (24) gives

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{2}{\pi} \int_0^\infty d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta \lambda^\beta} \left(\frac{\sin(\lambda r_+)}{r_+} - \frac{\sin(\lambda r_-)}{r_-} \right). \tag{25}$$

Let the positive charge be at the point with coordinates $(-l/2; 0; 0)$ and the negative at $(l/2; 0; 0)$. Then

$$r_\pm = \sqrt{(x \pm l/2)^2 + y^2 + z^2}, \quad \mathbf{l} = (-l; 0; 0), \tag{26}$$

$$r_\pm = \sqrt{x^2 + y^2 + z^2 \pm lx + l^2/4} = \sqrt{r^2 \pm lx + l^2/4} = r \cdot \sqrt{1 \pm \frac{lx}{r^2} + \frac{l^2}{4r^2}}. \tag{27}$$

Using the approximate formulas

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x, \quad \sin(\alpha(1-x)) \approx \sin\alpha - (\alpha x)\cos(\alpha), \tag{28}$$

for $x \to 0$ and Eq. (27), we get

$$r_\pm \approx r \left(1 \pm \frac{lx}{2r^2} \right), \tag{29}$$

$$\sin(\lambda r_\pm) = \sin\left(\lambda r \left(1 \pm \frac{lx}{2r^2} \right) \right) \approx \sin(\lambda r) \pm \lambda r \frac{lx}{2r^2} \cos(\lambda r). \tag{30}$$

Then

$$\frac{\sin(\lambda r_\pm)}{r_\pm} \approx \left(\sin(\lambda r) \pm \lambda r \frac{lx}{2r^2} \cos(\lambda r) \right) \left(\frac{1}{r} \mp \frac{lx}{2r^3} \right) \tag{31}$$

$$= \frac{\sin(\lambda r)}{r} \pm \frac{\lambda r l x}{2r^3} \cos(\lambda r) \mp \frac{lx}{2r^3} \sin(\lambda r). \tag{32}$$

As a result, we have

$$\frac{\sin(\lambda r_+)}{r_+} - \frac{\sin(\lambda r_-)}{r_-} \approx \frac{\lambda r l x}{2r^3} \cos(\lambda r) - \frac{lx}{2r^3} \sin(\lambda r). \tag{33}$$

Using $\mathbf{l} = (-l; 0; 0)$ and $\mathbf{r} = (x; y; z)$, we have $(\mathbf{l}, \mathbf{r}) = -lx$, and then we obtain

$$\frac{\sin(\lambda r_+)}{r_+} - \frac{\sin(\lambda r_-)}{r_-} \approx \frac{(\mathbf{l}, \mathbf{r})}{2r^3} \left(\sin(\lambda r) - (\lambda r) \cos(\lambda r) \right). \tag{34}$$

The electrostatic potential of an electric dipole in the media with PLSD is

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{(\mathbf{l}, \mathbf{r})}{r^3} A_{\alpha,\beta}(r), \tag{35}$$

where

$$A_{\alpha,\beta}(r) = \frac{1}{\pi}\int_0^\infty d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta \lambda^\beta}\left(\sin(\lambda r) - (\lambda r)\cos(\lambda r)\right). \tag{36}$$

The function $A_{\alpha,\beta}(r)$ describes the nonlocal properties of dipole field in PLSD-media. Product of the vector $\mathbf{l}$, which is drawn from the negative charge to positive, and the absolute value of the charge Q is the dipole moment $\mathbf{d} = Q\mathbf{l}$.

For $\alpha = 2$ and $a_\beta = 0$, Eq. (35) gives the well-known equation for the potential $\Phi(\mathbf{r})$ of electric dipole

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{(\mathbf{l},\mathbf{r})}{r^3}, \tag{37}$$

where ε_0 is the permittivity of free space.

5. Infinite Charged Wire

In this section, we consider the electric potential of a uniformly charged wire with a linear density τ. Let the wire be located along the z-axis. If we have an "infinitely long" uniformly linearly distributed charge, we can determine the potential by integration.

At a given point at distance R from the wire, the contribution to the potential from an infinitesimal section of wire of length dz' is

$$\frac{\tau}{4\pi\epsilon_0 r}dz' = \frac{1}{4\pi\epsilon_0}\frac{\tau}{\sqrt{R^2 + (z-z')^2}}dz', \tag{38}$$

where $\tau dz'$ is the charge on that section of wire.

The integration over z from $-\infty$ to $+\infty$ gives the potential of the infinite wire

$$\Phi(R) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{\tau}{\sqrt{R^2 + (z-z')^2}} \cdot C_{\alpha,\beta}(\sqrt{R^2 + (z-z')^2})dz'. \tag{39}$$

Using the explicit form of the function $C_{\alpha,\beta}$, which is given by Eq. (19), we get

$$\Phi(R) = \frac{\tau}{4\pi\varepsilon_0}\frac{2}{\pi}\int_0^\infty d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta\lambda^\beta}\left(\int_{-\infty}^{+\infty}\frac{\sin(\lambda\sqrt{R^2 + (z-z')^2})}{\sqrt{R^2 + (z-z')^2}}dz'\right). \tag{40}$$

Using the variable

$$r = \sqrt{R^2 + (z-z')^2}, \tag{41}$$

we have

$$\int_{-\infty}^{+\infty}\frac{\sin(\lambda\sqrt{R^2 + (z-z')^2})}{\sqrt{R^2 + (z-z')^2}}dz' = 2\int_R^{+\infty}\frac{\sin(\lambda r)}{\sqrt{r^2 - R^2}}dr. \tag{42}$$

Then we use the integral (see Sec. 2.5.6.2 of Ref. 30) of the form

$$\int_a^\infty (x^2 - a^2)^{\beta-1}\sin(bx)dx = \frac{\sqrt{\pi}}{2}\left(\frac{2a}{b}\right)^{\beta-1/2}\Gamma(\beta)J_{1/2-\beta}(ab), \tag{43}$$

where $a > 0$, $b > 0$, $0 < \beta < 1$, and $J_{1/2-\beta}$ is the Bessel function of the first kind. In our case, we have $a = R$, $b = \lambda$, $\beta = 1/2$, and

$$2\int_{R}^{+\infty} \frac{\sin(\lambda r)}{\sqrt{r^2 - R^2}} dr = \pi J_0(R\lambda) , \tag{44}$$

where

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2} . \tag{45}$$

Electric field of infinite wire in the media with PLSD has the form

$$\Phi(R) = \frac{\tau}{2\pi\varepsilon_0} \int_0^{\infty} \frac{\lambda J_0(\lambda R)}{\lambda^\alpha + a_\beta \lambda^\beta} d\lambda . \tag{46}$$

The electric field $\mathbf{E} = -\nabla\Phi$ at the points perpendicularly away from the wire, and is inversely proportional to the first power of the separation distance.

6. Uniformly Charged Disk

Let us consider a uniformly charged disk of radius a with the total charge Q. The potential at a distance h at the perpendicular to the disk plane passing through its center is

$$\Phi(h) = \frac{1}{4\pi\varepsilon_0} \int_S dxdy \frac{\sigma}{\sqrt{x^2 + y^2 + h^2}} C_{\alpha,\beta}(\sqrt{x^2 + y^2 + h^2}) , \tag{47}$$

where $\sigma = Q/(\pi a^2)$ is the charge density per unit area of the disk plane.

Using cylindrical coordinates (R, α, h) instead of the Cartesian (x, y, z), we get

$$dxdy = dS = RdRd\alpha, \quad R = \sqrt{x^2 + y^2} , \tag{48}$$

and

$$\Phi(h) = \frac{\sigma}{4\pi\varepsilon_0} dR \int_0^{2\pi} d\alpha \int_0^a \frac{R}{\sqrt{R^2 + h^2}} C_{\alpha,\beta}(\sqrt{R^2 + h^2}) . \tag{49}$$

Using Eq. (19) of the function $C_{\alpha,\beta}$, we get

$$\Phi(h) = \frac{Q}{2\pi a^2 \varepsilon_0} \frac{2}{\pi} \int_0^{\infty} d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta \lambda^\beta} \left(\int_0^a \frac{\sin(\lambda\sqrt{R^2 + h^2})}{\sqrt{R^2 + h^2}} RdR \right) . \tag{50}$$

Using the variable $r = \sqrt{R^2 + h^2}$, we obtain

$$\int_0^a \frac{\sin(\lambda\sqrt{R^2 + h^2})}{\sqrt{R^2 + h^2}} RdR = \int_0^{\sqrt{a^2+h^2}} \sin(\lambda r) dr = \frac{1 - \cos(\lambda\sqrt{a^2 + h^2})}{\lambda} . \tag{51}$$

As a result, we have the relation

$$\Phi(h) = \frac{Q}{\pi^2 a^2 \varepsilon_0} \int_0^{\infty} \frac{\sin^2(\lambda\sqrt{a^2 + h^2}/2)}{\lambda^\alpha + a_\beta \lambda^\beta} d\lambda , \tag{52}$$

where we used the equation $1 - \cos(x) = 2\sin^2(x/2)$. Equation (52) described the electric field of uniformly charged disk in the PLSD-media.

7. Capacitance of Spherical Capacitor

The capacitance for spherical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each.

The voltage between the spheres is

$$\Phi_2 - \Phi_1 = \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{C_{\alpha,\beta}(r_1)}{r_1} - \frac{C_{\alpha,\beta}(r_2)}{r_2}\right). \tag{53}$$

Using the explicit form of $C_{\alpha,\beta}$, we have

$$\Phi_2 - \Phi_1 = \frac{Q}{4\pi\varepsilon_0}\int_0^\infty d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta\lambda^\beta}\left(\frac{\sin(\lambda r_1)}{r_1} - \frac{\sin(\lambda r_2)}{r_2}\right). \tag{54}$$

From the definition, the capacitance of spherical capacitor is

$$C = \frac{Q}{\Phi_2 - \Phi_1} = \frac{4\pi\varepsilon_0 r_1 r_2}{r_2 C_{\alpha,\beta}(r_1) - r_1 C_{\alpha,\beta}(r_2)}, \tag{55}$$

where $C_{\alpha,\beta}(r)$ is defined by Eq. (19).

8. Multipole Expansion for Media with PLSD

We can consider an electric multipole expansion for a distribution of charged particles in the PLSD-media. Let $\mathbf{R}$ be a vector from a fixed reference point to the observation point, and $\mathbf{r} = x_k\mathbf{e}_k$ be a vector from the reference point to a point in the distribution of charges. The potential $\Phi(\mathbf{R})$ of the electric field for this distribution is defined by the equation

$$\Phi(\mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\int_W \frac{\rho(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|} C_{\alpha,\beta}(|\mathbf{R}-\mathbf{r}|) d^3r, \tag{56}$$

where $(\mathbf{R}-\mathbf{r})$ is a vector from a point in the distribution to the observation point. Using the theorem of cosines, we have

$$|\mathbf{R}-\mathbf{r}|^2 = r^2 + R^2 - 2rR\cos\theta, \tag{57}$$

where $r = |\mathbf{r}|$, $R = |\mathbf{R}|$, and θ is the polar angle, defined such that $\cos\theta = (\mathbf{r},\mathbf{R})/|\mathbf{r}||\mathbf{R}|$. Equation (57) gives

$$|\mathbf{R}-\mathbf{r}| = R\sqrt{1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}}. \tag{58}$$

Using the variables

$$\epsilon = \frac{r}{R}, \quad \xi = \cos\theta = \frac{(\mathbf{r},\mathbf{R})}{rR}, \tag{59}$$

we represent Eq. (58) in the form

$$|\mathbf{R}-\mathbf{r}| = R\sqrt{1 - 2\epsilon\xi + \epsilon^2}. \tag{60}$$

Substituting the expression (19) of the function $C_{\alpha,\beta}$ into (56), we get

$$\Phi(\mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\frac{2}{\pi}\int_0^\infty d\lambda \frac{\lambda}{\lambda^\alpha + a_\beta\lambda^\beta}\left(\int_W \frac{\rho(\mathbf{r})\sin(\lambda|\mathbf{R}-\mathbf{r}|)}{|\mathbf{R}-\mathbf{r}|} d^3r\right). \tag{61}$$

The right-hand side of Eq. (61) is the generating function for Legendre polynomials $P_n(\xi)$:

$$\frac{1}{|\mathbf{R}-\mathbf{r}|} = \frac{1}{R}\left(1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}\right)^{-1/2} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta)\,. \tag{62}$$

Using (62) and

$$\sin(\lambda|\mathbf{R}-\mathbf{r}|) = \sin\left(\lambda R\sqrt{1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}}\right)\,, \tag{63}$$

we obtain

$$\frac{\sin(\lambda|\mathbf{R}-\mathbf{r}|)}{|\mathbf{R}-\mathbf{r}|} \approx A_0 + A_1\epsilon + A_2\epsilon^2 + A_3\epsilon^3 + o(\epsilon^3), \tag{64}$$

where

$$A_0 = \frac{1}{R}\sin(\lambda R), \tag{65}$$

$$A_1 = \frac{1}{R}(\sin(\lambda R)\xi - \cos(\lambda R)\lambda R\xi), \tag{66}$$

$$\begin{aligned} A_2 = \frac{1}{2R}&\left(-\sin(\lambda R) + 3\sin(\lambda R)\xi^2 - \sin(\lambda R)\lambda^2 R^2\xi^2 - 3\cos(\lambda R)\lambda R\xi^2\right. \\ &\left.+\cos(\lambda R)\lambda R\right)\,, \end{aligned} \tag{67}$$

$$\begin{aligned} A_3 = \frac{1}{6R}&\left(-6\sin(\lambda R)\lambda^2R^2\xi^3 - 9\sin(\lambda R)\xi + 15\sin(\lambda R)\xi^3 + 3\sin(\lambda R)\lambda^2R^2\xi\right. \\ &\left.+9\cos(\lambda R)\lambda R\xi - 15\cos(\lambda R)\lambda R\xi^3 + \cos(\lambda R)\lambda^3R^3\xi^3\right)\,, \end{aligned} \tag{68}$$

and $\xi = \cos\theta = (\mathbf{r},\mathbf{R})/(rR)$.

Substitution of (64) with (65)–(68) into Eq. (61), we get the electric multipole expansion for distributions of charged particles in the PLSD-media.

9. Conclusion

In this paper, we consider new area of fractional nonlocal electrodynamics that deals with a description of electric and magnetic fields in complex media with PLSD. The fractional-order electromagnetic theory, which is considered in this paper, can be characterized by universal spatial behavior of electromagnetic fields in such media by analogy with the universal temporal behavior of low-loss dielectrics.[34–36]

References

1. V. P. Silin and A. A. Ruhadze, *Electromagnetic Properties of Plasmas and Plasma-Like Media* (Gosatomizdat, Moscow, 1961), 2nd edn. (USSR, Librikom, Moscow, 2012) (in Russian).
2. A. F. Alexandrov, L. S. Bogdankevich and A. A. Rukhadze, *Fundamentals of Plasma Electrodynamics* (Vysshaya Shkola, Moscow, 1978) (in Russian).
3. A. F. Alexandrov and A. A. Rukhadze, *Lectures on the Electrodynamics of Plasma-Like Media* (Moscow State University Press, 1999) (in Russian).

4. A. A. Rukhadze and V. P. Silin, *Sov. Phys. Usp.* **4**(3) (1961) 459–484.
5. V. L. Ginzburg and V. M. Agranovich, *Sov. Phys. Usp.* **5**(2) (1962) 323–346.
6. V. L. Ginzburg and V. M. Agranovich, *Sov. Phys. Usp.* **5**(4) (1963) 675–710.
7. N. B. Baranova and B. Ya. Zel'dovich, *Sov. Phys. Usp.* **22**(3) (1979) 143–159.
8. V. M. Agranovich and Yu. N. Gartstein, *Phys. Usp.* **49**(10) (2006) 1029–1044.
9. V. M. Agranovich and V. L. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons* (Interscience Publishers, 1966).
10. V. M. Agranovich and V. L. Ginzburg, *Crystal Optics with Spatial Dispersion and Excitons: An Account of Spatial Dispersion*, 2nd edn. (Springer-Verlag, Berlin, 1984).
11. M. V. Kuzelev and A. A. Rukhadze, *Methods of Waves Theory in Dispersive Media* (World Scientific, Zhurikh, 2009).
12. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics. Electrodynamics of Continuous Media*, Vol. 8, Chap. XII, 2nd edn. (Pergamon, Oxford, 1984).
13. P. Halevi (ed.), *Spatial Dispersion in Solids and Plasmas* (North-Holland, Amsterdam, New York, 1992).
14. V. E. Tarasov and J. J. Trujillo, *Ann. Phys.* **334** (2013) 1–23.
15. V. E. Tarasov, *Cent. Eur. J. Phys.* **11**(11) (2013) 1580–1588.
16. V. E. Tarasov, *ISRN Condens. Matter Phys.* **2014** (2014) 794097.
17. V. E. Tarasov, *Phys. Plasmas* **20**(10) (2013) 102110.
18. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
19. V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers. Background and Theory*, Vol. 1 (Springer/Higher Education Press, 2012).
20. M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers* (Springer, Netherlands, 2011).
21. R. E. Gutierrez, J. M. Rosario and J. A. Tenreiro Machado, *Math. Probl. Eng.* **2010** (2010) 375858.
22. D. Valerio, J. J. Trujillo, M. Rivero, J. A. Tenreiro Machado and D. Baleanu, *Eur. Phys. J. Spec. Topics* **222**(8) (2013) 1827–1846.
23. M. D. Ortigueira and J.A. Tenreiro Machado, *J Comput. Phys.* **293** (2015) 4–13.
24. V. E. Tarasov, *Commun. Nonlinear Sci. Numer. Simul.* **18**(11) (2013) 2945–2948.
25. V. E. Tarasov, *Commun. Nonlinear Sci. Numer. Simul.* **30**(1–3) (2016) 1–4.
26. J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado (eds.), *Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering* (Springer, Dordrecht, 2007).
27. V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, New York, 2011).
28. V. E. Tarasov, *Int. J. Mod. Phys. B* **27**(9) (2013) 1330005.
29. V. E. Tarasov, *Ann. Phys.* **323**(11) (2008) 2756–2778.
30. A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series. Special Functions*, Vol. 2 (Gordon Breach Science Publishers/CRC Press, 1988–1992).
31. V. E. Tarasov, *J. Phys. A* **39**(48) (2006) 14895–14910.
32. V. E. Tarasov, *J. Math. Phys.* **47**(9) (2006) 092901.
33. V. E. Tarasov, *Phys. A: Statist. Mech. Appl.* **421** (2015) 330–342.
34. A. K. Jonscher, *Nature* **267** (1977) 673–679.
35. A. K. Jonscher, *Universal Relaxation Law* (Chelsea Dielectrics, London, 1996).
36. V. E. Tarasov, *J. Phys.: Condens. Matter* **20**(17) (2008) 175223.