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WEYL QUANTIZATION OF DYNAMICAL SYSTEMS WITH FLAT PHASE SPACE

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The Weyl quantization rules that allow one to obtain quantum analogs of equations of motion for a wide class of dynamical systems with flat phase space are considered.

Let a system have n degrees of freedom and its phase space be a real linear space with dimensionality $2n$. Observables are functions $A(q,p)$, where $q,p \in R^n$. Quantization is usually understood as a single procedure [1, 2] where any classical observable, i. e., a real function $A(q,p)$, is associated with a relevant quantum observable, i. e., a self-adjoint operator $A(\hat{q},\hat{p})$. In this case, the function $A(q,p)$ itself is called a symbol of the operator $A(\hat{q},\hat{p})$.

We assume that evolution of a classical system in a flat phase space R^{2n} is described by the differential equation

$$\frac{d}{dt}A(q,p) = \mathcal{L}\left(q,p,\frac{\partial}{\partial q},\frac{\partial}{\partial p}\right)A(q,p), \tag{1}$$

where $A(q,p)$ is a smooth function defined in space R^{2n} and describing a classical observable, and $\mathcal{L}\left(q,p,\frac{\partial}{\partial q},\frac{\partial}{\partial p}\right)$ is a linear differential operator defined in the space of smooth functions. In order to obtain evolution equations for quantum systems, one should specify the rules making it possible to quantize classical equations of motion (1) of the corresponding differential operators. In defining these rules, one often tries to write the equations of motion in a Hamiltonian form [3–6], i. e. in terms of the Poisson bracket with a certain Hamilton function. However, in the general case, it is difficult to determine whether the Hamilton function exists, and, if it does, whether it is unique, and to find its explicit form if it exists and is unique [7]. Therefore, for classical dynamical systems of a general form, quantization is more conveniently carried out starting from equations of motion. In the present paper, we propose the rules for Weyl quantization of evolution equations of classical dynamical systems with flat phase space.

1. LIE–JORDAN ALGEBRAS

Let the set of observables form a linear space $\mathcal{M}_0$ over the field of real numbers R. For the observables from $\mathcal{M}_0$, we define two bilinear multiplication operations denoted by symbols $\cdot$ and $\circ$ and satisfying the conditions

(1) $<\mathcal{M}_0,\cdot>$ is a Lie algebra:

$$A\cdot B = -B\cdot A, \quad (A\cdot B)\cdot C + (B\cdot C)\cdot A + (C\cdot A)\cdot B = 0;$$

(2) $<\mathcal{M}_0,\circ>$ is a special Jordan algebra:

$$A\circ B = B\circ A, \quad ((A\circ A)\circ B)\circ A = (A\circ A)\circ(B\circ A);$$

(3) the differentiation identity for the Jordan algebra and the relation identity for associators are fulfilled:

$$A \cdot (B \circ C) = (A \cdot B) \circ C + B \circ (A \cdot C),$$

$$(A \circ B) \circ C - A \circ (B \circ C) = \frac{\hbar^2}{4}\Big((A \cdot B) \cdot C - A \cdot (B \cdot C)\Big).$$

In this case, the Lie–Jordan algebra is said to be defined [8, 9]. We shall also assume that there exists a unity I in $\mathcal{M}_0$ such that $A \circ I = A$ and $A \cdot I = 0$. We denote as $\mathcal{M}$ a free Lie–Jordan algebra over the field of real numbers R with unity I and generators q^k and p^k, where $k = 1, \ldots, n$ and

$$q^k \cdot p^l = \delta_{kl} I, \quad q^k \cdot q^l = 0, \quad p^k \cdot p^l = 0. \tag{2}$$

For the classical observables $A(q,p)$ and $B(q,p)$, these operations can be defined in terms of the Poisson bracket in R^{2n} and ordinary multiplication of functions,

$$\begin{aligned} A(q,p) \cdot B(q,p) &= \{A(q,p), B(q,p)\}, \\ A(q,p) \circ B(q,p) &= A(q,p)B(q,p). \end{aligned} \tag{3}$$

For the quantum observables $\hat{A} = A(\hat{q}, \hat{p})$ and $\hat{B} = B(\hat{q}, \hat{p})$ operations of Lie and Jordan multiplication are defined in the form of commutator and anticommutator,

$$\hat{A} \cdot \hat{B} = \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A}), \quad \hat{A} \circ \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}). \tag{4}$$

2. MULTIPLICATION ALGEBRA FOR THE LIE–JORDAN ALGEBRA

For any element $A \in \mathcal{M}$, we define two operators of left ($L_A^{\pm}$) and two operators of right ($R_A^{\pm}$) multiplications, which self-map $\mathcal{M}$ by the following rules:

$$L_A^+ C = A \circ C, \quad L_A^- C = A \cdot C,$$

$$R_A^+ C = C \circ A, \quad R_A^- C = C \cdot A$$

for any $C \in \mathcal{M}$. These maps are endomorphisms of the algebra $\mathcal{M}$ module. A subalgebra of the algebra of endomorphisms of module $\mathcal{M}$, generated by various operators $L_A^{\pm}$ and $R_A^{\pm}$, is called the multiplication algebra [10] of the Lie–Jordan algebra $\mathcal{M}$ and is denoted by $\mathcal{A}(\mathcal{M})$. Algebras generated by all operators of left and right multiplications of the Lie–Jordan algebra coincide with $\mathcal{A}(\mathcal{M})$ since $L_A^{\pm} = \pm R_A^{\pm}$.

The identities which the Lie–Jordan algebras meet lead to the relations for the multiplication operators. To obtain them, one should use the complete linearization [10] of the identities that define the algebra, and the properties of the Jordan multiplication commutativity and the Lie multiplication anticommutativity.

Theorem 1. *Multiplication algebra $\mathcal{A}(\mathcal{M})$ for the Lie–Jordan algebra $\mathcal{M}$ is defined by*

(1) *the Lie relations*

$$L^-_{A \cdot B} = L^-_A L^-_B - L^-_B L^-_A,$$

(2) *the Jordan relations*

$$L^+_{(A\circ B)\circ C} + L^+_B L^+_C L^+_A + L^+_A L^+_C L^+_B = L^+_{A\circ B} L^+_C + L^+_{B\circ C} L^+_A + L^+_{A\circ C} L^+_B,$$

$$L^+_{(A\circ B)\circ C} + L^+_B L^+_C L^+_A + L^+_A L^+_C L^+_B = L^+_C L^+_{A\circ B} + L^+_B L^+_{A\circ C} + L^+_A L^+_{B\circ C},$$

$$L^+_C L^+_{A\circ B} + L^+_B L^+_{A\circ C} + L^+_A L^+_{B\circ C} = L^+_{A\circ B} L^+_C + L^+_{B\circ C} L^+_A + L^+_{A\circ C} L^+_B,$$

(3) *the mixed relations*

$$L^+_{A \cdot B} = L^-_A L^+_B - L^+_B L^-_A, \quad L^-_{A\circ B} = L^+_A L^-_B + L^+_B L^-_A, \tag{5}$$

$$L^+_{A\circ B} = L^+_A L^+_B - \frac{\hbar^2}{4} L^-_B L^-_A, \quad L^+_B L^+_A - L^+_A L^+_B = -\frac{\hbar^2}{4} L^-_{A\cdot B}. \tag{6}$$

For the algebra of classical observables $\hbar = 0$.

From these relations, the following statements result.

Corollary 1. If the Jordan algebra is generated by the set $X = \{x^k\}$, then the corresponding multiplication algebra is generated by the set of operators $\{L^+_{x^k}, L^+_{x^k\circ x^m} \mid x^k, x^m \in X\}$.

Corollary 2. If the Lie algebra is generated by the set $X = \{x^k\}$, then the corresponding multiplication algebra is generated by the set of operators $\{L^-_{x^k} \mid x^k \in X\}$.

Corollary 3. If the Lie–Jordan algebra $\mathcal{M}$ is generated by the set $X = \{x^k\}$, then the corresponding multiplication algebra $\mathcal{A}(\mathcal{M})$ is generated by the set of operators $\{L^+_{x^k}, L^-_{x^k} \mid x^k \in X\}$.

Note that, in the latter statement, we mention the set of operators $\{L^+_{x^k}, L^-_{x^k} | x^k \in X\}$, and not the set $\{L^+_{x^k}, L^+_{x^k\circ x^l}, L^-_{x^k} | x^k, x^l \in X\}$, and this is due to first relation of (6).

Making use of the properties of generators (2) and relation (5), one easily proves the following theorem.

Theorem 2. *If the Lie–Jordan algebra $\mathcal{M}$ with unity I is generated by the set $X = \{q^k, p^k, I \mid k = 1, \ldots, n\}$, whose elements meet relations (2), the corresponding multiplication algebra $\mathcal{A}(\mathcal{M})$ is generated by the set of operators $\{L^\pm_{q^k}, L^\pm_{p^k}, L^\pm_I \mid q^k, p^k, I \in X\}$, that satisfy the commutation relations*

$$\begin{aligned} &[L^\pm_{q^k}, L^\mp_{p^l}] = \delta_{kl} L^+_I, \\ &[L^\pm_{q^k}, L^\pm_{q^l}] = [L^\pm_{q^k}, L^\pm_{p^l}] = [L^\pm_{p^k}, L^\pm_{p^l}] = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &[L^\mp_{q^k}, L^\pm_{q^l}] = [L^\mp_{p^k}, L^\pm_{p^l}] = 0, \\ &[L^\pm_{q^k}, L^\pm_I] = [L^\pm_{p^k}, L^\pm_I] = 0. \end{aligned} \tag{8}$$

Note that the given commutation relations for the generators $\{L^\pm_{q^k}, L^\pm_{p^k}, L^\pm_I \mid q^k, p^k, I \in X\}$ of the multiplication algebra $\mathcal{A}(\mathcal{M})$ are the same for classical and quantum observables. Relations (7), (8) define the Lie algebra, generated by operators $L^\pm_{q^k}, L^\pm_{p^k}, L^\pm_I$. The evident statement follows from Theorem 2: any element $\mathcal{L}$ of the multiplication algebra $\mathcal{A}(\mathcal{M})$ of the Lie–Jordan algebra $\mathcal{M}$ can be written as a polynomial $\mathcal{L}(L^\pm_{q^k}, L^\pm_{p^k}, L^\pm_I)$ in the multiplication operators $\{L^\pm_{q^k}, L^\pm_{p^k}, L^\pm_I\}$.

3. WEYL QUANTIZATION

Correspondence between operators $\hat{A} = A(\hat{q}, \hat{p})$ and symbols $A(q,p)$ is completely determined by the formulas that express the symbols of operators $\hat{q}^k\hat{A}$, $\hat{A}\hat{q}^k$, $\hat{p}^k\hat{A}$, $\hat{A}\hat{p}^k$ in terms of the symbol of the operator $\hat{A}$. Weyl quantization π_W is said to be defined if these formulas have the form

$$\begin{aligned} &\pi_W\left(\left(q^k + \frac{i\hbar}{2}\frac{\partial}{\partial p^k}\right)A(q,p)\right) = \hat{q}^k\hat{A}, \\ &\pi_W\left(\left(q^k - \frac{i\hbar}{2}\frac{\partial}{\partial p^k}\right)A(q,p)\right) = \hat{A}\hat{q}^k, \end{aligned} \tag{9}$$

$$\begin{aligned} &\pi_W\left(\left(p^k - \frac{i\hbar}{2}\frac{\partial}{\partial q^k}\right)A(q,p)\right) = \hat{p}^k\hat{A}, \\ &\pi_W\left(\left(p^k + \frac{i\hbar}{2}\frac{\partial}{\partial q^k}\right)A(q,p)\right) = \hat{A}\hat{p}^k \end{aligned} \tag{10}$$

for any $\hat{A} = \pi_W(A(q,p))$. The proof of validity of these formulas can be found in [11].

Definition (3) implies that the operators L^+_A and L^-_A acting on classical observables are defined by the formulas

$$\begin{aligned} &L^+_A B(q,p) = A(q,p)B(q,p), \\ &L^-_A B(q,p) = \{A(q,p), B(q,p)\}. \end{aligned} \tag{11}$$

By virtue of definition (4), the operators $\hat{L}_A^+$ and $\hat{L}_A^-$ are expressed as

$$\hat{L}_A^+\hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}), \quad \hat{L}_A^-\hat{B} = \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A}).$$

Making use of the multiplication operators $L_A^\pm$ and $\hat{L}_A^\pm$, we rewrite formulas (9) and (10) in the form

$$\pi_W(L_{q^k}^\pm A) = \hat{L}_{q^k}^\pm \hat{A}, \quad \pi_W(L_{p^k}^\pm A) = \hat{L}_{p^k}^\pm \hat{A}.$$

Since these relations are valid for any $\hat{A} = \pi_W(A)$, we can define the Weyl quantization of the multiplication operators $L_{q^k}^\pm$ and $L_{p^k}^\pm$ in the following manner:

$$\pi_W(L_{q^k}^\pm) = \hat{L}_{q^k}^\pm, \quad \pi_W(L_{p^k}^\pm) = \hat{L}_{p^k}^\pm. \tag{12}$$

These relations define the Weyl quantization of the generating operators of the multiplication algebra of the Lie–Jordan algebra of classical observables.

From definition (11) for classical observables, we get

$$L_{q^k}^+ A(q,p) = q^k A(q,p), \quad L_{p^k}^+ A(q,p) = p^k A(q,p) \tag{13}$$

and

$$L_{q^k}^- A(q,p) = \frac{\partial A(q,p)}{\partial p^k}, \quad L_{p^k}^- A(q,p) = -\frac{\partial A(q,p)}{\partial q^k}. \tag{14}$$

Expressions (13) and (14) enable the linear polynomial differentiation operator to be considered as an element of the multiplication algebra of the Lie–Jordan algebra of classical observables. The following theorem results.

Theorem 3. *The linear polynomial differentiation operator*

$$\mathcal{L}\left(q, p, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)$$

acting on the classical observables $A(q,p) \in \mathcal{M}$ is an element of the multiplication algebra $\mathcal{A}(\mathcal{M})$ of the Lie–Jordan algebra $\mathcal{M}$ of classical observables,

$$\mathcal{L}\left(q, p, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right) = \mathcal{L}(L_q^+, L_p^+, -L_p^-, L_q^-).$$

The proof evidently follows from the definition of the operators $L_{q^k}^\pm$ and $L_{p^k}^\pm$ specified in (13) and (14).

By virtue of (12) and commutation relations (7), (8), the Weyl quantization π_W associates the differential operator $\mathcal{L}\left(q, p, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)$ in the functional space and the operator $\mathcal{L}\left(\hat{L}_q^+, \hat{L}_p^+, -\hat{L}_p^-, \hat{L}_q^-\right)$, acting in the operator space. Thus, the Weyl quantization of differential equations with polynomial operators [12] that describe evolution of observables of dynamical system (1) is defined by the formula

$$\pi_W\left(\mathcal{L}\left(q, p, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)\right) = \mathcal{L}\left(\hat{L}_q^+, \hat{L}_p^+, -\hat{L}_p^-, \hat{L}_q^-\right).$$

Since the commutation relations for the operators $L_{q^k}^\pm$, $L_{p^k}^\pm$ and $\hat{L}_{q^k}^\pm$, $\hat{L}_{p^k}^\pm$ coincide, the following theorem takes place.

Theorem 4. *In the Weyl quantization, ordering of generating operators in the operator $\mathcal{L}\left(\hat{L}_q^+, \hat{L}_p^+, -\hat{L}_p^-, \hat{L}_q^-\right)$ is uniquely determined by ordering in the operator $\mathcal{L}\left(L_q^+, L_p^+, -L_p^-, L_q^-\right)$.*

Correspondence between polynomial differential operators and elements of the multiplication algebra of the Lie–Jordan algebra of classical observables can be extended up to correspondence between operators of a more general form and elements of a certain (normalized involute) multiplication algebra [12].

4. QUANTIZATION OF A LORENZ-TYPE SYSTEM

Consider an evolution equation for classical observable $A_t(q,p)$ for a classical dissipative Lorenz-type system [13] of the form

$$\frac{d}{dt}A_t(q,p) = (-\sigma q_1+\sigma p_1)\frac{\partial A_t(q,p)}{\partial q_1}+(rq_1-p_1-q_1p_2)\frac{\partial A_t(q,p)}{\partial p_1}+(\sigma p_2)\frac{\partial A_t(q,p)}{\partial q_2}+(-bp_2+q_1p_1)\frac{\partial A_t(q,p)}{\partial p_2}. \tag{15}$$

This equation written for the observables $x = q_1$, $y = p_1$, and $z = p_2$ describes the classical dissipative Lorenz model [14, 15],

$$x' = -\sigma x + \sigma y, \quad y' = rx - y - xz, \quad z' = -bz + xy,$$

where $x' = dx(t)/dt$. This model was proposed by Lorenz in [14], and it is the most known classical dissipative system with a strange attractor. This system demonstrates a chaotic behavior [14, 15] when parameters are chosen to be close to the values $\sigma = 10$, $r = 28$, $b = 8/3$.

The differential operator $\mathcal{L}$ for system (15) has the form

$$\mathcal{L}\left(q,p,\frac{\partial}{\partial q},\frac{\partial}{\partial p}\right) = (-\sigma q_1+\sigma p_1)\frac{\partial}{\partial q_1}+(rq_1-p_1-q_1p_2)\frac{\partial}{\partial p_1}+(\sigma p_2)\frac{\partial}{\partial q_2}+(-bp_2+q_1p_1)\frac{\partial}{\partial p_2}.$$

Rewriting this operator in terms of $L^{\pm}_{q^k}$ and $L^{\pm}_{p^k}$, we obtain

$$\mathcal{L} = -(-\sigma L^+_{q_1}+\sigma L^+_{p_1})L^-_{p_1}+(rL^+_{q_1}-L^+_{p_1}-L^+_{q_1}L^+_{p_2})L^-_{q_1}-(\sigma L^+_{p_2})L^-_{p_2}+(-bL^+_{p_2}+L^+_{q_1}L^+_{p_1})L^-_{q_2}.$$

The Weyl quantization of this operator leads to the following operator in the algebra of quantum observables:

$$\hat{\mathcal{L}} = -\left(-\sigma\hat{L}^+_{q_1}+\sigma\hat{L}^+_{p_1}\right)\hat{L}^-_{p_1}+\left(r\hat{L}^+_{q_1}-\hat{L}^+_{p_1}-\hat{L}^+_{q_1}\hat{L}^+_{p_2}\right)\hat{L}^-_{q_1}-\left(\sigma\hat{L}^+_{p_2}\right)\hat{L}^-_{p_2}+\left(-b\hat{L}^+_{p_2}+\hat{L}^+_{q_1}\hat{L}^+_{p_1}\right)\hat{L}^-_{q_2}.$$

Using the definition of the operators $\hat{L}^+_A, \hat{L}^-_A$, we get the quantum Lorenz-type equation

$$\frac{d}{dt}\hat{A}_t = \frac{i}{\hbar}\left[\frac{\sigma(\hat{p}_1^2+\hat{p}_2^2)}{2}-\frac{r\hat{q}_1^2}{2},\hat{A}_t\right]-\frac{i\sigma}{\hbar}\hat{q}_1\circ\left[\hat{p}_1,\hat{A}_t\right]+\frac{i}{\hbar}\hat{p}_1\circ\left[\hat{q}_1,\hat{A}_t\right]+\frac{i}{\hbar}b\hat{p}_2\circ\left[\hat{q}_2,\hat{A}_t\right]$$
$$+\frac{i}{\hbar}\hat{q}_1\circ\left(\hat{p}_2\circ\left[\hat{q}_1,\hat{A}_t\right]\right)-\frac{i}{\hbar}\hat{q}_1\circ\left(\hat{p}_1\circ\left[\hat{q}_2,\hat{A}_t\right]\right).$$

We notice that the Weyl quantization leads just to the following form of the last two terms: $\hat{q}_k\circ\left(\hat{p}_l\circ\left[\hat{q}_m,\hat{A}\right]\right)$, each of which is equal to $\hat{p}_l\circ\left(\hat{q}_k\circ\left[\hat{q}_m,\hat{A}\right]\right)$. However, they are not equal to $(\hat{q}_k\circ\hat{p}_l)\circ\left[\hat{q}_m,\hat{A}\right]$, which follows from (6).

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