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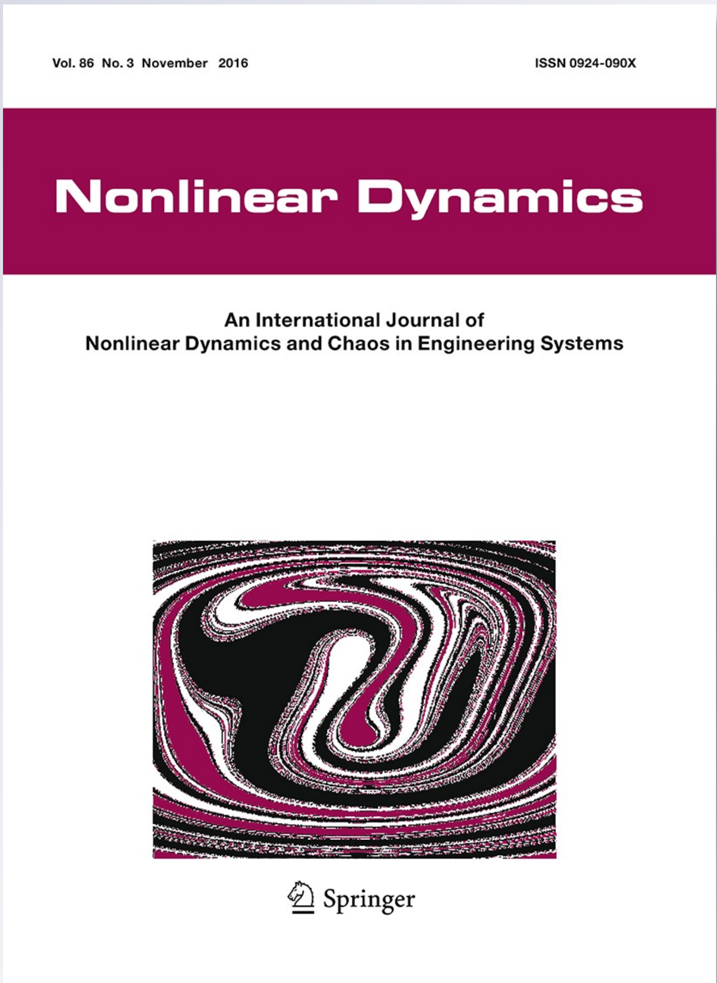

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ORIGINAL PAPER

Partial fractional derivatives of Riesz type and nonlinear fractional differential equations

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Abstract Generalizations of fractional derivatives of noninteger orders for N-dimensional Euclidean space are proposed. These fractional derivatives of the Riesz type can be considered as partial derivatives of noninteger orders. In contrast to the usual Riesz derivatives, the suggested derivatives give the usual partial derivatives for integer values of orders. For integer values of orders, the partial fractional derivatives of the Riesz type are equal to the standard partial derivatives of integer orders with respect to coordinate. Fractional generalizations of the nonlinear equations such as sine-Gordon, Boussinesq, Burgers, Korteweg–de Vries and Monge–Ampere equations for nonlocal continuum are considered.

Keywords Fractional calculus · Fractional derivative · Nonlocal continuum · Fractional dynamics · Nonlinear fractional equations

Mathematics Subject Classification 26A33

1 Introduction

Theory of fractional-order differentiation and integration has a big history [1–5]. It has been proposed various types and forms of fractional derivatives and integrals of noninteger orders. The most famous definitions were proposed by Riemann, Liouville, Grünwald, Letnikov, Marchaud, Riesz, Caputo [6–12]. All fractional-order derivatives have a set of interesting unusual characteristic properties [13–19] such as a violation of the usual Leibniz rule, a deformations of the usual chain rule, a violation of the semi-group property and other. For example, the derivatives and integrals of noninteger orders are noncommutative and nonassociative operators in general. The fractional derivatives violate the usual Leibniz rule, and this violation is a characteristic property for all types of fractional derivatives [20]. Fractional-order derivatives can be characterized by a deformations of the usual chain rule [14,15], which becomes more complicated.

Fractional calculus has a wide application in physics (for example see [21–27]), since it allows us to describe the behavior of systems and media that are characterized by power-law nonlocality and power-law long-term memory (hereditarity). The unusual mathematical properties of fractional-order operators give a possibility of building of new mathematical models for nontrivial and unusual complex continua systems, processes and media.

In papers [28,29], we propose the fractional derivatives with respect to coordinate x_j $\mathbb{D}^{\pm}_{C}\left[{\alpha \atop j}\right]$ of noninteger order α for N-dimensional Euclidean space. These fractional derivatives are given in two forms (even "+" and odd "−"). These derivatives cannot give the usual local derivatives for some integer values of α. The fractional derivatives of the Riesz type $\mathbb{D}^{+}_{C}\left[{\alpha \atop j}\right]$ give the par-

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tial derivatives of integer orders only for even values $\alpha = 2m > 0$, where $m \in \mathbb{N}$. The derivatives $\mathbb{D}^{-}_{C}\left[{\alpha \atop j}\right]$ give the partial derivatives of integer orders only for odd values $\alpha = 2m + 1 > 0$, where $m \in \mathbb{N}$. The derivatives $\mathbb{D}^{-}_{C}\left[{2m \atop j}\right]$ and $\mathbb{D}^{+}_{C}\left[{2m+1 \atop j}\right]$ are nonlocal differential operators of integer orders that cannot be represented as $\partial^{2m}/\partial x_j^{2m}$ and $\partial^{2m+1}/\partial x_j^{2m+1}$. As a result, the fractional generalization of local continuum models by using $\mathbb{D}^{\pm}_{C}\left[{\alpha_j \atop j}\right]$ cannot give a local description for some integer values of α_j. In addition, we have some arbitrariness in formulation of fractional nonlocal models that is caused by a possibility to use $\mathbb{D}^{+}_{C}\left[{\alpha_j \atop j}\right]$ and $\mathbb{D}^{-}_{C}\left[{\alpha_j \atop j}\right]$ in the same equation. To minimize this arbitrariness at fractional generalizations of local models, we suggested a rule of "fractionalization" in [29]. This rule suggests to replace the usual partial derivatives of odd orders with respect to x_j by the fractional derivatives $\mathbb{D}^{-}_{C}\left[{\alpha \atop j}\right]$, and the usual partial derivatives of even orders with respect to x_j by the fractional derivatives $\mathbb{D}^{+}_{C}\left[{\alpha \atop j}\right]$.

In this paper, we propose another way to minimize arbitrariness of fractional generalization of continuum models. We suggest new fractional-order derivatives $\mathbb{D}\left[{\alpha_j \atop j}\right]$ that are generalizations of the fractional derivatives $\mathbb{D}^{+}_{C}\left[{\alpha \atop j}\right]$ and $\mathbb{D}^{-}_{C}\left[{\alpha \atop j}\right]$. The main advantage of this generalized derivative is a direct connection with the usual partial derivatives for all integer orders. For integer values $\alpha = n \in \mathbb{N}$, these partial fractional derivatives of the Riesz type $\mathbb{D}\left[{\alpha \atop j}\right]$ are equal to the standard partial derivatives of integer orders n with respect to x_j

$$\mathbb{D}\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\partial^n}{\partial x_j^n} \quad (n \in \mathbb{N}).$$

This property allows us much easier to build fractional nonlocal models of continua and fields.

In this paper, we propose the partial fractional derivatives of the Riesz type. The Riesz fractional derivatives and potentials have been suggested in [30,31] (see also [6,32]). These operators also are considered in different works (see [6,8,9,33], [34–41,101] and [28,29,42–44]). There is a relation between the Riesz fractional derivative (see Sections 25 and 26 of [6]) and the left-sided and right-sided Liouville fractional derivatives, which is shown in Section 12 of [6] (see also [9]) for one-dimensional case. Another way of relating the Riesz derivative to other fractional derivatives is compositions of left-sided and right-sided Grünwald–Letnikov (see Section 20 of [6]), the Marchaud fractional derivative (see Section 5.4 of [6]) and the Caputo derivatives by using equations 2.4.6 and 2.4.7 of [8]. For example, the composition of left-sided and right-sided Grünwald–Letnikov gives the fractional Grünwald-Letnikov-Riesz derivatives (see Section 20 of [6]). The partial derivatives of this type are considered and applied in [45] (see also [46,47]).

It should be noted that the Grünwald-Letnikov fractional derivatives coincide with the Marchaud fractional derivatives for the function space $L_p(\mathbb{R})$, where $1 \leqslant p < \infty$ (see Theorem 20.4 in [6]). Therefore, the fractional Grünwald–Letnikov–Riesz derivatives can be considered as a composition of left-sided and right-sided regularized derivatives, one of which is the Marchaud fractional derivative (see Section 5.4 of [6], and about a regularized fractional derivative see also [48]).

In this paper, we develop an approach to formulation of partial fractional, which is proposed in [28,29]. The representation of the suggested partial fractional derivative is connected with representation of the Riesz fractional derivatives in the Lizorkin form, which is suggested in [33] and described in detail in Sections 25 and 26 of [6]. A main advantage of the suggested approach is based on the direct connection of these fractional derivatives with the lattice fractional derivatives that are proposed in [28,29]. Main disadvantage of the other approached, which can be based on the Liouville, Caputo, Marchaud fractional derivatives, is an absence of direct relationship with the lattice representation (see Sect. 5 of [28] and also [29]). An existence of a relationship with lattice models is more appropriate to get microstructural and nanostructured models of continuum and media with power-law nonlocality, where the long-range intermolecular and interatomic interactions are crucial in determining nonlocal properties. Using the suggested operators, we consider fractional generalizations of well-known nonlinear equations, such as sine-Gordon, Boussinesq, Burgers, Korteweg–de Vries and Monge–Ampere equations.

It should be noted that the suggested approach can be applied to the Riesz fractional derivatives, which are defined by the fractional m-dimensional differential operators [31,32]. In this case to define new partial fractional derivatives, we should use these operators in the one-dimensional form instead of the derivatives $\mathbb{D}^{+}_{C}\left[{\alpha \atop j}\right]$ in the corresponding equations.

2 Fractional derivatives of Riesz type

Let us give a definition of the fractional derivatives of the Riesz type that is proposed in [29].

Definition 1 The fractional derivative of the Riesz type of the order α with respect to x_j is defined by the equation

$$\mathbb{D}^+_C\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r}) := \frac{1}{d_1(m,\alpha)} \times \int_{\mathbb{R}^1} \frac{1}{|z_j|^{\alpha+1}} (\Delta^m_{z_j} f)(\mathbf{r})\, dz_j, \quad (0 < \alpha < m), \quad (1)$$

where $(\Delta^m_{z_j} f)(\mathbf{r})$ is a finite difference of order m of a function $f(\mathbf{r})$ with the vector step $\mathbf{z}_j = z_j\,\mathbf{e}_j \in \mathbb{R}^N$ for the point $\mathbf{r} \in \mathbb{R}^N$. The centered difference with respect to z_j

$$(\Delta^m_{z_j} f)(\mathbf{r}) := \sum_{n=0}^{m} (-1)^n \frac{m!}{n!\,(m-n)!} f(\mathbf{r}-(m/2-n)\, z_j\,\mathbf{e}_j). \quad (2)$$

The constant $d_1(m,\alpha)$ is defined by

$$d_1(m,\alpha) := \frac{\pi^{3/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma((1+\alpha)/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) := 2 \sum_{s=0}^{[m/2]} (-1)^{s-1} \frac{m!}{s!(m-s)!} (m/2-s)^\alpha$$

for the centered difference (2).

The constant $d_1(m,\alpha)$ is different from zero for all $\alpha > 0$ in the case of an even m and centered difference $(\Delta^m_j f)$ (see Theorem 26.1 in [6]). Note that the integral (1) does not depend on the choice of $m > \alpha$. Therefore, we can always choose an even number m so that it is greater than parameter α, and we can use the centered difference (2) for all positive real values of α.

Remark 1 Using (1), we can see that the partial fractional derivative $\mathbb{D}^+_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right] f(\mathbf{r})$ can be considered as the Riesz fractional derivative in the Lizorkin form [33] (see also Sections 25.4 and 26 of [6], and Section 2.10 of [8]) of the function $f(\mathbf{r})$ with respect to one component $x_j \in \mathbb{R}^1$ of the vector $\mathbf{r} \in \mathbb{R}^N$, i.e., the operator $\mathbb{D}^+_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ can be considered as a partial fractional derivative of the Riesz type.

An important property of the fractional derivatives of the Riesz type is the Fourier transform $\mathcal{F}$ of this operators.

Proposition 1 *The Fourier transform $\mathcal{F}$ of the fractional derivatives of the Riesz type with respect to x_j has the form*

$$\mathcal{F}\left(\mathbb{D}^+_C\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k}) = |k_j|^\alpha (\mathcal{F} f)(\mathbf{k}). \quad (3)$$

The property (3) is valid for functions $f(\mathbf{r})$ from for the Lizorkin space and the space $C^\infty(\mathbb{R}^1)$ of infinitely differentiable functions on $\mathbb{R}^1$ with compact support.

Proof The proof of this Proposition is a consequence of the properties of the Riesz fractional derivatives [6,8] for $\mathbb{R}^N$ with $N = 1$. □

Using that $(-i)^{2m} = (-1)^m$, the fractional derivatives of the Riesz type of even $\alpha = 2m$, where $m \in \mathbb{N}$, are connected with the usual partial derivative of integer orders $2m$ by the relation

$$\mathbb{D}^+_C\begin{bmatrix}2m\\ j\end{bmatrix} f(\mathbf{r}) = (-1)^m \frac{\partial^{2m} f(\mathbf{r})}{\partial x_j^{2m}}. \quad (4)$$

For $\alpha = 2$, the derivative of the Riesz type is the local operator $-\partial^2/\partial x_j^2$. The fractional derivatives $\mathbb{D}^+_C\left[\begin{smallmatrix}2m\\ j\end{smallmatrix}\right]$ for even orders α are local operators. Note that the derivative $\mathbb{D}^+_C\left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ cannot be considered as a derivative of first order with respect to x_j, i.e.,

$$\mathbb{D}^+_C\begin{bmatrix}1\\ j\end{bmatrix} f(\mathbf{r}) \neq \frac{\partial f(\mathbf{r})}{\partial x_j}, \quad (5)$$

and it is nonlocal operator.

Remark 2 The derivative $\mathbb{D}^+_C\left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ can be considered as the Hilbert transform of $|k_j|$ [9]. All derivatives of the Riesz type $\mathbb{D}^+_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for odd orders $\alpha = 2m+1$, where $m \in \mathbb{N}$, are nonlocal operators that cannot be considered as usual partial derivatives $\partial^{2m+1}/\partial x_j^{2m+1}$.

3 Conjugated fractional derivatives of Riesz type

Let us give a definition of the Riesz fractional integration, which is usually called the Riesz potential (see equations 25.31 and 25.32 of [6] and Section 2.10 of [8]).

Definition 2 The Riesz fractional integrals of order $\alpha > 0$ can be defined by the equation

$$\mathbf{I}^{\alpha}_{\mathbf{r}} f(\mathbf{r}) := \int_{\mathbb{R}^N} R_{\alpha,N}(\mathbf{r}-\mathbf{z})\, f(\mathbf{z})\, d^N\mathbf{z}, \quad (\alpha > 0), \tag{6}$$

where the Riesz kernel $R_{\alpha,N}(\mathbf{r})$ is defined by

$$R_{\alpha,N}(\mathbf{r}) = \frac{\Gamma((N-\alpha)/2)}{2^{\alpha}\, \pi^{N/2}\, \Gamma(\alpha/2)}\, |\mathbf{r}|^{\alpha-N} \tag{7}$$

if $\alpha \neq N+2n$ and $\alpha \neq N$, where $n, N \in \mathbb{N}$. For $\alpha = N+2n$ and $\alpha = N$, the Riesz kernel $R_{\alpha,N}(\mathbf{r})$ is

$$R_{\alpha,N}(\mathbf{r}) = -\frac{1}{\gamma_N(\alpha)}\, |\mathbf{r}|^{\alpha-N}\, \ln|\mathbf{r}|, \tag{8}$$

where

$$\gamma_N(\alpha) = \gamma_N(N+2n) = (-1)^n\, 2^{N+2n-1} \pi^{N/2}\, n!\, \Gamma(N/2+n), \tag{9}$$

and $n = 0, 1, 2, \ldots$.

The Riesz fractional integrals (6) have the Fourier transform $\mathcal{F}$ in the form

$$\mathcal{F}\Big(\mathbf{I}^{\alpha}_{\mathbf{r}} f(\mathbf{r})\Big) = |\mathbf{k}|^{-\alpha} (\mathcal{F} f)(\mathbf{k}). \tag{10}$$

Equation (10) holds for the operator (6) if the function $f(\mathbf{r})$ belongs to the Lizorkin space [6,8]. Note that the Lizorkin spaces are invariant with respect to application of the Riesz fractional integration. This means that the functions obtained by applying the Riesz integration to functions from the Lizorkin space also belong to this space.

In papers [28,29], we suggest to define the fractional integral $\mathbb{I}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ of the Riesz type as the Riesz potential of order α with respect to x_j.

Definition 3 Factional integral of the Riesz type with respect to x_j is defined by the equation

$$\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r}) = \int_{\mathbb{R}^1} R_{\alpha,1}(x_j - z_j) \times f(\mathbf{r} + (z_j - x_j)\,\mathbf{e}_j)\, dz_j, \quad (\alpha > 0), \tag{11}$$

where $\mathbf{r} = \sum_{j=1}^{N} x_j \mathbf{e}_j$ is the radius vector, $\mathbf{e}_j$ is the basis of the Cartesian coordinate system, and

$$f(\mathbf{r}) = f(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N),$$
$$f(\mathbf{r} + (z_j - x_j)\,\mathbf{e}_j) = f(x_1, x_2, \ldots, x_{j-1}, z_j, x_{j+1}, \ldots, x_N).$$

The Riesz kernel $R_{\alpha,1}(\mathbf{r})$ is defined by

$$R_{\alpha,1}(\mathbf{r}) = \frac{\Gamma((1-\alpha)/2)}{2^{\alpha}\, \pi^{1/2}\, \Gamma(\alpha/2)}\, |\mathbf{r}|^{\alpha-1} \tag{12}$$

if $\alpha \neq 2n+1$ where $n = 0, 1, 2, , \ldots$. For $\alpha = 2n+1$, the Riesz kernel $R_{\alpha,1}(\mathbf{r})$ is

$$R_{\alpha,1}(\mathbf{r}) = -\frac{1}{\gamma_1(\alpha)} |\mathbf{r}|^{\alpha-1} \ln|\mathbf{r}|, \tag{13}$$

where

$$\gamma_1(\alpha) = \gamma_1(2n+1) = (-1)^n\, 2^{2n}\, \pi^{1/2}\, n!\, \Gamma(n+1/2). \tag{14}$$

Note that the integration (11) can be considered as integration (6) for $N = 1$ with respect to one variable $x_j \in \mathbb{R}^1$. For example, Eq. (11) for $(x, y) \in \mathbb{R}^2$ gives

$$\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ x\end{bmatrix} f(x, y) = \int_{-\infty}^{+\infty} R_{\alpha,1}(x-z)\, f(z, y)\, dz, \quad (\alpha > 0), \tag{15}$$

$$\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ y\end{bmatrix} f(x, y) = \int_{-\infty}^{+\infty} R_{\alpha,1}(y-z)\, f(x, z)\, dz, \quad (\alpha > 0). \tag{16}$$

An important property of the Riesz fractional integrals of the Riesz type with respect to x_j is the Fourier transform $\mathcal{F}$ of this integrals in the form

$$\mathcal{F}\left(\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right) = |k_j|^{-\alpha}\, (\mathcal{F} f)(\mathbf{k}). \tag{17}$$

where $\mathbf{k} = \sum_{j=1}^{N} k_j\, \mathbf{e}_j$.

Let us note that the distinction between the fractional integral of the Riesz type $\mathbb{I}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ and the Riesz potential $\mathbf{I}^{\alpha}_{\mathbf{r}}$ is the use of $|k_j|^{-\alpha}$ instead of $|\mathbf{k}|^{-\alpha}$. The continuum integral $\mathbb{I}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ of the Riesz type is an integration of $f(\mathbf{r})$ with respect to one variable x_j instead of all N variables $x_1, \ldots x_N$ in $\mathbf{I}^{\alpha}_{\mathbf{r}}$.

If $f(\mathbf{r})$ belongs to the Lizorkin space as a function of x_j, then we have [6] the semi-group property

$$\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} \mathbb{I}^{+}_{C}\begin{bmatrix}\beta\\ j\end{bmatrix} f(\mathbf{r}) = \mathbb{I}^{+}_{C}\begin{bmatrix}\alpha+\beta\\ j\end{bmatrix} f(\mathbf{r}), \tag{18}$$

where $\alpha > 0$ and $\beta > 0$.

The fractional derivative $\mathbb{D}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ yields an operator inverse to the fractional integration $\mathbb{I}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for a special space of functions

$$\mathbb{D}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix}\mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r}) = f(\mathbf{r}), \quad (\alpha > 0). \tag{19}$$

Equation (19) holds for $f(\mathbf{r})$ belonging to the Lizorkin space of functions with respect to $x_i \in \mathbb{R}$. Moreover, this property is also valid for the fractional integration in the framework of a set of L_p-spaces $L_p(\mathbb{R}^1)$ where $1 \leqslant p < 1/\alpha$ (see Theorem 26.3 in [6]).

Using the property (10), we can see that the fractional derivative $\mathbb{D}^{+}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ with $\alpha = 1$ cannot be considered as the standard derivative of first order with respect to x_j. Therefore, we define new fractional derivative $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ of the Riesz type.

Definition 4 The conjugate Riesz fractional derivative of the order α with respect to x_j is defined by the equation

$$\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} := \begin{cases} \frac{\partial}{\partial x_j}\mathbb{D}^{+}_{C}\left[\begin{smallmatrix}\alpha-1\\ j\end{smallmatrix}\right] & \alpha > 1\\ \frac{\partial}{\partial x_j} & \alpha = 1\\ \frac{\partial}{\partial x_j}\mathbb{I}^{+}_{C}\left[\begin{smallmatrix}1-\alpha\\ j\end{smallmatrix}\right] & 0 < \alpha < 1. \end{cases} \tag{20}$$

For $0 < \alpha < 1$, the operator $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ is analogous to the conjugate Riesz derivative [10]. Therefore, the operator $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for all values $\alpha > 0$ is called conjugate derivative of the Riesz type [29].

Using (3) and (17), we can see that the Fourier transform $\mathcal{F}$ of the fractional derivative (20) is given by

$$\mathcal{F}\left(\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k}) = i\, k_j\, |k_j|^{\alpha-1}(\mathcal{F}f)(\mathbf{k}) = i\ \mathrm{sgn}(k_j)\, |k_j|^{\alpha}(\mathcal{F}f)(\mathbf{k}). \tag{21}$$

For the odd values of α, Eqs. (4) and (20) give the relation

$$\mathbb{D}^{-}_{C}\begin{bmatrix}2m+1\\ j\end{bmatrix} f(\mathbf{r}) = (-1)^m \frac{\partial^{2m+1} f(\mathbf{r})}{\partial x_j^{2m+1}}, \quad (m \in \mathbb{N}). \tag{22}$$

Equation (22) means that the fractional derivatives $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$ of the odd orders α are local operators represented by the usual derivatives of integer orders.

Note that the continuum derivative $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}2\\ j\end{smallmatrix}\right]$ cannot be considered as a local derivative of second order with respect to x_j. The derivatives $\mathbb{D}^{-}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for even orders $\alpha = 2m$, where $m \in \mathbb{N}$, are nonlocal operators that cannot be considered as usual partial derivatives $\partial^{2m}/\partial x_j^{2m}$.

4 Partial fractional derivatives of the Riesz type

Let us give a definition of the partial fractional derivatives.

Definition 5 The partial fractional derivatives of the Riesz type of the order α is the linear operator

$$\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} := \cos\left(\frac{\pi\,\alpha}{2}\right)\mathbb{D}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} + \sin\left(\frac{\pi\,\alpha}{2}\right)\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix}. \tag{23}$$

where $\mathbb{D}^{\pm}_{C}\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ are defined by Eqs. (1) and (20).

The partial fractional derivatives (23) have the following Fourier transform.

Proposition 2 *The Fourier transform $\mathcal{F}$ of the partial fractional derivatives of the Riesz type has the form*

$$\mathcal{F}\left(\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k}) = e^{i\,\pi\,\alpha\ \mathrm{sgn}(k_j)/2}|k_j|^{\alpha}(\mathcal{F}f)(\mathbf{k}), \tag{24}$$

where $\alpha > 0$. Equation (24) is valid for functions $f(\mathbf{r})$ from for the Lizorkin space and the space $C^{\infty}(\mathbb{R}^1)$ of infinitely differentiable functions on $\mathbb{R}^1$ with compact support.

Proof The proof is based on relations (3), (21) and the Euler's formula. Using (23), we get

$$\begin{aligned}\mathcal{F}\left(\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k}) &= \cos\left(\frac{\pi\,\alpha}{2}\right)\mathcal{F}\left(\mathbb{D}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k})\\ &\quad+\sin\left(\frac{\pi\,\alpha}{2}\right)\mathcal{F}\left(\mathbb{D}^{-}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})\right)(\mathbf{k})\\ &= \left(\cos\left(\frac{\pi\,\alpha}{2}\right)|k_j|^{\alpha} + i\ \mathrm{sgn}(k_j)\ \sin\left(\frac{\pi\,\alpha}{2}\right)|k_j|^{\alpha}\right)\\ &\quad\times(\mathcal{F}f)(\mathbf{k}) = e^{i\,\pi\,\alpha\ \mathrm{sgn}(k_j)/2}|k_j|^{\alpha}(\mathcal{F}f)(\mathbf{k}).\end{aligned}$$

□

The partial fractional derivatives of the Riesz type (23) of integer order $\alpha > 0$ are directly related with the partial derivatives of integer orders.

Proposition 3 *The partial fractional derivatives of the Riesz type of orders α for integer positive values $\alpha = m \in \mathbb{N}$ are usual partial derivative of integer orders m,*

$$\mathbb{D}\begin{bmatrix} m \\ j \end{bmatrix} f(\mathbf{r}) = \frac{\partial^m f(\mathbf{r})}{\partial x_j^m}, \quad (25)$$

where $m \in \mathbb{N}$.

Proof The proof of this proposition follows from relations (4) and (22). Let us first consider the even values of orders. For even $\alpha = 2m$, where $m \in \mathbb{N}$, the partial fractional derivatives of the Riesz type are given by

$$\mathbb{D}\begin{bmatrix} 2m \\ j \end{bmatrix} f(\mathbf{r}) = \cos(\pi m)\, \mathbb{D}^+_C \begin{bmatrix} 2m \\ j \end{bmatrix} f(\mathbf{r}) + \sin(\pi m)\, \mathbb{D}^-_C \begin{bmatrix} 2m \\ j \end{bmatrix} f(\mathbf{r}) = (-1)^m\, \mathbb{D}^+_C \begin{bmatrix} 2m \\ j \end{bmatrix} f(\mathbf{r}) = (-1)^{2m} \frac{\partial^{2m} f(\mathbf{r})}{\partial x_j^{2m}} = \frac{\partial^{2m} f(\mathbf{r})}{\partial x_j^{2m}}.$$

For odd $\alpha = 2m+1$, where $m \in \mathbb{N}$, the partial fractional derivatives of the Riesz type are

$$\mathbb{D}\begin{bmatrix} 2m+1 \\ j \end{bmatrix} f(\mathbf{r}) = \cos(\pi m + \pi/2)\, \mathbb{D}^+_C \begin{bmatrix} 2m+1 \\ j \end{bmatrix} f(\mathbf{r}) + \sin(\pi m + \pi/2)\, \mathbb{D}^-_C \begin{bmatrix} 2m+1 \\ j \end{bmatrix} f(\mathbf{r}) = \cos(\pi m)\, \mathbb{D}^-_C \begin{bmatrix} 2m+1 \\ j \end{bmatrix} f(\mathbf{r}) = (-1)^m\, \mathbb{D}^-_C \begin{bmatrix} 2m+1 \\ j \end{bmatrix} f(\mathbf{r}) = (-1)^{2m} \frac{\partial^{2m+1} f(\mathbf{r})}{\partial x_j^{2m+1}} = \frac{\partial^{2m+1} f(\mathbf{r})}{\partial x_j^{2m+1}}.$$

As a result, we have

$$\mathbb{D}\begin{bmatrix} \alpha \\ j \end{bmatrix} f(\mathbf{r}) = \frac{\partial^\alpha f(\mathbf{r})}{\partial x_j^\alpha} \quad (\alpha \in \mathbb{N}). \quad (26)$$

□

This property greatly simplifies the construction of fractional nonlocal generalizations of models of continua and fields.

Remark 3 Let us compare the suggested fractional derivatives with the regularised fractional derivative, which is proposed in [48], the unified fractional derivatives, which is considered in [49], the Marchaud derivative [6] and other. The derivatives $\mathbb{D}\begin{bmatrix} \alpha \\ j \end{bmatrix}$ have the following advantages compared with the above mentioned:

(1) The Marchaud and regularised fractional derivatives are defined as ordinary differential operators in the standard definitions. The proposed derivatives $\mathbb{D}\begin{bmatrix} \alpha \\ j \end{bmatrix}$ are the partial derivatives with respect to variable x_j of order α_j in general.

(2) The derivatives $\mathbb{D}\begin{bmatrix} \alpha \\ j \end{bmatrix}$ have analogs for physical lattices as opposed to the fractional centered derivatives, that is based on the central differences of noninteger orders (for details see Sect. 5 of [28]). The fractional derivatives, which are based on the fractional central differences of type 2, correspond to interaction of lattice particles with virtual particles with half-integer numbers that do not exist in the physical lattices.

(3) The regularised fractional derivatives [48] are based on the Grünwald–Letnikov fractional derivatives. These derivatives have the lattice and finite-difference analogs that can be considered as an asymptotic discretization only (for details see Section 2.2 of [29] and [50]). The Fourier transform $\mathcal{F}$ of the Grünwald–Letnikov fractional differences $\nabla^\alpha_{h,\pm}$ is given by
$$\mathcal{F}\{\nabla^\alpha_{h,\pm} f(x)\}(k) = (1 - \exp\{\pm ikh\})^\alpha \mathcal{F}\{f(x)\}(k)$$
for any function $f(x) \in L_1(\mathbb{R})$ (see Property 2.30 of [8]). Therefore, these fractional differences (and lattice derivatives) cannot be considered as an exact discretization of the fractional derivatives [50]. Note that the standard Leibniz rule does not hold for the Grünwald–Letnikov differences of integer order. The suggested fractional derivatives $\mathbb{D}\begin{bmatrix} \alpha \\ j \end{bmatrix}$ are based on the lattice derivatives ($\mathcal{T}$-differences) that can be considered as an exact discrete analogs of the fractional derivatives [50]. The Fourier transform $\mathcal{F}$ of the $\mathcal{T}$-ifference ${}^{\mathcal{T}}\Delta^\alpha_h$ of order α is given by
$$\mathcal{F}\{{}^{\mathcal{T}}\Delta^\alpha_h f(x)\}(k) = e^{i\pi\alpha\, \mathrm{sgn}(k_j)/2}\, |k_j|^\alpha\, \mathcal{F}\{f(x)\}(k),$$
which is analog of (24). The main characteristic property of these differences is that the standard Leibniz rule holds for the $\mathcal{T}$-differences of integer order (for details see [50,51]).

5 Properties of partial fractional derivatives of the Riesz type

Let us describe some properties of the partial fractional derivatives of the Riesz type. All these properties are similar to properties of the Riesz derivatives of non-integer orders [8,31,32]. As the properties of the partial fractional derivatives, we will consider the linearity, (non)commutativity, semigroup property and the violation of the chain and Leibniz rules. It should be noted that the violation of the standard Leibniz and chain rules can be considered as characteristic properties of the derivatives of noninteger orders [15,18–20]. The performance of commutative and semigroup properties cannot be considered as fundamental properties that any partial derivative should satisfy. To prove this fact, we demonstrate that these properties cannot be regarded as the fundamental properties even for standard derivatives of integer orders. For standard partial derivatives of integer orders and the suggested partial derivatives of noninteger orders, the implementation or violation of these two properties is dependent on the considered function spaces. Therefore, the suggested partial fractional derivatives of the Riesz type are reasonable to consider as operators that satisfy these main properties of partial derivatives for fractional and integer orders. The fact that the proposed fractional derivatives coincide with the usual partial derivative for integer orders and satisfy the same algebraic properties is a necessary and sufficient condition for operators to be considered an promising partial fractional derivatives.

1) Linearity.

The partial fractional derivatives of the Riesz type are the linear operators

$$\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix}\Big(a\, f(\mathbf{r})+b\, g(\mathbf{r})\Big) = a\,\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(\mathbf{r})+b\,\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} g(\mathbf{r}), \tag{27}$$

where $a, b \in \mathbb{R}$.

2) Commutative property.

In general, the partial fractional derivatives with respect to the same variable x_j are not commute

$$\mathbb{D}\begin{bmatrix}\alpha_1\\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_2\\ j\end{bmatrix} f(\mathbf{r}) \neq \mathbb{D}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_1\\ j\end{bmatrix} f(\mathbf{r}), \quad (\alpha_1 \neq \alpha_2). \tag{28}$$

The partial fractional derivatives of the Riesz type for different variables x_i and x_j, where $i \neq j$, commute

$$\mathbb{D}\begin{bmatrix}\alpha_1\\ i\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_2\\ j\end{bmatrix} f(\mathbf{r}) = \mathbb{D}\begin{bmatrix}\alpha_2\\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_1\\ i\end{bmatrix} f(\mathbf{r}) \quad (i \neq j) \tag{29}$$

for sufficiently good functions $f(\mathbf{r})$. The commutation relation (29) with $\alpha_1 = \alpha_2 = 1$ is

$$\mathbb{D}\begin{bmatrix}1\\ i\end{bmatrix}\mathbb{D}\begin{bmatrix}1\\ j\end{bmatrix} f(\mathbf{r}) = \mathbb{D}\begin{bmatrix}1\\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}1\\ i\end{bmatrix} f(\mathbf{r}), \quad (i \neq j) \tag{30}$$

that can be rewritten in the form

$$\frac{\partial^2 f(\mathbf{r})}{\partial x_i \partial x_j} = \frac{\partial^2 f(\mathbf{r})}{\partial x_j \partial x_i}. \tag{31}$$

It is well known that the commutation relation (31) may be broken for discontinuous functions $f(\mathbf{r})$ and if the partial derivatives of $f(\mathbf{r})$ are not continuous. For example, the commutativity may be broken if the function does not have differentiable partial derivatives. It is possible when the Clairaut's theorem on equality of mixed partial derivatives is not satisfied, i.e., when the second partial derivatives are not continuous.

Let us introduce the notation $x = x_i$ and $y = x_j$, where $i \neq j$, and a function $f(x, y) = g(\mathbf{r})$. In the general case, the partial derivatives are not commute, i.e.,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x, y) \neq \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x, y). \tag{32}$$

As an example, we can consider the function

$$f(x, y) := \frac{x\, y\, (x^2 - y^2)}{x^2 + y^2} \quad ((x, y) \neq (0, 0)),$$
$$f(x, y) := 0 \quad ((x, y) = 0). \tag{33}$$

The partial derivatives of the function (33) exist and everywhere continuous. The limit of difference quotients shows that

$$\frac{\partial f}{\partial x}(0, 0) = 0, \quad \frac{\partial f}{\partial y}(0, 0) = 0. \tag{34}$$

However, the second partial derivatives are not continuous at the point $(x, y) = (0, 0)$. Using that

$$\frac{\partial f}{\partial y}(x, 0) = x, \quad \frac{\partial f}{\partial x}(0, y) = -y, \tag{35}$$

we get

$$\frac{\partial^2 f}{\partial x\, \partial y}(0,0) = 1, \quad \frac{\partial f}{\partial y\, \partial x}(0,0) = -1. \tag{36}$$

As a result, we have inequality (32) at the point $(x, y) = (0, 0)$ since (36) gives $1 \neq -1$. Therefore, the commutative property for is not satisfied even for the partial derivatives of integer orders in the general case. The commutativity of partial derivatives is satisfied only on function spaces, where the Clairaut's theorem is satisfied. For example, the commutativity of partial derivatives holds for the space of polynomial functions.

3) Semigroup property.

In the general case, the semigroup property is not satisfied

$$\mathbb{D}\begin{bmatrix}\alpha_1 \\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_2 \\ j\end{bmatrix} f(\mathbf{r}) \neq \mathbb{D}\begin{bmatrix}\alpha_1+\alpha_2 \\ j\end{bmatrix} f(\mathbf{r}), \quad (\alpha_1, \alpha_2 > 0). \tag{37}$$

The property (37) leads to the fact that action of two repeated partial fractional derivatives of order α does not equivalent to the action of fractional derivative of double order 2α,

$$\mathbb{D}\begin{bmatrix}\alpha \\ j\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha \\ j\end{bmatrix} f(\mathbf{r}) \neq \mathbb{D}\begin{bmatrix}2\,\alpha \\ j\end{bmatrix} f(\mathbf{r}), \quad (\alpha_1 > 0). \tag{38}$$

It should be noted that the semigroup property is not satisfied for standard partial derivatives of integer order. Therefore, the semigroup property cannot be considered as fundamental property that should be satisfied for any partial derivative. To demonstrate a violation of the semigroup property, we consider standard partial derivatives of a function $f(x) = g(\mathbf{r})$ with respect to the variable $x = x_j$. Let us define the derivatives of integer order by the equation

$$\frac{\partial^n f}{\partial x^n}(x) := \lim_{h\to 0} \frac{{}^f\Delta^n f(x)}{h^n}, \tag{39}$$

where ${}^f\Delta^n f(x)$ is the standard forward finite difference of order $n \in \mathbb{N}$. This formula defines the derivative of order $n \in \mathbb{N}$ by a single passage to the limit. Equation (39) gives

$$\frac{\partial f}{\partial x}(x) := \lim_{h\to 0} \frac{1}{h}\Big(f(x_0+h) - f(x_0)\Big), \tag{40}$$

$$\frac{\partial^2 f}{\partial x^2}(x_0) := \lim_{h\to 0} \frac{1}{h^2}\Big(f(x_0+2\,h) - 2\, f(x_0+h) - f(x_0)\Big). \tag{41}$$

Using (40), the repeated action of the first derivatives is defined by the equation

$$\begin{aligned}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)(x_0) &:= \lim_{h_1\to 0} \frac{1}{h_1}\left(\frac{\partial f}{\partial x}(x_0+h_1) - \frac{\partial f}{\partial x}(x_0)\right) \\ &= \lim_{h_1\to 0} \lim_{h_2\to 0} \frac{1}{h_1\, h_2}\Big(f(x_0+h_1+h_2) \\ &\quad - f(x_0+h_1) - f(x_0+h_2) + f(x_0)\Big).\end{aligned} \tag{42}$$

Therefore, the repeated action of the first derivatives is represented by the double limit instead of the single passage to the limit in the second derivative (41).

In the general case, the repeated action of the first derivatives is not equal to the action of the first derivative (for example, see Section 122 of [52]), i.e.,

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)(x_0) \neq \frac{\partial^2 f}{\partial x^2}(x_0). \tag{43}$$

For example, we can consider the function

$$f(x) := x^3 \sin(x^{-1}) \quad (x \neq 0), \quad f(x) := 0 \quad (x = 0). \tag{44}$$

For this function, the first derivative exists

$$\frac{\partial f}{\partial x}(x) = 3\,x^2 \sin(x^{-1}) - x \cos(x^{-1}) \quad (x \neq 0), \qquad \frac{\partial f}{\partial x}(0) := 0, \tag{45}$$

but the second derivative at $x = x_0 = 0$ does not exist since the following limit

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{\partial f}{\partial x}(x_0+h) - \frac{\partial f}{\partial x}(x_0)\right) \tag{46}$$

does not exist at $x_0 = 0$, where

$$\frac{1}{h}\Big(\frac{\partial f}{\partial x}(0+h) - \frac{\partial f}{\partial x}(0)\Big) = \frac{1}{h}\Big(3\,h^2 \sin(h^{-1}) - h \cos(h^{-1})\Big) = 3\,h \sin(h^{-1}) - \cos(h^{-1}).$$

At the same time, the following limit at $x = x_0 = 0$

$$\lim_{h\to 0} \frac{1}{h^2}\Big(f(x_0 + 2h) - 2 f(x_0 + h) + f(x_0)\Big)$$
$$= \lim_{h\to 0} \frac{1}{h^2}\Big(8h^3 \sin(2h^{-1}) - 2h^3 \sin(2h^{-1})\Big)$$
$$= \lim_{h\to 0}\Big(8h \sin(2h^{-1}) - 2h \sin(h^{-1})\Big) = 0 \qquad (47)$$

exists at $x_0 = 0$.

4) The Leibniz rule.

It should be noted that the Leibniz rule for the suggested partial fractional derivative of order $\alpha \neq 1$ does not satisfied

$$\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix}(f\, g) \neq g\, \mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} f + f\, \mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} g \quad (\alpha > 0, \quad \alpha \neq 1). \qquad (48)$$

This property is a characteristic property of all types of fractional-order derivatives [19,20].

5) Chain rule.

It should be noted that the standard chain rule for the partial fractional derivative of order $\alpha \neq 1$ does not satisfied

$$\mathbb{D}\begin{bmatrix}\alpha\\ j\end{bmatrix} f(g(\mathbf{r})) \neq (\mathbb{D}_{\mathbf{g}}\mathbf{f}(\mathbf{g}))_{\mathbf{g}=\mathbf{g}(\mathbf{r})}\mathbb{D}\begin{bmatrix}\alpha\\ \mathbf{j}\end{bmatrix}\mathbf{g}(\mathbf{r})$$
$$(\alpha > 0, \quad \alpha \neq 1), \qquad (49)$$

where $\mathbb{D}_g$ is a derivative of integer or noninteger order with respect to the variable g. This property is can be considered as a characteristic property of fractional-order derivatives [15].

6 Nonlinear fractional-order differential equations

The partial fractional derivatives of the Riesz type can be used in the partial fractional-order differential equations of continua and fields with power-law nonlocality [53–58]. In this section, we consider fractional generalizations of nonlinear equations, such as sine-Gordon, Boussinesq, Burgers, Korteweg–de Vries and Monge–Ampere equations. We can use the derivatives $\mathbb{D}\left[\begin{smallmatrix}\alpha_j\\ j\end{smallmatrix}\right]$ with different values α_j ($\alpha_i \neq \alpha_j$ for $i \neq j$) to describe the anisotropy in nonlocal media.

Let us give some comments about motivation of fractional nonlocal generalization of the nonlinear differential equations. This physical motivation is based on a direct connection of the suggested fractional differential equations and lattice (discrete) models with long-range interactions. We can state that the suggested nonlinear fractional differential equations with the Riesz fractional derivatives can be considered as an continuum limit of the lattice models with long-range interactions of power-law type.

From a mathematical point of view, the standard lattice models are based on mathematical approach that is used the forward, backward and central finite differences. From the physical point of view, this approach assumes a short-ranged and nearest-neighbor approximation. However, there exist different physical cases, when continuum cannot be described in the framework of this approximation. For example, the excitation transfer [59] is due to the transition dipole-dipole interaction; the vibron energy transport [60] is due to the transition dipole-dipole interaction that corresponds to the interaction of the type $1/|n-m|^3$, where $n, m \in \mathbb{Z}$. In systems, where the dispersion curves of two elementary excitations are intersect or close, there is an effective long-range transfer. Such situations, which are called the polariton effects [59], arise for excitons, phonons and photons in semiconductors and molecular crystals. Polyatomic molecules contain charged groups with a long-range Coulomb interaction that corresponds to $1/|n-m|^1$ between them. One of the most widely used type of long-range interactions [61–66], which are considered for nonlinear differential equations of corresponding nonlocal continuum, is given by the interaction kernel

$$K_\alpha(n-m) = \frac{1}{|n-m|^\alpha}, \qquad (50)$$

where α is a positive real number. In this case, we have nonlocal coupling given by the power-law function (50) with a physical relevant parameter α, which characterizes the weakening of the interaction. Some integer values of α correspond to the well-known physical situations that correspond to the Coulomb potential for $\alpha = 1$ and the dipole-dipole interaction for $\alpha = 3$.

Quantum and classical lattice systems with long-range interactions have been actively investigated beginning with the Dyson's papers [67,68], where the interaction kernels of the form (50) are applied. The long-range interactions of different types have been studied in lattice systems as well as in their continuous analogs. Different lattice and chain models with long-range interactions have been considered in [67–75]. An infinite one-dimensional Ising model with long-range interactions is presented in [67–69]. The d-dimensional classical Heisenberg model with long-range interaction is described in [74], and their quantum generalization has been proposed in [70–73]. Kinks in lattice models with long-range interactions are described in [76]. Time periodic spatially localized solutions, which are called breathers, on lattices and chains with long-range interactions are studied in [77–79]. Decay properties of breathers for discrete systems with long-range interactions, which are described by the discrete nonlinear Schrodinger equations, are described in [61–66,80]. Properties of dynamical chaos and synchronization in lattice models with long-range interaction of the type (50) are considered in works [81–83]. We can note that a characteristic property of the lattice models with power-like long-range interactions is that the solutions of differential equations have power-like tails [77,79,81–84]. Similar characteristic property has the corresponding continuum models that are described by fractional differential equations with Riesz derivatives of noninteger orders. In the works [28,29,85,86], we prove that the equations with the Riesz fractional derivatives are directly related to lattice models with long-range interactions of power-law type.

The three-dimensional fractional-order generalizations of the well-known nonlinear differential equations for nonlocal anisotropic continuum can be written in the following forms.

(1) The fractional-order differential equation of nonlinear waves in nonlocal continuum with nonlocality can be presented in the form

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \, \mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(\mathbf{r},t) + N(f(\mathbf{r},t)) = 0, \tag{51}$$

where $N(f)$ is the nonlinear term. The integer-order differential equation is defined by $\alpha_j = 2$ for (51), and we obtain

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \frac{\partial^2 f(\mathbf{r},t)}{\partial x_j^2} + N(f(\mathbf{r},t)) = 0. \tag{52}$$

As an example, we can consider $N(f) = \sin(f)$. In this case, equation (51) has the form of the fractional three-dimensional sine-Gordon equation

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \, \mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(\mathbf{r},t) + \sin(f(\mathbf{r},t)) = 0, \tag{53}$$

The corresponding integer-order differential equation is defined by $\alpha_j = 2$ such that

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \frac{\partial^2 f(\mathbf{r},t)}{\partial x_i^2} + \sin(f(\mathbf{r},t)) = 0. \tag{54}$$

(2) In general, the nonlinear term can contain fractional-order derivatives [107–111]. The fractional three-dimensional Boussinesq equation is

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \, \mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(\mathbf{r},t) + \sum_{j=1}^{3} B_j \, \mathbb{D}\begin{bmatrix}\beta_j \\ j\end{bmatrix} f(\mathbf{r},t) + \sum_{j=1}^{3} C_j \, \mathbb{D}\begin{bmatrix}\gamma_j \\ j\end{bmatrix} f^2(\mathbf{r},t) = 0. \tag{55}$$

The third term of equation (55) describes the gradient contribution to the dynamics of continuum. For $\alpha_j = 2$, $\beta_x = 4$, $\gamma_x = 2$, we get the integer-order differential equation in the form

$$\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \sum_{j=1}^{3} A_j \frac{\partial^2 f(\mathbf{r},t)}{\partial x_j^2} + \sum_{j=1}^{3} B_j \frac{\partial^4 f(\mathbf{r},t)}{\partial x_j^4} + \sum_{j=1}^{3} C_j \frac{\partial^2 f^2(\mathbf{r},t)}{\partial x_j^2} = 0. \tag{56}$$

In the one-dimensional case, Eq. (56) gives nonlinear partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} - A_x \frac{\partial^2 f(x,t)}{\partial x^2} + B_x \frac{\partial^2 f^2(x,t)}{\partial x^2} + C_x \frac{\partial^4 f(x,t)}{\partial x^4} = 0. \tag{57}$$

Equation (57) has been derived by analysis of long waves in shallow water. This equation is also applied to problems in the percolation of water in porous subsurface strata.

(3) Complex nonlinear media can be described by differential equations of the first order in time [102–106]. The fractional three-dimensional Burgers equation can be considered in the form

$$\frac{\partial f(\mathbf{r},t)}{\partial t} - \sum_{j=1}^{3} A_j \, \mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(\mathbf{r},t) + \sum_{j=1}^{3} B_j \, \mathbb{D}\begin{bmatrix}\beta_j \\ j\end{bmatrix} f^2(\mathbf{r},t) = 0. \quad (58)$$

For $\alpha_j = 2$ and $\beta_i = 1$, Eq. (58) is the second-order differential equation is defined by in the form

$$\frac{\partial f(\mathbf{r},t)}{\partial t} - \sum_{j=1}^{3} A_j \frac{\partial^2 f(\mathbf{r},t)}{\partial x_j^2} + \sum_{j=1}^{3} B_j \frac{\partial f^2(\mathbf{r},t)}{\partial x_j} = 0. \quad (59)$$

The special form of one-dimensional fractional Burgers equation has been discussed in [87]. In one-dimensional case, Eq. (59) gives

$$\frac{\partial f(x,t)}{\partial t} - A_1 \frac{\partial^2 f(x,t)}{\partial x^2} + B_1 \frac{\partial f^2(x,t)}{\partial x} = 0. \quad (60)$$

This is the well-known Burgers Eq. [88], that is applied in fluid dynamics as simplified mathematical model for turbulence, mass transport, shock wave and boundary layer behavior.

(4) The fractional generalization of the three-dimensional Korteweg–de Vries equation can be considered in the from

$$\frac{\partial f(\mathbf{r},t)}{\partial t} + \sum_{j=1}^{3} A_j \, \mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(\mathbf{r},t) - \sum_{j=1}^{3} B_j \, \mathbb{D}\begin{bmatrix}\beta_j \\ j\end{bmatrix} f^2(\mathbf{r},t) = 0. \quad (61)$$

The one-dimensional fractional KdV equation has been suggested in papers [89,90]. For $\alpha_j = 3$ and $\beta_j = 1$, equation (61) gives the third-order differential equation in the form

$$\frac{\partial f(\mathbf{r},t)}{\partial t} + \sum_{j=1}^{3} A_j \frac{\partial^3 f(\mathbf{r},t)}{\partial x_j^3} - \sum_{j=1}^{3} B_j \frac{\partial f^2(\mathbf{r},t)}{\partial x_j} = 0. \quad (62)$$

For one-dimensional case, Eq. (62) has the form

$$\frac{\partial f(x,t)}{\partial t} + A_x \frac{\partial^3 f(x,t)}{\partial x^3} - B_x \frac{\partial f^2(x,t)}{\partial x} = 0. \quad (63)$$

This is the well-known one-dimensional Korteweg–de Vries equation that is derived in the analysis of shallow waves in canals. Now the Korteweg–de Vries equations are used in a different physical phenomena as models for solitons, traveling waves, shock wave formation, boundary layer behavior, mass transport and turbulence. The three-dimensional generalization of the KdV equation has been considered in [91]. The three-dimensional KdV equation with power functions is considered in [92].

(5) The fractional generalization of the Monge–Ampere equation can be considered in the form

$$A\left(\mathbb{D}\begin{bmatrix}\beta_1 \\ 1\end{bmatrix} f(x,y)\mathbb{D}\begin{bmatrix}\beta_2 \\ 2\end{bmatrix} f(x,y) - \left(\mathbb{D}\begin{bmatrix}\alpha_1 \\ 1\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_2 \\ 2\end{bmatrix} f(x,y)\right)^2\right) + \sum_{i,j=1}^{2} A_{ij} \, \mathbb{D}\begin{bmatrix}\alpha_i \\ i\end{bmatrix}\mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f(x,y) + C = 0, \quad (64)$$

where A, $B_{ij} = B_{ji}$, C are functions depending on x, y, f, and $\mathbb{D}\begin{bmatrix}\alpha_j \\ j\end{bmatrix} f$. We also use notation $x_1 = x$ and $x_2 = y$. For $\alpha_j = 1$ and $\beta_j = 2$, equation (64) gives the well-known Monge–Ampere equation

$$A\left(\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right) + \sum_{i,j=1}^{2} B_{ij} \frac{\partial^2 f(x,y)}{\partial x_i \partial x_j} + C = 0. \quad (65)$$

The Monge–Ampere Eq. (65) often appears in differential geometry. For example, this equation appear in the Minkowski problem and in the Weyl problem in differential geometry of surfaces. Note that there are a lot of problems to formulate a fractional-order generalization of differential geometry [17,18], since the usual chain rule [14,15], which defines the coordinate transformations, becomes much more complicated.

For computer simulations of the fractional-order differential equations with the partial fractional derivatives of the Riesz type we can used methods suggested in [93–97] and [28,29,50]. Let us note the main differences between the methods of lattice fractional calculus, which is recently proposed in [28,29,50], and the numerical approach for fractional differential equations that are considered in [93–97].

The discrete models, which are proposed in papers [28,29], correspond to the continuum models exactly [50], and these models can be considered as microstructural basis in the form of physical lattices. They are not asymptotically equivalent, i.e., they are not an approximation. Equations of lattice models exactly correspond to fractional differential equations without any approximation. (For details about exact and asymptotic connections of lattice and continuum models, see [28]).

The numerical methods for partial fractional differential equations with the Riesz space fractional derivatives, which are considered in [93,94], replace the Riesz fractional derivatives by the finite differences with power-law weights (the finite-difference approximation). The same type of replacements is used in the finite difference methods [95–97]. The lattice fractional calculus approach is based on special type of infinite fractional differences that describe long-range interactions in physical lattices. In general, the finite differences correspond to models with nearest-neighbor and next-nearest-neighbor interactions [98,99]. Nonlocal continuum theory is based on the assumption that the forces between particles are a long-range type, thus reflecting the long-range character of interatomic and intermolecular forces. In papers [28,29,50], we suggest physical lattice models with long-range interactions of power-law type. Discrete (lattice) models with long-range type of interactions are important in fractional nonlocal models. The long-range interactions, which are represented by infinite differences, describe nonlocality at micro- and nanoscales. Therefore, the suggested lattice models with long-range interactions more correctly describe the continuum media with nonlocality of power-law type.

In the works [28,29,50] have been given exact discrete (lattice) analogs of the Riesz fractional derivatives of integer and noninteger orders that have the form (50). The computer simulations are actively used for the linear and nonlinear systems with long-range interactions of the form (50) with integer and noninteger α (for example, see [61,63–66,79,83,84]). Therefore, we can assume that computer simulations of nonlinear differential equations with the suggested fractional-order derivatives can be successfully realized. We assume that the suggested nonlinear fractional differential equations with the derivatives of the Riesz type can demonstrate new effects such discrete solitons, kinks, breathers, dynamical chaos and synchronization by computer simulations.

7 Conclusion

In this paper, we propose new partial fractional-order derivatives that are generalizations of the Riesz fractional derivatives, which are suggested in [28,29]. The main advantage of these partial fractional derivatives of the Riesz type is that these derivatives for integer orders are equal to the usual partial derivatives of integer orders n with respect to x_j. This property allows us much easier to build fractional models of nonlocal continua. The other advantage of the suggested approach is a direct connection [28,29,50,85,86] of the suggested fractional-order derivatives with equations of lattice models with long-range interactions and lattice fractional calculus that is proposed in recent papers [28,29]. The lattice approach can be more appropriate to get microstructural models of media with power-law nonlocality, where the intermolecular and interatomic interactions are crucial in determining continuum properties. We assume that the suggested Riesz fractional derivatives can be used in the fractional vector calculus [28,100]. For computer simulations of the fractional partial differential equations with generalized Riesz space derivatives, the finite-difference methods [93,94] and the lattice calculus [29,50] can be used.

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