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Stationary states of dissipative quantum systems

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Abstract

In this Letter we consider stationary states of dissipative quantum systems. We discuss stationary states of dissipative quantum systems, which coincide with stationary states of Hamiltonian quantum systems. Dissipative quantum systems with pure stationary states of linear harmonic oscillator are suggested. We discuss bifurcations of stationary states for dissipative quantum systems which are quantum analogs of classical dynamical bifurcations. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The dissipative quantum systems are of strong theoretical interest [1]. As a rule, any microscopic system is always embedded in some (macroscopic) environment and therefore it is never really isolated. Frequently, the relevant environment is in principle unobservable or it is unknown [2,3]. This would render theory of dissipative quantum systems a fundamental generalization of quantum mechanics [4].

Spohn [5–7] derives sufficient condition for existence of an unique stationary state for dissipative quantum system described by Lindblad equation. The irreducibility condition given by [8] defines stationary state of dissipative quantum systems. An example, where the stationary state is unique and approached by all states for long times is considered by Lindblad [9] for Brownian motion of quantum harmonic oscillator. The stationary solution of Wigner function evolution equation for dissipative quantum system was discussed in [10,11]. Quantum effects in the steady states of the dissipative map are considered in [12].

2. Definition of stationary states

In general case, the time evolution of quantum state ρ_t is described by Liouville–von Neumann equation

$$\frac{d}{dt}\rho_t = \hat{\Lambda}\rho_t, \tag{1}$$

where $\hat{\Lambda}$ is a quantum Liouville operator. For Hamiltonian systems quantum Liouville operator has the form

$$\hat{\Lambda}\rho_t = -\frac{i}{\hbar}[H, \rho_t], \tag{2}$$

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where $H = H(q, p)$ is a Hamilton operator. If quantum Liouville operator $\hat{\Lambda}$ cannot be represented in the form (2), then quantum system is called non-Hamiltonian or dissipative quantum system. Stationary state is defined by the condition

$$\hat{\Lambda}\rho_t = 0.$$

For Hamiltonian systems this condition has the form

$$[H, \rho_t] = 0. \tag{3}$$

3. Pure stationary states of Hamiltonian systems

A pure state $\rho_\Psi = |\Psi\rangle\langle\Psi|$ is a stationary state of Hamiltonian quantum system, if $|\Psi\rangle$ is an eigenvector of Hamilton operator $H = H(q, p)$. Using $\langle\Psi|\Psi\rangle = 1$, we get the equality (3) in the form

$$H|\Psi\rangle = |\Psi\rangle E, \tag{4}$$

where $E = \langle\Psi|H|\Psi\rangle$. Eq. (4) defines pure stationary states $|\Psi\rangle$ of Hamiltonian systems. Eigenvalues of Hamilton operator are identified with the energy of the system. It is known, that Hamilton operator for linear harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{5}$$

Eq. (4) has the solution if

$$E_n = \frac{1}{2}\hbar\omega(2n+1). \tag{6}$$

In coordinate representation stationary states of linear harmonic oscillator are

$$\Psi_n(q) = \frac{1}{q_0}\exp\left(-\frac{q^2}{2q_0^2}\right)\mathcal{H}_n\left(\frac{q}{q_0}\right), \quad q_0 = \sqrt{\frac{\hbar}{m\omega}}, \tag{7}$$

where $\mathcal{H}_n(q/q_0)$ is Hermitian polynomial of order n.

4. Pure stationary states of dissipative systems

Let us consider Liouville–von Neumann equation (1) of the form

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \sum_{k=1}^{s} \hat{F}_k N_k\big(\hat{L}_H, \hat{R}_H\big)\rho_t. \tag{8}$$

Here $\hat{F}^k$ are operators act on operator space, $\hat{L}_A$ and $\hat{R}_A$ are operators of left and right multiplication [13] defined by

$$\hat{L}_A B = AB, \qquad \hat{R}_A B = BA,$$

for all operators B.

Let $\rho_\Psi = |\Psi\rangle\langle\Psi|$ is a pure state with eigenvector $|\Psi\rangle$ of the Hamilton operator H. If Eq. (4) is satisfied, then the state $\rho_\Psi = |\Psi\rangle\langle\Psi|$ is a stationary state of Hamilton system

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t], \tag{9}$$

associated with dissipative system (8).

If vector $|\Psi\rangle$ is eigenvector of H, then Liouville–von Neumann equation (8) for pure state $\rho_\Psi = |\Psi\rangle\langle\Psi|$ has the form

$$\frac{d}{dt}\rho_\Psi = \sum_{k=1}^{s} N_k(E, E)\hat{F}_k\rho_\Psi,$$

where the functions $N_k(E, E)$ are defined by

$$N_k(E, E) = \langle\Psi|\big(N_k^\dagger\big(\hat{L}_H, \hat{R}_H\big)I\big)|\Psi\rangle.$$

Operator $N_k^\dagger(\hat{L}_H, \hat{R}_H)$ is adjoint operator on operator space defined by

$$\big(N_k^\dagger\big(\hat{L}_H, \hat{R}_H\big)A\big|B\big) = \big(A\big|N_k\big(\hat{L}_H, \hat{R}_H\big)B\big),$$

where $(A|B) = \mathrm{Tr}(A^\dagger B)$. If all functions $N_k(E, E)$ are equal to zero

$$N_k(E, E) = 0, \tag{10}$$

then the stationary state of Hamiltonian quantum system (9) is stationary state of dissipative quantum system (8).

Note, that functions $N_k(E, E)$ are eigenvalues and $|\Psi\rangle$ is eigenvector of operators $N_k(H, H) = N_k^\dagger(\hat{L}_H, \hat{R}_H)I$, since

$$N_k(H, H)|\Psi\rangle = |\Psi\rangle N_k(E, E).$$

Therefore stationary states of dissipative quantum system (8) are defined by zero eigenvalues of operators $N_k(H, H) = N_k^\dagger(\hat{L}_H, \hat{R}_H)I$.

5. Dissipative systems with oscillator stationary states

In this section we consider simple examples of dissipative quantum systems (8).

(1) Let us consider nonlinear oscillator with friction defined by the equation

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H_{\rm nl}, \rho_t] + \frac{i}{\hbar}\beta\big[q^2, p^2 \circ \rho_t\big], \qquad (11)$$

where Hamilton operator $H_{\rm nl}$ is

$$H_{\rm nl} = \frac{p^2}{2m} + \frac{m\Omega^2 q^2}{2} + \frac{\gamma q^4}{2},$$

and

$$A \circ B = \frac{1}{2}(AB + BA).$$

Eq. (11) can be rewritten in the form

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \frac{2im\beta}{\hbar}\left[q^2, \left(\frac{p^2}{2m} + \frac{\gamma q^2}{2m\beta} - \frac{\Delta}{4\beta}I\right) \circ \rho_t\right], \qquad (12)$$

where $\Delta = \Omega^2 - \omega^2$, and H is Hamilton operator of linear harmonic oscillator (5). Eq. (12) has the form (8), where

$$\hat{F} = \frac{2im\beta}{\hbar}\big(\hat{L}_{q^2} - \hat{R}_{q^2}\big),$$

$$N\big(\hat{L}_H, \hat{R}_H\big) = \frac{1}{2}\big(\hat{L}_H + \hat{R}_H\big) - \frac{\Delta}{2\beta}\hat{L}_I,$$

$$N(E, E) = \langle\Psi|H - \frac{\Delta}{2\beta}I|\Psi\rangle = E - \frac{\Delta}{2\beta}.$$

Let $\gamma = \beta m^2\omega^2$. The dissipative system (11) has one stationary state (7) of harmonic oscillator with energy $E_n = (\hbar\omega/2)(2n + 1)$, if

$$\Delta = 2\beta\hbar\omega(2n + 1),$$

where n is an integer non-negative number. This stationary state is one of stationary states of linear harmonic oscillator with the mass m and frequency ω. In this case we can have the quantum analog [14] of dynamical Hopf bifurcation [15,16].

(2) Let us consider dissipative system described by evolution equation

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \frac{i}{\hbar}\big[q, N\big(\hat{L}_H, \hat{R}_H\big)\rho_t\big], \qquad (13)$$

where the Hamilton operator is defined by (5) and

$$N\big(\hat{L}_H, \hat{R}_H\big) = \cos\left(\frac{\pi}{2\varepsilon_0}\big(\hat{L}_H + \hat{R}_H\big)\right) = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{i\pi}{2\varepsilon_0}\right)^{2m} \big(\hat{L}_H + \hat{R}_H\big)^{2m}. \qquad (14)$$

The operator $\hat{F}$ on operator space is

$$\hat{F} = \frac{i}{\hbar}\big(\hat{L}_q - \hat{R}_q\big).$$

The function $N(E, E)$ has the form

$$N(E, E) = \cos\left(\frac{\pi E}{\varepsilon_0}\right) = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{i\pi E}{\varepsilon_0}\right)^{2m}.$$

The stationary state condition (10) has the solution

$$E = \frac{\varepsilon_0}{2}(2n + 1),$$

where n is an integer number. If parameter ε_0 is equal to $\hbar\omega$, then quantum system (13), (14) has stationary states (7) with the energy (6). As the result stationary states of dissipative quantum system (13) coincide with stationary states (7) of the linear harmonic oscillator.

If the parameter ε_0 is equal to $\hbar\omega(2l + 1)$, then quantum system (13), (14) has stationary states (7) with $n(k, m) = 2kl + k + l$ and

$$E_{n(k,l)} = \frac{\hbar\omega}{2}(2k + 1)(2l + 1).$$

(3) Let us consider the operators $N_k(\hat{L}_H, \hat{R}_H)$ in the form

$$N_k\big(\hat{L}_H, \hat{R}_H\big) = \frac{1}{2\hbar}\sum_{n,m} v_{kn}v^*_{km}\big(2\hat{L}^n_H\hat{R}^m_H - \hat{L}^{n+m}_H - \hat{R}^{n+m}_H\big),$$

and $\hat{F}_k = \hat{L}_I$. In this case, Liouville–von Neumann equation (8) has the form of Lindblad equation [17–19]:

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \frac{1}{2\hbar}\sum_{k=1}^{m}\big(\big[V_k\rho_t, V_k^\dagger\big] + \big[V_k, \rho_t V_k^\dagger\big]\big), \qquad (15)$$

with operators

$$V_k = \sum_n v_{kn} H^n, \qquad V_k^\dagger = \sum_m v_{km}^* H^m.$$

If $\rho_\Psi = |\Psi\rangle\langle\Psi|$ is a pure stationary state, then $N_k(E, E) = 0$ and this state is a stationary state of the dissipative quantum system (15).

6. Dynamical bifurcations and catastrophes

Let us consider a special case of dissipative quantum systems (8) such that the function $N_k(E, E)$ be a potential function, i.e., we have a potential $V(E)$ such that

$$\frac{\partial V(E)}{\partial E_k} = N_k(E, E),$$

where $E_k = \langle\Psi|H_k|\Psi\rangle$, and

$$N_k(E, E) = \langle\Psi|\big(N_k^\dagger\big(\hat{L}_H, \hat{R}_H\big)I\big)|\Psi\rangle,$$

$$H = \sum_{k=1}^{s} H_k, \qquad H_k|\Psi\rangle = |\Psi\rangle E_k.$$

In this case, the stationary condition (10) for dissipative system (8) is defined by critical points of the potential $V(E)$. If the system has one variable E, then the function $N(E, E)$ is always potential function. In general case, the functions $N_k(E, E)$ are potential, if

$$\frac{\partial N_k(E, E)}{\partial E_l} = \frac{\partial N_l(E, E)}{\partial E_k}.$$

Stationary states of dissipative quantum system (8) with potential functions $N_k(E, E)$ are depend on critical points of potential $V(E)$. This allows to use theory of bifurcations and catastrophes for parametric set of functions $V(E)$. Note that a bifurcation in a space of variables $E = \{E_k \mid k = 1, \ldots, s\}$ is a bifurcation in the space of eigenvalues of Hamilton operator H_k.

For polynomial operators $N_k(\hat{L}_H, \hat{R}_H)$ we have

$$N_k\big(\hat{L}_H, \hat{R}_H\big)\rho = \sum_{n=0}^{N}\sum_{m=0}^{n} a_{n,m}^{(k)} H^m \rho H^{n-m}.$$

In general case, m and n are multi-indexes. The function $N_k(E, E)$ is a polynomial

$$N_k(E, E) = \sum_{n=0}^{N} \alpha_n^{(k)} E^n,$$

where

$$\alpha_n^{(k)} = \sum_{m=0}^{n} a_{n,m}^{(k)}.$$

We can define the variable $x = E - a$, such that function $N_k(E, E) = N_k(x + a, x + a)$ has no the term x^{n-1}.

$$N_k(x + a, x + a) = \sum_{n=0}^{N} \alpha_n^{(k)}\big(x + a^{(k)}\big)^n = \sum_{n=0}^{N}\sum_{m=0}^{n} \alpha_n^{(k)} \frac{n!}{m!(n-m)!} x^m \big(a^{(k)}\big)^{n-m}.$$

If the coefficient of the term x^{n-1} is equal to zero

$$\alpha_n^{(k)} \frac{n!}{(n-1)!} a^{(k)} + \alpha_{n-1}^{(k)} = \alpha_n^{(k)} n a^{(k)} + \alpha_{n-1}^{(k)} = 0,$$

then we have

$$a^{(k)} = -\frac{\alpha_{n-1}^{(k)}}{n\alpha_n^{(k)}}.$$

If we change parameters $\alpha_n^{(k)}$, then can arise stationary states of dissipative quantum systems. For example, the bifurcation with birth of linear oscillator stationary state is a quantum analog of dynamical Hopf bifurcation [15,16].

If the function $N(E, E)$ is equal to

$$N(E, E) = \pm\alpha_n E^n + \sum_{j=1}^{n-1} \alpha_j E^j, \quad n \geqslant 2,$$

then potential $V(x)$ is

$$V(x) = \pm x^{n+1} + \sum_{j=1}^{n-1} a_j x^j, \quad n \geqslant 2,$$

and we have catastrophe of type $A_{\pm n}$.

If we have s variables E_k, where $k = 1, 2, \ldots, s$, then quantum analogous of elementary catastrophes

$A_{\pm n}$, $D_{\pm n}$, $E_{\pm 6}$, E_7 and E_8 can be realized. Let us write the full list of typical set of potentials $V(x)$, which leads to elementary catastrophes (zero-modal) defined by $V(x) = V_0(x) + Q(x)$, where

$$A_{\pm n}: \quad V_0(x) = \pm x_1^{n+1} + \sum_{j=1}^{n-1} a_j x_1^j, \quad n \geqslant 2,$$

$$D_{\pm n}: \quad V_0(x) = x_1^2 x_2 \pm x_2^{n-1} + \sum_{j=1}^{n-3} a_j x_2^j + \sum_{j=n-2}^{n-1} x_1^{j-(n-3)},$$

$$E_{\pm 6}: \quad V_0(x) = \left(x_1^3 \pm x_2^4\right) + \sum_{j=1}^{2} a_j x_2^j + \sum_{j=3}^{5} a_j x_1 x_2^{j-3},$$

$$E_7: \quad V_0(x) = x_1^3 + x_1 x_2^3 + \sum_{j=1}^{4} a_j x_2^j + \sum_{j=5}^{6} a_j x_1 x_2^{j-5},$$

$$E_8: \quad V_0(x) = x_1^3 + x_2^5 + \sum_{j=1}^{3} a_j x_2^j + \sum_{j=4}^{7} a_j x_1 x_2^{j-4}.$$

Here $Q(x)$ is nondegenerate quadratic form with variables $x_2, x_3, \ldots, x_s$ for $A_{\pm n}$ and parameters $x_3, \ldots, x_s$ for other cases.

7. Fold catastrophe

In this section, we suggest an example of catastrophe A_2 called fold.

Let us consider Liouville–von Neumann equation for nonlinear quantum oscillator with friction

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \alpha_0 \frac{i}{\hbar}[q, p \circ \rho_t] + \frac{i}{\hbar}\alpha_1\big[q, p \circ (H \circ \rho_t)\big] + \frac{i}{\hbar}\alpha_2\big[q, p \circ \big(H \circ (H \circ \rho_t)\big)\big], \tag{16}$$

where H is Hamilton operator defined by (5).

In this case, we have

$$\hat{F} = \frac{i}{\hbar}\big(\hat{L}_q - \hat{R}_q\big)\big(\hat{L}_p + \hat{R}_p\big),$$

$$N\big(\hat{L}_H, \hat{R}_H\big) = \alpha_0 \hat{L}_I + \frac{\alpha_1}{2}\big(\hat{L}_H + \hat{R}_H\big) + \frac{\alpha_2}{4}\big(\hat{L}_H + \hat{R}_H\big)^2,$$

$$N(E, E) = \langle\Psi|N(H, H)|\Psi\rangle = \alpha_0 + \alpha_1 E + \alpha_2 E^2.$$

Stationary state $\rho_\Psi = |\Psi\rangle\langle\Psi|$ of harmonic oscillator is stationary state of dissipative quantum system (16), if

$$\alpha_0 + \alpha_1 E + \alpha_2 E^2 = 0.$$

If we define the variable x and parameter λ by

$$x = E - a, \quad a = -\frac{\alpha_1}{2\alpha_2}, \qquad \lambda = \frac{4\alpha_0\alpha_2 - \alpha_1^2}{4\alpha_2^2},$$

then we have stationary condition $N(E, E) = 0$ in the form

$$x^2 - \lambda = 0.$$

If $\lambda \leqslant 0$, then we have no stationary states. If $\lambda > 0$, then we have stationary states for discrete set of parameter values λ. If the parameters a and λ are equal to

$$a = \frac{\hbar\omega}{2}(n_1 + n_2 + 1), \qquad \lambda = \hbar^2\omega^2 \frac{(n_1 - n_2)^2}{4},$$

where n_1 and n_2 are non-negative integer numbers, then dissipative quantum system has two stationary state (7) of linear harmonic oscillator. The energy of these states is equal to

$$E_{n_1} = \hbar\omega\left(n_1 + \frac{1}{2}\right), \qquad E_{n_2} = \hbar\omega\left(n_2 + \frac{1}{2}\right).$$

8. Conclusion

Dissipative quantum systems can have stationary states. Stationary states of non-Hamiltonian and dissipative quantum systems can coincide with stationary states of Hamiltonian systems. As an example we suggest quantum dissipative systems with pure stationary states of linear harmonic oscillator. Using (8), it is easy to get dissipative quantum systems with stationary states of hydrogen atom. For a special case of dissipative systems we can use usual bifurcation and catastrophe theory. It is easy to derive quantum analogous of classical dynamical bifurcations.

Dissipative quantum systems with two stationary states can be considered as qubits. It allows to consider quantum computer with dissipation as nondissipative quantum computer. In general case, we can consider dissipative n-qubit quantum systems as quantum computer with mixed states and quantum operations, not necessarily unitary, as gates [20,21]. A mixed state (operator of density matrix) of n two-level quantum system is an element of 4^n-dimensional operator Hilbert space. It allows to use quantum computer model with 4-valued logic [21]. The gates of this model are general quantum operations which act on the mixed state.

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References

[1] U. Weiss, Quantum Dissipative Systems, World Scientific, Singapore, 1993.
[2] V.S. Maskevich, Laser Kinetics, Elsevier, Amsterdam, 1967.
[3] V.E. Tarasov, Phys. Lett. B 323 (1994) 296.
[4] I. Prigogine, From Being to Becoming, Freeman, San Francisco, 1980.
[5] H. Spohn, Rep. Math. Phys. 10 (1976) 189.
[6] H. Spohn, Lett. Math. Phys. 2 (1977) 33.
[7] H. Spohn, J.L. Lebowitz, Commun. Math. Phys. 54 (1977) 97.
[8] E.P. Davies, Commun. Math. Phys. 19 (1970) 83.
[9] G. Lindblad, Rep. Math. Phys. 10 (1976) 393.
[10] C. Anastopoulous, J.J. Halliwell, Phys. Rev. D 51 (1995) 6870.
[11] A. Isar, A. Sandulescu, W. Scheid, quant-ph/9605041.
[12] T. Dittrich, R. Craham, Europhys. Lett. 4 (1987) 263.
[13] V.E. Tarasov, Moscow Univ. Phys. Bull. 6 (2001) 6.
[14] V.E. Tarasov, Phys. Lett. A 288 (2001) 173.
[15] J.M.T. Thompson, T.S. Lunn, Appl. Math. Modelling 5 (1981) 143.
[16] J. Marsden, M. McCracken, The Hopf Bifurcation and its Applications, Springer, Berlin, 1976.
[17] V.E. Tarasov, Mathematical Introduction to Quantum Mechanics, MAI, Moscow, 2000.
[18] G. Lindblad, Commum. Math. Phys. 48 (1976) 119.
[19] R. Alicki, K. Lendi, Quantum Dynamical Semigroups and Applications, Springer, Berlin, 1987.
[20] D. Aharonov, A. Kitaev, N. Nisan, Quantum circuits with mixed states, in: Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computation, STOC, 1997, pp. 20–30, quant-ph/9806029.
[21] V.E. Tarasov, Preprint SINP MSU, 2001-31/671;
V.E. Tarasov, quant-ph/0112025;
V.E. Tarasov, quant-ph/0201033.