Continuous medium model for fractal media

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Abstract

We consider the description of the fractal media that uses the fractional integrals. We derive the fractional generalizations of the equation that defines the medium mass. We prove that the fractional integrals can be used to describe the media with non-integer mass dimensions. The fractional equation of continuity is considered.
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1. Introduction

The application of fractals in physics [1–3] is far ranging, from the dynamics of fluids in porous media to resistivity networks in electronics. The cornerstone of fractals is the meaning of dimension, specifically the fractal dimension. Fractal dimension can be best calculated by box counting method which means drawing a box of size R and counting the mass inside. Fractal models of media are enjoying considerable popularity. This is due in part to the relatively small number of parameters that can define a fractal medium of great complexity and rich structure.

Derivatives and integrals of fractional order have found many applications in recent studies of scaling phenomena [3–9]. It is interesting to use fractional integration to consider the properties of the fractional media. It is interesting to find the connection between the fractional integrals and fractals. The natural questions arise: what could be the physical meaning of the fractional integration? This physical meaning can be following: the fractional integration can be considered as an integration in some fractional space. In order to use this interpretation we must define a fractional space. The first interpretation of the fractional space is connected with fractional dimension space. The fractional dimension interpretation follows from the formulas for dimensional regularizations. If we use the well-known formulas for dimensional regu-

larizations [11], then we get that the fractional integration can be considered as an integration in the fractional dimension space [10] up to the numerical factor $\Gamma(D/2)/(2\pi^{D/2}\Gamma(D))$. This interpretation was suggested in Ref. [10].

In this Letter we consider the second physical interpretation of the coordinate fractional integration. This interpretation follows from the fractional measure of space [10] that is used in the fractional integrals. We consider the mass fractional dimension and the fractional generalizations of the equation that defines the mass of the medium. We prove that the fractional integrals can be used to describe fractal media with non-integer mass dimensions. The fractional integrals can be used not only to calculate the mass dimensions of fractal media. Fractional integration can be used to describe the dynamical processes in the fractal media. Using fractional integrals, we can derive the fractional generalization of the dynamical equations. The fractional generalization of the Liouville equation was suggested in Ref. [10]. Using the Fourier transform, we can introduce fractional dynamical equations with coordinate fractional derivatives.

In Section 2, the definition of mass dimension and the fractional generalization of the medium mass equation is considered. In Section 3, we consider the properties of the fractal media. We define fractality and homogeneity properties of the media. In Section 4, we discuss the local density of the fractal media. In Section 5, the fractional generalizations of the equations are considered. We prove that these equations can be used to describe fractal media. Finally, a short conclusion is given in Section 7.

2. Mass fractal dimension

Equations that define the fractal dimensions have the passage to the limit. This passage makes difficult the practical application to the real fractal media. The other dimensions, which can be easy calculated from the experimental data, are used in empirical investigations. For example, the mass fractional dimension [13,14] can be easy measured.

The properties of the fractal media like mass obeys a power law relation

$$M(R) = kR^{D_m},\qquad(1)$$

where M is the mass of fractal medium, R is a box size (or a sphere radius), and D_m is a mass fractal dimension. Amount of mass of a medium inside a box of size R has a power law relation (1).

Fractal dimension can be calculated by box counting method which means drawing a box of size R and counting the mass inside. To calculate the mass fractal dimension, take the logarithm (1) both sides

$$\ln(M) = D_m \ln(R) + \ln k.$$

Log–log plot of M and R gives us the slope D_m, the fractal dimension. When we graph $\ln(M)$ as a function of $\ln(R)$, we get a value of about D_m which is the fractal dimension of fractal media.

The power law relation (1) can be naturally derived by using the fractional integral.

In order to describe the fractal media, we suggest to use the space with fractional measure. In this case, we can use the constant density distribution for homogeneous fractal media. If we use $\rho(x) = \rho_0 = \text{const}$, then we get (1). In the next sections, we prove that the mass fractal dimension is connected with the order of fractional integrals. Therefore the fractional integrals can be used to describe fractal media with non-integer mass dimensions.

3. Fractal media

Let us consider the region W_A in 3-dimensional Euclidean space E^3, where A is the midpoint of this region. The volume of the region W_A is denoted by $V(W_A)$. If the region W_A is a ball with the radius R_A, then the midpoint A is a center of the ball, and the volume $V(W_A) = (4/3)\pi R_A^3$.

The mass of the region W_A in the fractal media is denoted by $M(W_A)$. Let $\bar{\rho}(W_A)$ is an average mass density of the region W_A of the fractal medium. This density is defined by the equation

$$\bar{\rho}(W_A) = M(W_A)/V(W_A).$$

In the general case, the fractal media cannot be considered as continuous media. There are points and domains that do not filled by the medium particles. These domains can be called the porous. The fractal media can be considered as continuous media for the scales much more than mean value of the pore size R_p.

The fractal media can be characterized by the following property:

If the volume $V(W_A)$ of the region W_A increase, then the average mass density $\bar{\rho}(W_A)$ decrease.

This property is satisfied for all points A and all regions W_A if the volume of region is much more the average value of porous volume ($V(W_A) \gg V_p$). For the ball region of the fractal media, this property can be described by the power law

$$\bar{\rho}(W_A) = \rho_0 (R/R_p)^{D-3},$$

where ρ_0 is a constant value; R is the radius of the ball W_A. Here $R \gg R_p$, where R_p is a mean radius of the porous sphere.

The fractality of medium means that the mass of this medium in any region of Euclidian space E^3 increase more slowly that the volume of this region. For the ball region W, we have

$$M(R) \sim R^D, \quad D < 3.$$

The characteristic property of fractal media can be considered in the following forms:

For all regions W_A and W_B in the fractal media such that $W_A \subset W_B$ and $V(W_A) < V(W_B)$, we have that the corresponding average mass densities satisfy the inequality $\bar{\rho}(W_A) > \bar{\rho}(W_B)$, i.e.,

$$W_A \subset W_B, \quad V(W_A) < V(W_B)$$
$$\implies \bar{\rho}(W_A) > \bar{\rho}(W_B). \tag{2}$$

We would like to consider the fractal medium that have some homogeneous property. Now we shall give the definition of homogeneous fractal media.

The fractal media is called homogeneous if the following property is satisfied:

For all regions W_A and W_B of the homogeneous fractal media such that the volumes are equal $V(W_A) = V(W_B)$, we have average densities of these regions are equal too $\bar{\rho}(W_A) = \bar{\rho}(W_B)$, i.e.,

$$V(W_A) = V(W_B) \implies \bar{\rho}(W_A) = \bar{\rho}(W_B). \tag{3}$$

Fractal media is called a homogeneous fractal media if the average density value of the region does not depends on the translation and rotation of this region.

Note that the wide class of the fractal media satisfies the homogeneous property. In many cases, we can consider the porous media [15,16], polymers [17],

colloid aggregates [18], and aerogels [19] as homogeneous fractal media. The dendrites cannot be considered as a homogeneous fractal medium.

We can generalize the fractal property (2) for the homogeneous fractal media. This generalization is connected with consideration of two different points and two different regions of the media. For the homogeneous fractal media we can remove the restriction $W_A \subset W_B$ in Eq. (2). These regions can satisfy the condition $W_A \cap W_B = \emptyset$. Therefore we have the following property:

For all regions W_A and W_B of the homogeneous fractal medium in the Euclidean space E^3 that satisfies the inequality $V(W_A) < V(W_B)$, we have that the average densities are connected by the inequality $\bar{\rho}(W_A) > \bar{\rho}(W_B)$.

As the result, we have the following properties for all regions W_A and W_B in the Euclidean space E^3 with homogeneous fractal media:

(1) *Fractality*: if $V(W_A) < V(W_B)$, then $\bar{\rho}(W_A) > \bar{\rho}(W_B)$.
(2) *Homogeneity*: if $V(W_A) = V(W_B)$, then $\bar{\rho}(W_A) = \bar{\rho}(W_B)$ and $M(W_A) = M(W_B)$.

4. Local density of fractal media

In the general case, the fractal media cannot be considered as continuous media. There are points and domains that do not filled of the medium particles. These domains can be called the porous. The fractal media can be considered as continuous media for the scales much more than mean value of the pore size R_p. In this case, we can use the integration and differentiation.

In many cases, the fractal media are described by the equations with the integer integration. However, this description can be incorrect for fractal media. For example, the mass of the medium is derived by the following equation

$$M(W) = \int\limits_W \rho(\mathbf{r}) \, d^3\mathbf{r}. \tag{4}$$

If we would like to use this equation with the integer integral, then we cannot define the local mass density $\rho(\mathbf{r})$ for the homogeneous fractal media.

In order to prove this statement, we consider the points A, B, and C, such that $R_A = |AC| > R_B = |BC|$. Obviously, we require that $R_A \gg R_p$ and $R_B \gg R_p$. Let us describe the mass of the media using the integer integrals. The mass of the regions W_A and W_B of the homogeneous fractal media are defined by the equations

$$M(W_A) = \int_{W_A} \rho_A(\mathbf{r}) \, d^3\mathbf{r},$$

$$M(W_A) = \int_{W_A} \rho_B(\mathbf{r}) \, d^3\mathbf{r}. \qquad (5)$$

We use the index A and B in the local densities, since we would like to prove that $\rho_A(\mathbf{r}_C) \neq \rho_B(\mathbf{r}_C)$. Obviously, that the local densities ρ_A and ρ_B of the continuous medium satisfy the condition $\rho_A(\mathbf{r}_C) = \rho_B(\mathbf{r}_C)$.

In order to satisfy the fractal property of the medium and Eq. (5), we must use the following local densities:

$$\rho_A(\mathbf{r}) = \rho_0 \big(|\mathbf{r} - \mathbf{r}_A|/R_p \big)^{D-3},$$

$$\rho_B(\mathbf{r}) = \rho_0 \big(|\mathbf{r} - \mathbf{r}_B|/R_p \big)^{D-3}. \qquad (6)$$

Let us consider the point C such that $|AC| > |BC|$. Using Eq. (6), we have the local density in this point in the form

$$\rho_A(\mathbf{r}_C) = \rho_0 \big(|AC|/R_p \big)^{D-3},$$

$$\rho_B(\mathbf{r}_C) = \rho_0 \big(|BC|/R_p \big)^{D-3}.$$

Since $|AC| > |BC|$, it follows that $\rho_A(\mathbf{r}_C) < \rho_B(\mathbf{r}_C)$. Obviously, that we have two different values for one point C of fractal medium. Therefore we cannot use Eq. (4) with the integer integrals for the homogeneous fractal media. We can use Eq. (4) with the local density $\rho(\mathbf{r}) \sim |\mathbf{r}|^{D-3}$ only for the non-homogeneous distribution of the medium mass that has the single out point. This point is defined by $\mathbf{r} = 0$.

As the result, we have that the fractality and homogeneity properties of the medium cannot be satisfied simultaneously by Eq. (4) with the integer integrals.

In order to satisfy these properties, we must use the generalization of Eq. (4) such that fractality and homogeneity properties can be realized in the form:

(1) *Fractality*: the mass of the region W of fractal media obeys a power law relation

$$M(W) = M_0 \left(\frac{(V(W))^{1/3}}{R_p} \right)^D, \qquad (7)$$

where $D < 3$. For the ball region, we have

$$M(R) = M_0(R/R_p)^D,$$

where $R = |\mathbf{r} - \mathbf{r}_A|$.

(2) *Homogeneity*: the local density of homogeneous fractal media are translation and rotation invariant value that have the form

$$\rho(\mathbf{r}) = \rho_0 = \text{const}. \qquad (8)$$

We can realize these requirements by the equation

$$M(W) = \int_W \rho(\mathbf{r}) \, d\mu_D(\mathbf{r}),$$

where μ_D is a new measure of the space E^3. Using Eq. (8), we have

$$M(W) = \rho_0 \int_W d\mu_D(\mathbf{r}) = \rho_0 V_D(W),$$

where $V_D(W)$ is the volume of the region W. Therefore we get that the homogeneous fractal media are described by the measure space (E, μ) such that

$$V_D(W) = \frac{\rho_0}{M_0} \left(\frac{V(W)}{R_p} \right)^{D/3},$$

where $V(W)$ is the usual volume of the region W, i.e., we have $V_D(W) \sim (V(W))^{D/3}$. If we consider the ball region, we get

$$V_D(W) = V_0(R/R_p)^D$$

where $D < 3$, and $V_0 = (4\pi/3)^{D/3}(\rho_0/M_0)$. In the next section, we prove that the natural generalization of Eq. (4) uses the fractional integration.

5. Fractional equations

Let us prove that the fractional integration allows us to generalize Eq. (4) such that the fractality and homogeneity conditions are realized in the form

$$M(W) \sim \big(V(W) \big)^{D/3}, \qquad V_D(W) \sim \big(V(W) \big)^{D/3},$$

$\rho(\mathbf{r}) = \text{const}$

are satisfied.

Let us define the fractional integral in Euclidean space E^3 in the Rietz form [12] by the equation

$$\left(I^D \rho\right)(\mathbf{r}_0) = \gamma_3^{-1}(D) \int\limits_W \frac{\rho(\mathbf{r})\, d^3\mathbf{r}}{|\mathbf{r} - \mathbf{r}_0|^{3-D}} \qquad (9)$$

where $\mathbf{r}_0 \in W$, and we use

$$|\mathbf{r} - \mathbf{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2},$$

$$\gamma_3(D) = \frac{2^D \pi^{3/2} \Gamma(D/2)}{\Gamma(1/2)}.$$

Using $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$, we can be rewritten Eq. (9) in an equivalent form

$$\left(I^D \rho\right)(\mathbf{r}_0) = \gamma_3^{-1}(D) \int\limits_W \rho(\mathbf{R} + \mathbf{r}_0)|\mathbf{R}|^{D-3}\, d^3\mathbf{R}.$$

Note that $\gamma_3(3-0) \neq 1$, where

$$\gamma_3(3-0) = \lim_{D \to 3-0} \gamma_3(D) = \frac{2^3 \pi^{3/2} \Gamma(3/2)}{\Gamma(1/2)}.$$

Using notations (9), we can write Eq. (4) in the form

$$M(W) = \gamma_3(3-0)\left(I^3 \rho\right)(\mathbf{r}_0).$$

The fractional generalization has the form

$$M(W) = \gamma_3(3-0)\left(I^D \rho\right)(\mathbf{r}_0).$$

We use the factor $\gamma_3(3-0)$ to derive Eq. (4) in the limit $D \to 3-0$. The fractional equation for $M(W)$ can be written in the equivalent form

$$M(W) = \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)} \int\limits_W \rho(\mathbf{R} + \mathbf{r}_0)|\mathbf{R}|^{D-3}\, d^3\mathbf{R}, \qquad (10)$$

where we use

$$\gamma_3(3-0)\gamma_3^{-1}(D) = \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)}.$$

Here and later we use the initial points in the integrals are set to zero.

If we have $\rho(\mathbf{r}) = \rho_0 = \text{const}$ and the ball region W, then

$$M(W) = \rho_0 \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)} \int\limits_W |\mathbf{R}|^{D-3}\, d^3\mathbf{R}.$$

Using the spherical coordinates, we get

$$M(W) = \frac{\pi 2^{5-D} \Gamma(3/2)}{\Gamma(D/2)} \rho_0 \int\limits_W R^{D-1}\, dR$$

$$= \frac{2^{5-D} \pi \Gamma(3/2)}{D \Gamma(D/2)} \rho_0 R^D,$$

where $R = |\mathbf{R}|$. As the result, we have $M(W) \sim R^D$, i.e., we derive Eq. (7) up to the numerical factor.

We can suppose that Eq. (10) can be used for non-homogeneous fractal media. In this case, the distribution depends on $R = |\mathbf{R}|$, the angels, and the single out point, which is defined by the vector $\mathbf{r}_0$.

Let us consider the spherical-homogeneous fractal media that have the single out point. These media are defined by the following properties:

(1) For all regions W_A and W_B in the Euclidian space E^3 such that $|OA| = |OB| = R$ and R much more than the size of the regions ($R_p^3 \ll V(W_A) = V(W_B) \ll R^3$), we have the relation $\bar{\rho}(W_A) = \bar{\rho}(W_B)$.
(2) For all regions W_A such that the size of the region is much smaller than distance $|OA| = R$ ($R_p^3 \ll V(W_A) \ll R^3$), we have that the density $\bar{\rho}(W_A)$ depends on the distance $R = |OA|$.

For example, we can consider the power law in the form

$$\bar{\rho}(W_A) \sim |OA|^\beta. \qquad (11)$$

The example of the spherical-homogeneous fractal media we can be realized in nature as dendrites.

If we consider the non-homogeneous fractal media, then the local density $\rho = \rho(\mathbf{R})$ have the single out point. For the spherical symmetric case, we have $\rho(\mathbf{R}) = \rho(|\mathbf{R}|)$ and the equation can be represented in the form

$$M(W) = \frac{\pi 2^{5-D} \Gamma(3/2)}{\Gamma(D/2)} \int\limits_0^R \rho(R) R^{D-1}\, dR,$$

where $R = |\mathbf{R}|$. For the simple case (11), we have the distribution in the form

$$\rho(\mathbf{R}) = \rho\left(|\mathbf{R}|\right) = c_0 |\mathbf{R}|^\beta.$$

Therefore the mass of fractal medium in the region W is

$$M(W) = c_0 \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int\limits_W |\mathbf{R}|^{D+\beta-3} d^3\mathbf{R}.$$

Using the spherical coordinates, we have

$$M(W) = c_0 \frac{\pi 2^{5-D}\Gamma(3/2)}{\Gamma(D/2)} \int\limits_0^R R^{D+\beta-3} dR$$

$$= \frac{\pi 2^{5-D}\Gamma(3/2)}{(D+\beta)\Gamma(D/2)} R^{D+\beta}.$$

In this case, the mass-dimension is equal to $D_m = D + \beta$.

To calculate β, we can use the definition of the spherical-homogeneous fractal media and the generalization of the box counting method. This generalization must use the box regions W that satisfy the conditions $V(W) \ll R$ from the definition of spherical-homogeneous fractal media.

6. Fractional equation of continuity

The fractional integrals can be used not only to calculate the mass dimensions of fractal media. Fractional integration can be used to describe the dynamical processes in the fractal media. Using fractional integrals, we can derive the fractional generalization of dynamical equations [10]. Let us derive the fractional analog of the equation of continuity for the fractional media.

Let us consider a domain W_0 for the time $t = 0$. In the Hamilton picture we have

$$\int\limits_{W_t} \rho(\mathbf{R}_t, t) d\mu_D(\mathbf{R}_t) = \int\limits_{W_0} \rho(\mathbf{R}_0, 0) d\mu_D(\mathbf{R}_0),$$

where we use the following notation

$$d\mu_D(\mathbf{R}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{R}|^{D-3} d^3\mathbf{R}.$$

Using the replacement of variables $\mathbf{R}_t = \mathbf{R}_t(\mathbf{R}_0)$, where $\mathbf{R}_0$ is a Lagrangian variable, we get

$$\int\limits_{W_0} \rho(\mathbf{R}_t, t)|\mathbf{R}_t|^{D-3} \frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0} d\mathbf{R}_0$$

$$= \int\limits_{W_0} \rho(\mathbf{R}_0, 0)|\mathbf{R}_0|^{D-3} d\mathbf{R}_0.$$

Since W_0 is an arbitrary domain we have

$$\rho(\mathbf{R}_t, t) d\mu_D(\mathbf{R}_t) = \rho(\mathbf{R}_0, 0) d\mu_D(\mathbf{R}_0),$$

or

$$\rho(\mathbf{R}_t, t)|\mathbf{R}_t|^{D-3} \frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0} = \rho(\mathbf{R}_0, 0)|\mathbf{R}_0|_0^{D-3}.$$

Differentiating this equation in time t, we obtain

$$\frac{d\rho(\mathbf{R}_t, t)}{dt}|\mathbf{R}_t|^{D-3} \frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0}$$

$$+ \rho(\mathbf{R}_t, t)\frac{d}{dt}\left(|\mathbf{R}_t|^{D-3} \frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0}\right) = 0,$$

or

$$\frac{d\rho(\mathbf{R}_t, t)}{dt} + \Omega_D(\mathbf{R}_t, t)\rho(\mathbf{R}_t, t) = 0, \qquad (12)$$

where d/dt is a total time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\mathbf{R}_t}{dt} \frac{\partial}{\partial \mathbf{R}_t}.$$

The function

$$\Omega_D(\mathbf{R}_t, t) = \frac{d}{dt} \ln\left(|\mathbf{R}_t|^{D-3} \frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0}\right)$$

describes the velocity of volume change. Eq. (12) is a fractional Liouville equation in the Hamilton picture. Using the equation

$$\frac{d\mathbf{R}_t}{dt} = \mathbf{V}_t(x),$$

then the function Ω_D is defined by

$$\Omega_D(\mathbf{R}_t, t) = \frac{d}{dt}\left(\ln|\mathbf{R}_t|^{D-3} + \ln\frac{\partial \mathbf{R}_t}{\partial \mathbf{R}_0}\right)$$

$$= (D-3)\frac{1}{|\mathbf{R}_t|} \frac{d|\mathbf{R}_t|}{dt} + \frac{\partial}{\partial \mathbf{R}_t} \frac{d\mathbf{R}_t}{dt}.$$

As the result we have

$$\Omega_D(\mathbf{R}_t, t) = \frac{(D-3)(\mathbf{R}_t, \mathbf{V}_t)}{|\mathbf{R}_t|^2} + \frac{\partial \mathbf{V}_t}{\partial \mathbf{R}_t},$$

where we use $\partial|\mathbf{R}_t|/\partial\mathbf{R}_t = \mathbf{R}_t/|\mathbf{R}_t|$.

In the general case ($D \neq 3$), the function Ω_D is not equal to zero ($\Omega_D \neq 0$) for solenoidal velocity

fields (div $\mathbf{V} = 0$). If $D = 3$, we have $\Omega_D \neq 0$ only for non-solenoidal velocity field. Therefore the flow in the fractional medium is similar to the non-solenoidal flow.

As the result, we have the following equation of continuity for fractal media:

$$\frac{\partial \rho(\mathbf{R}_t, t)}{\partial t} + \mathbf{V}_t \frac{\partial \rho(\mathbf{R}_t, t)}{\partial \mathbf{R}_t} + \frac{(D-3)(\mathbf{R}_t, \mathbf{V}_t)}{|\mathbf{R}_t|^2} \rho(\mathbf{R}_t, t)$$
$$+ \rho(\mathbf{R}_t, t) \frac{\partial \mathbf{V}_t}{\partial \mathbf{R}_t} = 0.$$

For the homogemeous media, we have $\rho = \text{const}$ and the equation of continuity leads us to the equation

$$\frac{(D-3)(\mathbf{R}_t, \mathbf{V}_t)}{|\mathbf{R}_t|^2} \rho(\mathbf{R}_t, t) + \rho(\mathbf{R}_t, t) \frac{\partial \mathbf{V}_t}{\partial \mathbf{R}_t} = 0.$$

Therefore, we get the non-solenoidal flow of the velocity $\text{div}_{\mathbf{R}} \mathbf{V} \neq 0$.

7. Conclusion

The application of fractals in physics is far ranging, from the dynamics of fluids in porous media to resistivity networks in electronics. The cornerstone of fractals is the meaning of dimension, specifically the fractal dimension. Fractal dimension can be best calculated by box counting method which means drawing a box of size R and counting the mass inside. When we graph $\ln(M)$ as a function of $\ln(R)$ we get a value of about D_m which is the fractal dimension of fractal media. The mass fractal dimensions of the media can be easy measured by experiments. The experimental determination of the mass fractal dimension can be realized by the usual box counting methods. The mass dimensions were measured for porous materials [15,16], polymers [17], and colloid aggregates [18]. For the porous materials the fractal dimension is proportional to porosity of the medium. Fractal models of porous media are enjoying considerable popularity [20–25]. This is due in part to the relatively small number of parameters that can define a fractal porous medium of great complexity and rich structure.

We suppose that the concept of fractional integration provides an alternative approach to describe the fractal media. In this Letter we prove that mass fractal media can be described by using the fractional integrals. We consider the fractional generalizations of the equation that defines the medium mass. This fractional generalization allows us to realize the consistent description of the fractal mass dimension of the media.

The fractional integrals can be used in order to describe the dynamical processes in the fractal media. The physics of fractal media can be successful described by the fractional integration. Fractional integration approach is potentially more useful for physics of fractal media than traditional methods that use the integer integration.

Note that the Liouville equation is a cornerstone of the statistical mechanics [26–28]. The fractional generalization of the Liouville equation was suggested in Ref. [10]. This fractional Liouville equation was derived from the fractional generalization of the normalization condition [10] that uses fractional integrals. The fractional generalization of Liouville equation allows us to derive the fractional analogs of Bogoliubov equations [29], Fokker–Planck equation, Enskog transport equation, and hydrodynamic equations. Using the Fourier transform, we can get the fractional dynamical equations with coordinate fractional derivatives. Using the methods suggested in Refs. [30–32], we can realize the Weyl quantization of the suggested fractional equations.

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